# Quantum Metrology and Unruh Effect



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*Para Ana, Davi e Henrique.*


*Para meu pai.*

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# Abstract

The Unruh effect is one of the most important results from quantum field theory in curved spacetimes. It states that, if we have a field in vacuum state in Minkoski spacetime, while an inertial observer detects no particle, an accelerated observer in the same spacetime would detect a thermal bath with temperature proportional to its acceleration. However, it is necessary an acceleration of order of $10^{20} m/s^2$ so we can observe a temperature of order $1K$. We can not detect the Unruh effect with current technology. An experiment to verify Unruh effect would be an important milestone in quantum field theory experiments. As important as verifying the existence of the effect, is verifying if the theory predicts the relation between temperature and acceleration correctly. Therefore, temperature estimation precision is a main concern for any experimental scheme to test the Unruh effect. Quantum metrology is a field of physics dedicated to exploit quantum resources, as quantum coherence and quantum entanglement, to improve parameter estimation in quantum systems. In this thesis we investigate the estimation of Unruh temperature in the context of relativistic quantum metrology. We study two cases: $(i)$ one uniformly accelerated Unruh-DeWitt detector which undergoes acceleration for a finite amount of proper time, so we can investigate the influence of coherence in temperature estimation precision; and $(ii)$ two entangled Unruh-DeWitt detectors, one inertial and the other uniformly accelerated, so we can study the role of entanglement of the ancilla in the analogous estimation process. We find that in both cases the possible precision of estimation takes its maximum for accelerations of order of units of detector energy gap, the signal-to-noise ratio takes its maximum for accelerations between two and three orders greater than detector energy gap and that the quantum coherence (or entanglement of the ancilla) are not quantum resources for Unruh temperature precision estimation. Our results represent an additional challenge to Unruh effect experimental verification. Moreover, we translate our results to estimation of Hawking temperature in the vicinity of a Schwarzschild black hole event horizon. We obtain the suitable Unruh-DeWitt detector energy gap so one can realize trustworthy temperature measurement in this context.

**Keywords**: Quantum Metrology, Unruh Effect, Quantum Information, Quantum Fisher Information, Relativistic Quantum Metrology, Relativistic Quantum Information, Unruh Temperature Estimation Precision.

# Resumo

O efeito Unruh é um dos principais resultados vindos da teoria quântica de campos em espaços curvos. Ele diz que, se tivermos um campo no estado de vácuo no espaçotempo de Minkowski, enquanto um observador inercial não detecta nenhuma partícula, um observador acelerado no mesmo espaçotempo detecta um banho térmico de temperatura proporcional a sua aceleração. No entanto é necessária uma aceleração da ordem de $10^{20}m/s^2$ para que seja observada uma temperatura de Unruh da ordem de $1K$. Não somos capazes de detectar o efeito Unruh com a tecnologia atual. Um experimento capaz de verificar o efeito Unruh seria um marco importante entre os experimentos sobre teoria quântica de campos. Tão importante quanto verificar a existência do efeito, é verificar se a teoria prevê a relação entre aceleração e temperatura corretamente. Portanto, a precisão da estimativa da temperature é uma questão central para qualquer esquema experimental que visa testar o efeito Unruh. Metrologia quântica é um campo da física que se dedica a explorar recursos quânticos, como coerência e emaranhamento, para melhorar a estimativa de parâmetros em sistemas quânticos. Nesta tese investigamos a estimativa da temperatura de Unruh no contexto de metrologia quântica relativistica. Nós estudamos dois casos: $(i)$ um detector de Unruh-DeWitt uniformemente acelerado por um tempo finito, de forma que possamos investigar a influência da coerência na precisão da estimativa de temperatura; e $(ii)$ dois detectores de Unruh-DeWitt emaranhados, um inercial e outro uniformemente acelerado, para que possamos estudar o papel do emaranhamento da ancilla no processo análogo de estimativa . Em ambos os casos, descobrimos que a precisão possível da estimativa possui valor máximo para acelerações da ordem de unidades do *gap* de energia do detector, a razão sinal-ruído toma seu valor máximo entre duas e três ordens de grandeza maiores que a do *gap* de energia do detector e que a coerência quântica (ou emaranhamento da ancilla) não contituem recursos quânticos para a precisão da estimativa da temperatura de Unruh. Nossos resultados representam um desafio adicional para a verificação experimental do efeito Unruh. Além disso, traduzimos nossos resultados para a estimativa da temperatura de Hawking nas proximidades de um horizonte de eventos de um buraco negro de Schwarzchild. Obtemos o *gap* de energia adequado do detector de Unruh-DeWitt para que se possa realizar medições confiáveis de temperatura neste contexto.

**Palavras chave**: Metrologia Quântica, Efeito Unruh, Informação Quântica, Informação de Fisher Quântica, Metrologia Quântica Relativística, Informação Quântica Relativística, Estimativa de Precisão da Temperatura de Unruh.

# Contents

# Chapter 1

# Introduction

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Quantum field theory in curved spacetime (QFTCS) [1–3] is the most conservative attempt to obtain low-energy quantum gravity insight. It is a theory of quantum fields constructed in a classical relativistic spacetime background which gives us the only results we have for quantum gravity in low-energy regime. The two main results one could obtain from QFTCS is the Hawking radiation [5,6] and the Unruh effect [7,8]. The Hawking radiation phenomenon states that black holes create and emit particles at the rate of a body with temperature $\kappa/(2\pi)$, where $\kappa$ denotes the surface gravity [4] (we use natural units along all this thesis $c = \hbar = K_B = G = 1$). The Unruh effect was discovered by Bill Unruh in 1976 in an attempt to gain insight about Hawking radiation. It states that an observer with constant acceleration in Minkowski vacuum will see himself immersed in a thermal bath. One of the main consequences of Unruh effect is that it shows that the concept of particle is observer dependent.

Some physicists contest the existence of the Unruh effect. In Ref. [9] Belinskiĭ *et al.* claim that, in the original derivation of Unruh effect, the quantization of the Rindler modes assumes vanishing fields at the boundaries of Rindler manifold and that would lead to the problem of quantizing a field with Hamiltonian which is not equivalent to quantizing the free field in Minkowski spacetime. In Ref. [10] Narozhy *et al.* claim the Unruh quantization procedure sets a boundary equivalent to cutting off the origin of Minkowski spacetime, changing the spacetime topology and symmetry group, and prohibiting any mode that crosses that point. This implies that procedures would be based in an incomplete set of modes in Minkowski spacetime and, then, the Unruh effect would come from a non rigorous derivation. In Ref. [11] Ford and O'Connell claim that a detector uniformly accelerated in vacuum thermalizes at Unruh temperature but do not irradiate. In Ref. [12] Buchholz and Verch claim that macroscopically, an accelerated observer will not realize a measurement of any global thermal effects in Minkowski vacuum. Authors also claim that microscopic probes used as Unruh temperature detectors response to acceleration is not due to Unruh effect but it is induced by measurement process. However, last reference is contradicted by Ref. [13] where Lima *et al.* claim that an extended body uniformly accelerated in Minkowski vacuum evolves to a Gibbs thermal state with local

temperature given by the Unruh temperature.

The above challenges to the Unruh effect existence are based in the claim that the set of modes chosen by Unruh in his original derivation is not a complete set of modes. This critic does not apply, however, to the derivation obtained by Sewell [14] using axiomatic quantum field theory, based in Bisognano-Whichmann theorem [15, 16]. It does not apply to the derivation of Unruh effect obtained by Unruh and Weiss [17] by Feynman integral formulation neither. Moreover, Unruh and Wald have demonstrated in 1984 [18] that the Unruh effect keeps the consistency of field theory in uniformly accelerated frames solving apparent paradoxes regarding energy conservation and causality. In this scenario, Unruh effect stands solid as a credible quantum field theory prediction.

The Unruh effect was not experimentally verified since it predicts it is necessary an acceleration of order of $10^{20}m/s^2$ so we can observe a temperature of order $1K$. The engineering required to obtain linear acceleration to reach a detectable temperature was not achieved yet. Still, it is important to invest energy and resources to reach the Unruh effect experimental verification. Besides settle any possible debate about the effect's existence, an experimental verification of the Unruh effect would be an important experiment of quantum field theory fundamentals, would change the Unruh effect status from theoretical prediction to (dis)confirmed phenomenon, it would increase the role of observations we have from nature and would give us more data we can work on.

Despite the engineering challenge involved with the Unruh effect experimental verification, there are some proposals for the Unruh effect observation. In Refs. [19, 20] Bell and Leinaas study the depolarization of electrons in magnetic field to estimate Unruh temperature. They conclude this effect is not observable for linear acceleration real experiments but it can be observed in ideal storage rings. In Ref. [21] Rogers proposes to detect radiation due to Unruh effect in accelerated electrons orbiting in a Penning trap. Such proposes involving electrons constrained to rings have the problem that you can not formally derive Unruh effect in these cases since the effect is derived for accelerated observers for whom we can associate a time translation symmetry. This is not possible for circularly moving observers. In Ref. [22] Chen and Tajima claim it is possible to detect Unruh radiation through electrons accelerated by high-intensity lasers. In Ref. [23] Schützhold, Schaller and Habs show that, for electrons accelerated by electric field of ultra-strong lasers, the resulting Unruh effect radiation consists of maximally entangled photons pairs. They conclude the Unruh effect radiation can be discriminated from the classical (Larmor) radiation due to different spectral and angular distributions. However such ultra-intense lasers were not realized yet and such observations can not be made. In Ref. [24] Matsas and Vanzella study the decay of protons and neutrons in accelerated regimes and conclude that dominant processes of high accelerated protons and neutrons are different from inertial and small accelerated ones. However the lifetime of protons in real accelerators is still very long and such observation is not possible. In Ref. [25] Scully *et al.* propose an experiment to detect Unruh radiation using ground state atoms in cavities. In Ref. [26] Martín-Martínez,

Fuentes and Mann claim accelerated detectors, under certain conditions, obtain a Berry phase which depends on Unruh temperature and differs from Berry phase for inertial detectors in contact with thermal baths. They claim this effect can be observed for accelerations of $10^9$ times smaller than usual proposals to detect Unruh temperature by linearly accelerated detectors and the acceleration would need to be sustained only by times of nanoseconds. However engineering needed to implement proposes of Refs. [25] and [26] are not available yet. In Ref. [27] Cozzella *et al.* finds dependency over Unruh temperature of long wavelengths radiation spectrum from circularly moving charges accelerated in direction perpendicular to circular trajectory plane. This is interesting because the authors propose to verify Unruh effect using standard classical electrodynamics. In Ref. [28] Hu *et al.* implement a simulation of Unruh effect using cesium Bose-Einstein condensate and observe data which agrees with Unruh prediction.

Despite the fact the proposes above are very distinct from each other, they have in common the fact that non of them regard the possible precision of Unruh temperature estimation. About estimation precision, there is a whole estimation theory concerned with the problem of estimation of parameters [29–34]. The estimation protocol has four steps: (1) preparation of probe state; (2) interaction of probe state with the system of interest; (3) measurement of the probe; (4) decode information about the parameter of interest from measured data. Related to estimation theory there is the field of physics known as quantum metrology [35] which is dedicated to employ quantum resources as entanglement and coherence to enhance precision in estimation of parameters from quantum systems. We shall highlight some achievements from the field. In Ref. [36] Giovannetti, Lloyd and Marccone show how to employ quantum squeezing and quantum entanglement to beat standard quantum limit. In Ref. [37] Micadei *et al.* show that measuring probes simultaneously in entangled basis and measuring probes sequentially hold the same precision for classically correlated probes while the simultaneous measurement holds better precision for noisy multipartite quantum systems. Quantum metrology also provided developments in quantum illumination. This is a technique which consists in illuminating objects of interest with entangled light in order to enhance detecting and imaging those objects. In Ref. [38] Lloyd shows that if entangled light is used for photodetection, when one employ a sign with $n$ bits of entanglement it is possible to increase the effective signal-to-noise ratio by a factor of $2^n$. In Ref. [39] Aguilar *et al.* implement an experimental scheme for quantum target detection by linear optical measurement devices. They found that polarization-entangled photon pairs improve over separable polarization states when the signal to noise ratio is greater than $1/40$. In Ref. [40] Barzanjeh *et al.* implement an experiment demonstrating quantum illumination at microwave frequencies for a room-temperature object. We also may highlight the role of quantum metrology in quantum thermometry [41]. This field concerns about controlling and measuring the temperature systems which operates in quantum regime and search for employments of quantum entanglement and quantum coherence to enhance temperature measurement. In Ref. [42] Cavina *et al.* generalize, for single-qubit thermometry, the relation between temperature uncertainty and the heat capacity of the qubit in a non-equilibrium

regime using quantum properties as coherence. We also highlight developments of quantum metrology in gravitational effects in quantum systems. In Ref. [43] Pang and Cheng develop a quantum treatment of interactions between gravitational waves and optomechanical detectors. In Ref. [44] Xu *et al.* report the result of an experiment in which they create a pair of entangled photons, send one to the quantum satellite Micius and retain the other on Earth. They find no decorrelation between the pair of photons. The results contradict a quantum gravity theory called event formalism.

As we can see, there is a wide range of applications for quantum metrology. Going back to discussion about Unruh effect detection, we should emphasize that, not only the detection of Unruh effect is important, but the precise estimation of Unruh temperature is needed, since we should verify if QFTCS gives the right prediction about its value. In this scenario, it seems a good strategy to apply quantum metrology protocols to investigate Unruh temperature estimation. In this context we developed the studies which based this PhD thesis. We investigate the precision we can estimate Unruh temperature. We do so by analyzing an important quantity for estimation theory: the quantum Fisher information [45] in the Unruh effect context.

Some studies about quantum metrology of Unruh effect were already realized. In Ref. [46] Tian *et al.* use master equation approach for open quantum systems to solve the dynamics of the state of the system. They calculate Fisher information for the population measurement and compare it to quantum Fisher information to obtain the optimal population measurement according to optimal quantum measurement condition. The authors concluded that, in the long time regime, the precision of the estimation of the temperature does not depend on the initial state preparation. In Ref. [47] Tian *et al.* study the evolution of a detector treated as an open quantum system in two distinct cases: inertial and uniformly accelerated. The authors compared the evolved states of each case and concluded that when the probe is initially entangled with a static detector they could enhance the distinguishability between the two distinct cases: the evolution of a static detector and the evolution of accelerated probe. In Ref. [48], Wang *et al.* studied the dependency of the detector energy gap and the strength of the interaction between the detector and the scalar field for the precision in the estimation of the Unruh temperature. The authors derive quantum Fisher information and studied the role of entanglement, concluding that it helps improving the precision of Unruh temperature estimation.

We want to go forward on development of quantum metrology for Unruh effect and answer the question: Can we utilize quantum entanglement and quantum coherence to enhance Unruh temperature estimation precision? To provide this answer we utilize Unruh-DeWitt detectors in flat spacetime couped to an external massless field in vacuum state and investigate two distinct cases: one accelerated detector in order to search for a relation between coherence and quantum Fisher information of detector; and the case of two detectors, one uniformly accelerated, one inertial, so we can study the influence of entanglement on temperature estimation precision. At last, we translate our results for the estimation of Hawking temperature at the vicinity of a black hole event horizon.

We find that quantum coherence of initial state of the single probe and quantum entanglement of initial state of the pair of probes play no role in precision of Unruh temperature. The most interesting result we find is a relation between maximum sign-to-noise ratio acceleration value and entanglement sudden death acceleration value [49]. When the pair of detectors are maximally entangled, the maximum sign-to-noise ratio acceleration value coincides with the entanglement sudden death acceleration value.

This thesis is organized in the following way: in the first chapter we present the Unruh Effect. We discuss basic theory of the free scalar field in curved spacetimes, we talk about Bogoliubov transformations, we present the Rindler spacetime and finally, we present the Unruh effect derivation. In the second chapter we present some metrology quantities which are going to be useful for us. We discuss classical and quantum Fisher information and sign-to-noise ratio. In the third and last chapter we present our original results. We first describe our detector model, and then we show the relation between precision and coherence, the relation between precision and entanglement, and finally, the precision of Hawking radiation estimation in the vicinity of a black hole event horizon.

# Chapter 2

# Unruh Effect

In this chapter we present the Unruh effect. In section 2.1 we present the basic formalism of quantum field theory in curved spacetime. In section 2.2 we present how to obtain the relation between different choices of positive frequency modes of Klein-Gordon equation. In section 2.3 we present the coordinate system adapted to constant acceleration. Finally we present in section 2.4 the Unruh Effect.

## 2.1 Free Scalar Fields in Curved Spacetime

Unruh effect is a phenomenon originally derived in Minkowski spacetime. However, the formalism we use to derive it lies in quantum field theory in curved spacetime. This is due to the fact that accelerated observers are better described by trajectories in Rindler spacetime, and the tools we use to change description from a quantum field theory in Minkowski to Rindler spacetime belong to quantum field theory in curved spacetime. We aim to obtain the relation between inertial (Minkowski) description of a system and the accelerated (Rindler) one. Our first step is to present in this section the theory of free scalar fields in curved spacetime. We do not pretend to make a detailed and wide exposition here. For a complete discussion we suggest [1–3].

A quantum field theory can be properly defined in spacetime with some restrictions. Our spacetime will be modeled by a 4-dimensional pseudo-riemannian manifold $(\mathcal{S}, g_{ab})$ with signature $(-+++)$. We assume $(\mathcal{S}, g_{ab})$ is time-orientable *i. e.* it is possible to define continuously a distinction of non-spacelike vectors into future-directed and past-directed vectors [50]. The problem of the differential equation with initial value in $\mathcal{S}$ must be well defined, and this holds if $(\mathcal{S}, g_{ab})$ is globally hyperbolic.

**Definition 2.1.1.** *Let $A \subset \mathcal{S}$. We denote $D(A) = \{p \in \mathcal{S} | $ every non-spacelike curve through $p$ intersects $A\}$. If there exists a hypersurface $\Sigma \subset \mathcal{S}$ such that $D(\Sigma) = \mathcal{S}$, we call $\Sigma$ a Cauchy surface and we say that $\mathcal{S}$ is* **globally hyperbolic**.

If $\mathcal{S}$ is globally hyperbolic, it is possible to define a coordinate system such that the hypersurface of constant time coordinate is a Cauchy surface [51]. If we know the initial state of a system in a given moment, then we can predict the state of the system in any time of the future or the past, and then, the problem of initial value in $\mathcal{S}$ is well defined [50].

For our purposes, it is sufficient to present the free scalar field in a static spacetime. A static spacetime is a spacetime where one can define a coordinate system $(t,\boldsymbol{x})$ which metric line element takes the form:

$$ds^2 = -g_{tt}(\boldsymbol{x})dt^2 + h_{ij}d\boldsymbol{x}^i d\boldsymbol{x}^j, \tag{2.1}$$

where $g_{tt}(\boldsymbol{x}) > 0$. Here we denote the space coordinates as bold letters and spacetime coordinates as usual letters.

Now we have established the restrictions to our spacetime, we can begin to talk about the scalar field. In spacetime $(\mathcal{S},g_{ab})$ the Klein-Gordon lagrangian density is

$$\mathcal{L} = \frac{\sqrt{-g}(\nabla_a\phi\nabla^a\phi + m^2\phi^2)}{2}. \tag{2.2}$$

Where $\nabla_a$ is the riemannian covariant derivative, $g \equiv det(g_{ab})$ and $m$ is the mass of the field. The Euler-Lagrange equations in $\mathcal{S}$ are:

$$\frac{\partial\mathcal{L}}{\partial\phi} - \nabla_a\frac{\partial\mathcal{L}}{\partial(\nabla_a\phi)} = 0, \tag{2.3}$$

which result in the Klein-Gordon (KG) equation:

$$(\nabla_a\nabla^a - m^2)\phi = 0. \tag{2.4}$$

Klein-Gordon equation describes the behaviour of the scalar field $\phi$ of mass $m$ in the spacetime $(\mathcal{S},g_{ab})$. In order to describe the space of solutions of Eq. ( 2.4), we shall look for a complete set of solutions of Klein-Gordon equation. Let $f_1(x)$ and $f_2(x)$ be two complex solutions of Eq. ( 2.4), we define a quantity in the space of solutions of Klein-Gordon equation which we will call the Klein-Gordon inner product:

$$(f_1,f_2)_{KG} \equiv i\int_{\Sigma_t} d^3\boldsymbol{x}\sqrt{h}(f_1^*\nabla_a f_2 - f_2\nabla_a f_1^*)n^a, \tag{2.5}$$

where $h \equiv det(h_{ij})$, $\Sigma_t$ is the hypersurface of constant $t$, $f_1^*$ is the complex conjugate of $f_1$ and $n^a$ is the unit normal vector to $\Sigma_t$ pointed to future. Note that Klein-Gordon inner product is not positive-definite, so call it an inner product is a abuse of nomenclature we allow ourselves because it is largely used in literature. The integral above does not depend on any specific $t$, since, using Eq. ( 2.4), we can show that the time derivative of the expression above vanishes.

Now let $\{f_i, f_i^*\}_{i \in \mathcal{I}}$ be a complete set of Klein-Gordon solutions such that:

$$\widetilde{f}_i(s, \boldsymbol{x}) = \int dt \, e^{ist} f_i(t, \boldsymbol{x}), \ \widetilde{f}_i(s < 0, \boldsymbol{x}) = 0 \tag{2.6}$$

$$(f_i, f_j)_{KG} = -(f_i^*, f_j^*)_{KG} = \delta_{ij}, \tag{2.7}$$

$$(f_i, f_j^*)_{KG} = (f_i^*, f_j)_{KG} = 0. \tag{2.8}$$

We are going to call the elements of the family $\{f_i, f_i^*\}_{i \in \mathcal{I}}$ the modes. Equation ( 2.6) shows that $f_i$ are the positive frequency solutions, so $f_i^*$ are negative frequency solutions. Equation ( 2.7) states that the family $\{f_i, f_i^*\}_{i \in \mathcal{I}}$ is normalized and Eq. ( 2.8) states that positive frequency solutions are orthogonal to negative frequency solution relative to Klein-Gordon inner product. We assume that the set of labels $\mathcal{I}$ is discrete, but the generalization to continuous case is straightforward.

The set $\{f_i\}$ endowed with KG inner product $(\cdot, \cdot)_{KG}$ spans the Hilbert space $\mathcal{H}_{KG}$. Also the set of $f_i^*$ endowed with $-(\cdot, \cdot)_{KG}$ spans the Hilbert space $\overline{\mathcal{H}}_{KG}$ [3]. Our particle notion is going to be related to states in the Fock space

$$\mathcal{F}(\mathcal{H}_{KG}) = \bigoplus_{i=0}^{\infty} \left[ S \left( \bigotimes_{j=0}^{i} \mathcal{H}_{KG} \right) \right], \tag{2.9}$$

where S denotes the simmetrization operator meaning that $S \left( \bigotimes_{j=0}^{i} \mathcal{H}_{KG} \right)$ denotes the space of completely symmetric tensor of rank $i$ of $\mathcal{H}_{KG}$.

Now we have defined the Hilbert space $\mathcal{H}_{KG}$ and its associated Fock space $\mathcal{F}(\mathcal{H}_{KG})$, we can move forward and quantize the Klein Gordon field. First step is to define the conjugate momentum of a field. Given a solution $\phi$ to Eq. ( 2.4), we define the conjugate momentum of the field as:

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \frac{\sqrt{h} \partial_t \phi}{\sqrt{g_{tt}}}. \tag{2.10}$$

The quantization procedure consists of associating to $\phi$ and $\pi$ the operators $\widehat{\phi}$ and $\widehat{\pi}$ that acts on $\mathcal{F}(\mathcal{H}_{KG})$ and must satisfies the canonical equal-time commutation relations:

$$\left[ \widehat{\phi}(t, \boldsymbol{x}), \widehat{\pi}(t, \boldsymbol{x}') \right] = i \delta(\boldsymbol{x}, \boldsymbol{x}') \tag{2.11}$$

$$\left[ \widehat{\phi}(t, \boldsymbol{x}), \widehat{\phi}(t, \boldsymbol{x}') \right] = [\widehat{\pi}(t, \boldsymbol{x}), \widehat{\pi}(t, \boldsymbol{x}')] = 0. \tag{2.12}$$

where the $\delta$ is the 3-dimensional Dirac function.

Since $\{f_i, f_i^*\}_{i \in \mathcal{I}}$ is an orthonormal base of the space of solutions of Klein-Gordon equation, we can expand any solution $\phi$ of Eq. ( 2.4) in terms of $f_i$ and $f_i^*$. After the quantization procedure we can still employ $\{f_i, f_i^*\}_{i \in \mathcal{I}}$ to expand the field operator $\widehat{\phi}$ using the the operators

$a$ and $a^\dagger$ which are called annihilation and creation operators respectively [1,2]:

$$\widehat{\phi}(x) = \sum_{i \in \mathcal{I}} a(f_i^*)f_i + a^\dagger(f_i)f_i^*. \tag{2.13}$$

From the above expansion we obtain:

$$a(f_i^*) = (f_i,\widehat{\phi})_{KG}, \ a^\dagger(f_i) = (\widehat{\phi},f_i^*)_{KG}. \tag{2.14}$$

The canonical commutations relations in Eqs. ( 2.11) and ( 2.12) can be written in terms of operators $a$ and $a^\dagger$ as

$$\left[a(f_i^*),a^\dagger(f_j)\right] = (f_i,f_j)_{KG}\mathbb{1} = \delta_{ij}\mathbb{1}, \tag{2.15}$$

$$\left[a(f_i^*),a(f_j^*)\right] = \left[a^\dagger(f_i),a^\dagger(f_j)\right] = 0, \tag{2.16}$$

where $\mathbb{1}$ denotes the identity operator on $\mathcal{F}(\mathcal{H}_{KG})$.

The vacuum state $|0\rangle$ of $\mathcal{F}(\mathcal{H}_{KG})$ is defined as the state that $a(f_i^*)|0\rangle = 0$ for all modes $f_i$. Note that the definition of the vacuum depends on the modes.

Now, let's say we have two different observers, $A$ and $B$, of the same system modeled by a scalar quantum field. Observer $A$ chooses a complete set of orthonormal modes $\{f_i,f_i^*\}_{i \in \mathcal{I}}$ and $B$ describes the system with a different complete set of modes $\{b_j,b_j^*\}_{j \in \mathcal{J}}$. Given a particular state of the system, as $A$ will describe the state in terms of $\{f_i,f_i^*\}$ while $B$ describes it in terms of $\{b_j,b_j^*\}$, $A$ and $B$ may interpret the same state differently. In particular, what is the vacuum state for $A$ may be interpreted not as a vacuum by $B$. This is the basis of Unruh Effect. To understand properly the relations between different complete sets of modes we need to present Bogoliubov transformations. We will discuss it in next section.

## 2.2   Bogoliubov Transformations

Let $\{f_i,f_i^*\}_{i \in \mathcal{I}}$ and $\{b_j,b_j^*\}_{j \in \mathcal{J}}$ be two different complete sets of orthonormal modes of solution space $\mathbb{S} = \mathcal{H}_{KG} \oplus \overline{\mathcal{H}}_{KG}$. Since they are both complete we can write:

$$b_j = \sum_{i \in \mathcal{I}} \alpha_{ji}f_i + \beta_{ji}f_i^*, \tag{2.17}$$

$$b_j^* = \sum_{i \in \mathcal{I}} \alpha_{ji}^*f_i^* + \beta_{ji}^*f_i. \tag{2.18}$$

But, it's also true that:

$$f_i = \sum_{j \in \mathcal{J}} (b_j, f_i)_{KG} \, b_j - (b_j^*, f_i)_{KG} \, b_j^*, \tag{2.19}$$

$$f_i^* = \sum_{j \in \mathcal{J}} -(b_j^*, f_i^*)_{KG} \, b_j^* + (b_j, f_i^*)_{KG} \, b_j. \tag{2.20}$$

We get from Eq. ( 2.17), Eq. ( 2.18) and from the property $(f_1^*, f_2^*)_{KG} = -(f_2, f_1)_{KG}$:

$$\alpha_{ji} = (f_i, b_j)_{KG} = (b_j, f_i)_{KG}^*, \tag{2.21}$$

$$\beta_{ji} = (b_j^*, f_i)_{KG} = -(b_j, f_i^*)_{KG}^*. \tag{2.22}$$

Combining the above expressions with Eqs. ( 2.19) and ( 2.20), we get:

$$f_i = \sum_{j \in \mathcal{J}} \alpha_{ji}^* b_j - \beta_{ji} b_j^*, \tag{2.23}$$

$$f_i^* = \sum_{j \in \mathcal{J}} \alpha_{ji} b_j^* - \beta_{ji}^* \, b_j \tag{2.24}$$

We can expand field operator as

$$\widehat{\phi} = \sum_{i \in \mathcal{I}} a(f_i^*) f_i + a^\dagger(f_i) f_i^* = \sum_{j \in \mathcal{J}} a(b_j^*) b_j + a^\dagger(b_j) b_j^*. \tag{2.25}$$

Combining the above expression with Eqs. ( 2.17) and ( 2.18), we get:

$$\sum_{i \in \mathcal{I}} a(f_i^*) f_i + a^\dagger(f_i) f_i^* =$$

$$= \sum_{j \in \mathcal{J}} a(b_j^*) \left( \sum_{i \in \mathcal{I}} \alpha_{ji} f_i + \beta_{ji} f_i^* \right) + a^\dagger(b_j) \left( \sum_{i \in \mathcal{I}} \alpha_{ji}^* f_i^* + \beta_{ji}^* f_i \right)$$

$$= \sum_{j \in \mathcal{J}} \sum_{i \in \mathcal{I}} \left( a(b_j^*) \alpha_{ji} + a^\dagger(b_j) \beta_{ji}^* \right) f_i + \left( a(b_j^*) \beta_{ji} + a^\dagger(b_j) \alpha_{ji}^* \right) f_i^*$$

$$= \sum_{i \in \mathcal{I}} \left( \sum_{j \in \mathcal{J}} a(b_j^*) \alpha_{ji} + a^\dagger(b_j) \beta_{ji}^* \right) f_i + \left( \sum_{j \in \mathcal{J}} a(b_j^*) \beta_{ji} + a^\dagger(b_j) \alpha_{ji}^* \right) f_i^*. \tag{2.26}$$

and by comparison we conclude that:

$$a(f_i^*) = \sum_{j \in \mathcal{J}} \alpha_{ji} a(b_j^*) + \beta_{ji}^* a^\dagger(b_j), \tag{2.27}$$

$$a^\dagger(f_i) = \sum_{j \in \mathcal{J}} \beta_{ji} a(b_j^*) + \alpha_{ji}^* a^\dagger(b_j). \tag{2.28}$$

By analogous calculations we obtain:

$$a(b_j^*) = \sum_{i \in \mathcal{I}} \alpha_{ji}^* a(f_i^*) - \beta_{ji}^* a^\dagger(f_i), \tag{2.29}$$

$$a^\dagger(b_j) = \sum_{i \in \mathcal{I}} -\beta_{ji} a(f_i^*) + \alpha_{ji} a^\dagger(f_i). \tag{2.30}$$

The transformations given in Eqs. ( 2.27) to ( 2.30) are called *Bogoliubov transformations*. They show how annihilation and creation operators from different complete set of orthonormal modes of a same space are related.

We define the vacuum states of each orthonormal positive frequency modes:

$$a(f_i^*)|0_f\rangle = 0 \; \forall i \in \mathcal{I}, \tag{2.31}$$

$$a(b_i^*)|0_b\rangle = 0 \; \forall j \in \mathcal{J}. \tag{2.32}$$

Let $N_i^f = a^\dagger(f_i)a(f_i^*)$ and $N_j^b = a^\dagger(b_j)a(b_j^*)$ be the number operator to each family $\{f_i, f_i^*\}_{i \in \mathcal{I}}$ and $\{b_j, b_j^*\}_{j \in \mathcal{J}}$. Naturally, expectation value of $N_i^f$ in $|0_f\rangle$ vanishes just like $N_j^b$ in $|0_b\rangle$. However we find:

$$\langle 0_b | N_i^f | 0_b \rangle = \sum_{j \in \mathcal{J}} |\beta_{ji}|^2, \tag{2.33}$$

$$\langle 0_f | N_j^b | 0_f \rangle = \sum_{i \in \mathcal{I}} |\beta_{ji}|^2. \tag{2.34}$$

Equations ( 2.33) and ( 2.34) show that the vacuum state of an observer who defines $\{f_i\}_{i \in \mathcal{I}}$ as his positive frequency modes may not be "empty" from the point of view of an observer that defines $\{b_j\}_{j \in \mathcal{J}}$ as his positive frequency modes and vice versa. Thats the core of Unruh effect. An accelerated observer does not see the inertial vacuum as an empty space because he defines his positive frequency modes differently from an inertial observer. In the next section we show how an accelerated system can be described using Rindler coordinates.

## 2.3 Rindler Coordinates

In this section we present Rindler spacetime. It provides a very suitable way to describe accelerated worldlines in Minkowski spacetime. As we will see, there are Rindler coordinate systems in which trajectories of constant space coordinates describe timelike curves in Minkowski spacetime with constant proper acceleration. Let $(t,x,y,z)$ be Minkowski coordinates. Minkowski metric in these coordinates is given by:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2. \tag{2.35}$$

Consider the following coordinate transformations:

$$t(\tau,\xi,y,z) = \frac{1}{a}e^{a\xi}\sinh a\tau,$$

$$x(\tau,\xi,y,z) = \frac{1}{a}e^{a\xi}\cosh a\tau,$$

$$y(\tau,\xi,y,z) = y, \ z(\tau,\xi,y,z) = z, \tag{2.36}$$

where $a$ is a fixed parameter that has a physical meaning which we will discuss later. Notice that the new coordinates cover only the region that satisfies $x > |t|$. This region is called right Rindler wedge, or, as we'll call it, wedge I. If we cover the region $x < -|t|$ with the coordinates:

$$t(\widetilde{\tau},\widetilde{\xi},\widetilde{y},\widetilde{z}) = \frac{1}{a}e^{a\widetilde{\xi}}\sinh a\widetilde{\tau},$$

$$x(\widetilde{\tau},\widetilde{\xi},\widetilde{y},\widetilde{z}) = -\frac{1}{a}e^{a\widetilde{\xi}}\cosh a\widetilde{\tau},$$

$$y(\widetilde{\tau},\widetilde{\xi},\widetilde{y},\widetilde{z}) = \widetilde{y}, \ z(\widetilde{\tau},\widetilde{\xi},\widetilde{y},\widetilde{z}) = \widetilde{z}, \tag{2.37}$$

then, we describe the region $|x| > |t|$. Trajectories of fixed $\xi$, $y$ and $z$ are integral curves of the Killing vector field $B = a(x\partial_t + t\partial_x)$ while trajectories of fixed $\widetilde{\xi}$, $\widetilde{y}$ and $\widetilde{z}$ of coordinates given in Eq. ( 2.37) are integral curves of the Killing vector field $\widetilde{B} = -a(x\partial_t + t\partial_x)$. These trajectories are hyperboles, since:

$$x^2 - t^2 = \frac{e^{2a\xi}}{a^2}. \tag{2.38}$$
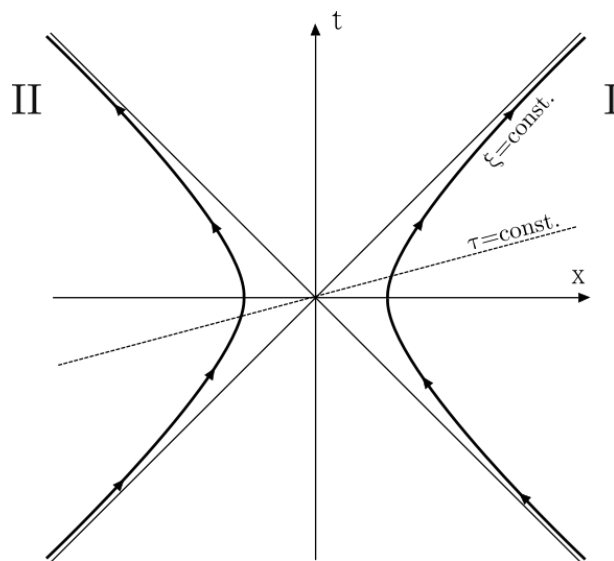


**Figure 2.1** | The hyperbole is the trajectory of a constant $\xi$ while curves of constant $\tau$ are straight lines through the origin. Trajectories of constant $\xi$ correspond to worldlines of points that move with constant proper acceleration.

The metric tensor field of wedge I in coordinates given in Eqs. ( 2.36) is:

$$ds^2 = e^{2a\xi}(-d\tau^2 + d\xi^2) + dy^2 + dz^2. \tag{2.39}$$

In wedge II we get the same expression but with coordinates with tilde. For shorthand, in some moments, we won't differentiate both coordinate systems but difference will be clear by context. The metric is static in both wedges. Curves of constant $\xi$ are timelike in wedges I and II and spacelike in the other 2 wedges corresponding to region $|x| < |t|$. Wedges I and II are causally disconnected and each one of them is globally hyperbolic.

Since the proper time is $\mathfrak{t} = e^{a\xi}\tau$, the four-velocity, in Minkowski coordinates, is

$$u^i = (\pm\cosh a\tau, \pm\sinh a\tau, 0, 0). \tag{2.40}$$

It satisfies condition $u^i u_i = -1$. Their proper four-acceleration is

$$u^i\nabla_i u^j = (\frac{a}{e^{a\xi}}\sinh a\tau, \frac{a}{e^{a\xi}}\cosh a\tau, 0, 0), \tag{2.41}$$

which has magnitude $\frac{a}{e^{a\xi}}$. Then, for $\xi = 0$, the hyperboles describe accelerated points in Minkowski spacetime with proper constant acceleration $a$.

Therefore, suppose we solve Klein-Gordon equation in Minkowski spacetime and in Rindler spacetime, define in each case a complete set of orthonormal modes and find the Bogoliubov transformations which relates them. As Rindler coordinates with $\xi = 0$ describes worldlines of uniformly accelerated observers with proper acceleration $a$, what we get from this specific Bogoliubov transformations is how to interpret observations made by an inertial observer from the point of view of an accelerated one. In particular, we will be able to interpret the vacuum seen by an inertial observer from the point of view of an accelerated observer, and from this system will emerge Unruh effect, which we will discuss in next section.

## 2.4 The Unruh Effect

The Unruh effect is a phenomenon discovered by Bill Unruh in 1976 [7] in an attempt to gain insight on recently discovered Hawking radiation [5,6]. The phenomenon lies in the distinct perceptions different observers can have of a same quantum field state. In particular, it is about how an uniformly accelerated observer perceives a vacuum state seen by an inertial observer. We will show that, while an inertial observer just sees vacuum, the accelerated observer sees himself immersed in a thermal bath. We will present this effect in details in this section.

First step to reach Unruh effect is to solve Klein-Gordon equation and define the complete set of orthonormal modes for different observers. We begin by the inertial observer considering Klein-Gordon equation in Minkowski spacetime and, then, we consider the same equation in Rindler spacetime and, in this case the complete set of orthonormal modes will be related to the

accelerated observer.

Consider the massless Klein-Gordon equation:

$$\nabla^i \nabla_i \phi = 0. \tag{2.42}$$

The natural choice of orthonormal modes in Minkowski spacetime is [74, 75] $\{v_{\boldsymbol{k}}\}$, where

$$v_{\boldsymbol{k}} = \frac{1}{\sqrt{16\pi^3 k_0}} \mathrm{e}^{i(\boldsymbol{k}\boldsymbol{x} - k_0 t)}, \tag{2.43}$$

and $k_0 = |\boldsymbol{k}|$.

Meanwhile, in order to describe constant accelerating frames we adopt the coordinates shown in Eq. ( 2.36) considering wedge I. In such coordinates Eq. ( 2.42) is written as [8]:

$$\left[ \frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \xi^2} - \mathrm{e}^{2a\xi} \left( \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \right] \phi = 0, \tag{2.44}$$

and the orthonormal complete set of solutions of positive frequency is $\{u^I_{\omega \boldsymbol{k}_\perp} = \varphi_{\omega \boldsymbol{k}_\perp}(\xi, \boldsymbol{x}_\perp) \mathrm{e}^{-i\omega\tau}\}$, where $\boldsymbol{x}_\perp = (y, z)$, $\boldsymbol{k}_\perp = (k_y, k_z)$, and [8, 76]:

$$\varphi_{\omega \boldsymbol{k}_\perp}(\xi, \boldsymbol{x}_\perp) = \left( \frac{\sinh(\pi\omega/a)}{4\pi^4 a} \right) K_{i\omega/a} \left( \frac{k_\perp \mathrm{e}^{a\xi}}{a} \right) \mathrm{e}^{i\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp}, \tag{2.45}$$

where $k_\perp = |\boldsymbol{k}_\perp|$, $K_j(z)$ is modified Bessel function.

These are positive frequency modes for Rindler wedge I. If we develop the same treatment to wedge II using the coordinates given in Eq. ( 2.37), we get the same differential equation as in Eq. ( 2.44) and, consequently, solutions which are basically the same expression of $u^I_{\omega \boldsymbol{k}_\perp}$ but in coordinates of wedge II. We denote it by $u^{II}_{\omega \boldsymbol{k}_\perp}$. We can extend solutions $u^I_{\omega \boldsymbol{k}_\perp}$ to all the region $|x| > |t|$ by defining $u^I_{\omega \boldsymbol{k}_\perp} \equiv 0$ over wedge II and, analogously, $u^{II}_{\omega \boldsymbol{k}_\perp} \equiv 0$ over wedge I. This way, if $\{u^I_{\omega \boldsymbol{k}_\perp}\}$ span the positive frequency Hilbert space of solutions of Eq. ( 2.42) on wedge I which we denote by $\mathcal{H}^I_{KG}$, and $\{u^{II}_{\omega \boldsymbol{k}_\perp}\}$ span $\mathcal{H}^{II}_{KG}$. This way, the set $\mathcal{H}^I_{KG} \oplus \mathcal{H}^{II}_{KG} \oplus \overline{\mathcal{H}}^I_{KG} \oplus \overline{\mathcal{H}}^{II}_{KG}$ is the space of solutions on region $|x| > |t|$. It's also possible to extend these solutions to all Minkowski spacetime [7, 8] such that $\{u^I_{\omega \boldsymbol{k}_\perp}, u^{II}_{\omega \boldsymbol{k}_\perp}\}$ becomes a complete set of modes of the space of solutions of Eq. ( 2.42) on Minkowski spacetime. Therefore, we can write any field operator in Minkowski spacetime as:

$$\widehat{\phi}(x) = \int_0^\infty d\omega \int d\boldsymbol{k}_\perp \left[ a(u^{I*}_{\omega \boldsymbol{k}_\perp}) u^I_{\omega \boldsymbol{k}_\perp} + a^\dagger(u^I_{\omega \boldsymbol{k}_\perp}) u^{I*}_{\omega \boldsymbol{k}_\perp} + a(u^{II*}_{\omega \boldsymbol{k}_\perp}) u^{II}_{\omega \boldsymbol{k}_\perp} + a^\dagger(u^{II}_{\omega \boldsymbol{k}_\perp}) u^{II*}_{\omega \boldsymbol{k}_\perp} \right]. \tag{2.46}$$

We denote $|0_M\rangle$ the state such that $a(v^*_{\boldsymbol{k}})|0_M\rangle = 0 \ \forall \boldsymbol{k}$ and $|0_R\rangle$ the state such that $a(u^{I*}_{\omega \boldsymbol{k}_\perp})|0_R\rangle = a(u^{II*}_{\omega \boldsymbol{k}_\perp})|0_R\rangle = 0 \ \forall \omega, \boldsymbol{k}_\perp$.

To obtain Bogoliubov coefficients we expand $\{u^I_{\omega \boldsymbol{k}_\perp}, u^{II}_{\omega \boldsymbol{k}_\perp}\}$ in terms of $\{v_{\boldsymbol{k}}\}$. As Bogoliubov

coefficients vanish for different $\boldsymbol{k}_\perp$ the relation gets:

$$u^I_{\omega \boldsymbol{k}_\perp} = \int_{-\infty}^{\infty} \frac{dk_x}{\sqrt{4\pi k_0}} \left( \alpha^I_{\omega k_x \boldsymbol{k}_\perp} \mathrm{e}^{i(k_x x - k_0 t)} + \beta^I_{\omega k_x \boldsymbol{k}_\perp} \mathrm{e}^{-i(k_x x - k_0 t)} \right) \frac{\mathrm{e}^{i\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp}}{2\pi}, \qquad (2.47)$$

$$u^{II}_{\omega \boldsymbol{k}_\perp} = \int_{-\infty}^{\infty} \frac{dk_x}{\sqrt{4\pi k_0}} \left( \alpha^{II}_{\omega k_x \boldsymbol{k}_\perp} \mathrm{e}^{i(k_x x - k_0 t)} + \beta^{II}_{\omega k_x \boldsymbol{k}_\perp} \mathrm{e}^{-i(k_x x - k_0 t)} \right) \frac{\mathrm{e}^{i\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp}}{2\pi}. \qquad (2.48)$$

We can calculate Bogoliubov coefficients explicitly [8, 77]:

$$\alpha^I_{\omega k_x \boldsymbol{k}_\perp} = \alpha^{II}_{\omega -k_x \boldsymbol{k}_\perp} = \frac{\mathrm{e}^{\pi\omega/2a}}{\sqrt{4\pi k_0 a \sinh(\pi\omega/a)}} \left( \frac{k_0 + k_x}{k_0 - k_x} \right)^{-i\omega/2a}, \qquad (2.49)$$

$$\beta^I_{\omega k_x \boldsymbol{k}_\perp} = \beta^{II}_{\omega -k_x \boldsymbol{k}_\perp} = -\frac{\mathrm{e}^{-\pi\omega/2a}}{\sqrt{4\pi k_0 a \sinh(\pi\omega/a)}} \left( \frac{k_0 + k_x}{k_0 - k_x} \right)^{-i\omega/2a}. \qquad (2.50)$$

From Eqs. ( 2.49) and ( 2.50), the following relations hold:

$$\beta^I_{\omega k_x \boldsymbol{k}_\perp} = -\mathrm{e}^{-\pi\omega/a} \alpha^{II*}_{\omega k_x \boldsymbol{k}_\perp}, \qquad (2.51)$$

$$\beta^{II}_{\omega k_x \boldsymbol{k}_\perp} = -\mathrm{e}^{-\pi\omega/a} \alpha^{I*}_{\omega k_x \boldsymbol{k}_\perp}. \qquad (2.52)$$

The above relations show that the Bogoliubov coefficients related to negative energy frequencies of Minkowski modes ($v^*_{\boldsymbol{k}}$) in one wedge is proportional to conjugated coefficient related to positive energy frequencies of Minkoswki modes ($v_{\boldsymbol{k}}$) in the oposite wedge. This have an interesting implication. Consider the following modes:

$$g_{-\omega \boldsymbol{k}_\perp} \equiv \frac{u^I_{\omega \boldsymbol{k}_\perp} + \mathrm{e}^{-\pi\omega/a} u^{II*}_{\omega -\boldsymbol{k}_\perp}}{\sqrt{1 - \mathrm{e}^{-2\pi\omega/a}}}, \qquad (2.53)$$

$$g_{+\omega \boldsymbol{k}_\perp} \equiv \frac{u^{II}_{\omega \boldsymbol{k}_\perp} + \mathrm{e}^{-\pi\omega/a} u^{I*}_{\omega -\boldsymbol{k}_\perp}}{\sqrt{1 - \mathrm{e}^{-2\pi\omega/a}}}. \qquad (2.54)$$

If we substitute Eqs. ( 2.47) and ( 2.48) into Eqs. ( 2.53) and ( 2.54) and use Eqs. ( 2.51) and ( 2.52), we get:

$$g_{-\omega \boldsymbol{k}_\perp} = \int_{-\infty}^{\infty} \frac{dk_x}{\sqrt{4\pi k_0}} \alpha^I_{\omega k_x \boldsymbol{k}_\perp} \sqrt{1 - \mathrm{e}^{-2\pi\omega/a}} \mathrm{e}^{i(k_x x - k_0 t)} \frac{\mathrm{e}^{i\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp}}{2\pi}, \qquad (2.55)$$

$$g_{+\omega \boldsymbol{k}_\perp} = \int_{-\infty}^{\infty} \frac{dk_x}{\sqrt{4\pi k_0}} \alpha^{II}_{\omega k_x \boldsymbol{k}_\perp} \sqrt{1 - \mathrm{e}^{-2\pi\omega/a}} \mathrm{e}^{i(k_x x - k_0 t)} \frac{\mathrm{e}^{i\boldsymbol{k}_\perp \cdot \boldsymbol{x}_\perp}}{2\pi}. \qquad (2.56)$$

Therefore, $g_{-\omega \boldsymbol{k}_\perp}$ and $g_{+\omega \boldsymbol{k}_\perp}$ are purely positive frequency solutions in Minkowski spacetime.

The considerations above enable us to notice an important relation between positive frequency solutions of Klein-Gordon equation regarding Mikowski and Rindler time isometries. Let $\psi^I_\omega$ be a positive frequency solution, regarding Rindler time $\tau$, in wedge I, that vanishes in wedge II. From Eq. ( 2.45) and the definition of $u^I_{\omega \boldsymbol{k}_\perp}$, $\psi^{II}_\omega = (\psi^I_\omega \circ \zeta)^*$, where $\zeta(t, x, y, z) = (-t, -x, y, z)$, is a positive frequency solution regarding $\tau$ in wedge II that vanish in wedge I.

We have that

$$\phi_\omega^1 = \psi_\omega^I + e^{-\pi\omega/a}\psi_\omega^{II*}, \tag{2.57}$$

$$\phi_\omega^2 = \psi_\omega^{II} + e^{-\pi\omega/a}\psi_\omega^{I*}, \tag{2.58}$$

are positive frequency solutions regarding Minkowski time isometry.

Now, from Eqs. ( 2.53) to ( 2.56) we conclude that $g_{-\omega k_\perp}$ and $g_{+\omega k_\perp}$ are proportional to $v_{\boldsymbol{k}}$, such that $a(v_{\boldsymbol{k}}^*)$ is proportional to $a(u_{\omega k_\perp}^{I*}) - e^{-\pi\omega/a}a^\dagger(u_{\omega-k_\perp}^{II})$ and $a(u_{\omega k_\perp}^{II*}) - e^{-\pi\omega/a}a^\dagger(u_{\omega-k_\perp}^{I})$. Therefore, the expression bellow holds:

$$\left[a(u_{\omega k_\perp}^{I*}) - e^{-\pi\omega/a}a^\dagger(u_{\omega-k_\perp}^{II})\right]|0_M\rangle = 0, \tag{2.59}$$

$$\left[a(u_{\omega k_\perp}^{II*}) - e^{-\pi\omega/a}a^\dagger(u_{\omega-k_\perp}^{I})\right]|0_M\rangle = 0. \tag{2.60}$$

It is time to introduce a more rigorous treatment of the Rindler modes. The family of solutions $\{u_{\omega k_\perp}^I, u_{\omega k_\perp}^{II}\}$ is not normalizable. This way they can not belong to Hilbert spaces $\mathcal{H}_{KG}^I$ and $\mathcal{H}_{KG}^{II}$. In order to fix that we assume that the modes are wrapped by wave packages sharply peaked about some discrete frequencies $\omega_i$, $i \in \mathbb{N}$, such that we manage the substitution $u_{\omega k_\perp}^I \to u_{\omega_i k_\perp}^I$. This way we ensure consistency [1]. For simplicity of notation we will denote in this section $a(u_{\omega_i k_\perp}^{I*}) = a_i^I$ and analogously the other annihilations and creation operators.

Using Eq. ( 2.59) and commutators among operators $a_i^I, a_i^{I\dagger}, a_i^{II}$ and $a_i^{II\dagger}$, we find:

$$\langle 0_M|a_i^{I\dagger}a_i^I|0_M\rangle = e^{-2\pi\omega_i/a}\langle 0_M|a_i^{II\dagger}a_i^{II}|0_M\rangle + e^{-2\pi\omega/a}. \tag{2.61}$$

Now, using Eq. ( 2.60) and the same commutators, we find:

$$\langle 0_M|a_i^{II\dagger}a_i^{II}|0_M\rangle = e^{-2\pi\omega_i/a}\langle 0_M|a_i^{I\dagger}a_i^I|0_M\rangle + e^{-2\pi\omega/a}. \tag{2.62}$$

Solving Eqs. ( 2.61) and ( 2.62), we obtain:

$$\langle 0_M|a_i^{I\dagger}a_i^I|0_M\rangle = \langle 0_M|a_i^{II\dagger}a_i^{II}|0_M\rangle = \frac{1}{e^{2\pi\omega/a} - 1}. \tag{2.63}$$

Hence, the expectation value of Rindler particle number operator is that of a Bose-Einstein particle in a thermal bath of temperature $T = a/2\pi$. This suggests that Minkowski vacuum can be expressed as a thermal state in Rindler wedge with boost generator as Hamiltonian.

To look for the $|0_M\rangle$ in terms of Rindler modes we multiply Eq. ( 2.59) by $a_i^{I\dagger}$ and Eq. ( 2.60) by $a_i^{II\dagger}$ to find:

$$(a_i^{I\dagger}a_i^I - a_i^{II\dagger}a_i^{II})|0_M\rangle = 0. \tag{2.64}$$

Hence, for each frequency, there is the same number of particles in I and II. Then we can write:

$$|0_M\rangle \propto \prod_i \sum_{n_i=0}^{\infty} \frac{C_{n_i}}{n_i!} (a_i^{I\dagger} a_i^{II\dagger})^{n_i} |0_R\rangle. \tag{2.65}$$

The equation above says that Minkoski vacuum will be proportional to a general state which, for the same mode frequency $\omega_i$, will have the same number of excited Rindler modes in wedge I and wedge II. The product runs over different frequencies because creation operators of modes with different frequencies commute.

Using Eqs. ( 2.59) and ( 2.60) we can get the recursion relation to $C_{n_i}$:

$$C_{n_i+1} - e^{-\pi\omega_i/a} C_{n_i} = 0, \tag{2.66}$$

which implies $C_{n_i} = e^{-n_i \pi \omega_i/a} C_0$. So we get to our final relation:

$$|0_M\rangle = \prod_i \left[ \sqrt{1 - e^{-2\pi\omega_i/a}} \sum_{n_i=0}^{\infty} e^{-n_i\pi\omega_i/a} |n_i^I\rangle \otimes |n_i^{II}\rangle \right]. \tag{2.67}$$

If we calculate the density matrix of Minkowski vacuum and decide to observe just wedge I (since I and II are causally disconnected) we get:

$$\rho^I = \prod_i \left[ (1 - e^{-2\pi\omega_i/a}) \sum_{n_i=0}^{\infty} e^{-2n_i\pi\omega_i/a} |n_i^I\rangle\langle n_i^I| \right]. \tag{2.68}$$

This is the density operator of a free boson system with temperature $T = a/2\pi$. The above expression implies that an observer accelerated in Minkowski spacetime, where an inertial observer can only observe vacuum, sees himself immersed in a thermal bath with temperature dependent on acceleration. This is the Unruh Effect.

This result is quite counter-intuitive. First implication is that the very notion of particle is observer dependent. Where an inertial observer can only perceive the vacuum, an accelerated observer will witness a bath of particles. This phenomenon is an evidence that the real fundamental physical entity is the quantum field, not the particles related to it.

Unruh effect also implies that a human observer in the intergalactic space where, by good enough approximation, there is nothing except the observer, will freeze since the temperature will be nearly the absolute zero. However, if by any means the observer accelerates, he will feel a temperature which can be as high as the nucleus of the sun, if he accelerates enough. Acceleration in empty space would be the difference between dying frozen or burned.

The effect is quite subtle, however. If we abandon for a while natural units and add back the physical constants to the Unruh temperature we obtain:

$$T = \frac{a\hbar}{2\pi K_B c}, \tag{2.69}$$

where $K_B$ is the Boltzmann constant and $c$ is light velocity. The Above expression tells us that, in the international system of units, it is required an acceleration of order of $10^{20} m/s^2$ to obtain an Unruh temperature of order $1K$. Such acceleration is not achievable with our current technology. So, Unurh effect was not directly observed yet. This difficulty related to Unruh temperature detection is the obstacle to achieve this experimental observation so important to physics fundamentals. Theoretical efforts to overcome technological limitations are certainly welcome. In this context we hope that quantum metrology can give us an insight about the most suitable experimental scheme to look for Unruh effect. With this hope we conducted the research that based this thesis.

# Chapter 3

# Metrology Quantities

## 3.1 Classical Fisher Information

Fisher Information raises from the context of the theory of estimation, founded by Ronald Aylmer Fisher in a series of fundamental papers [29–32]. For any experiment we run, we get as an answer a sample of observed data. Frequently we have some previous knowledge about the studied system so we can assume a probability distribution depending on some unknown parameters to model the experiment results. A standard problem of theory of estimation is to determine the parameters of a probability distribution from a sample of data.

We model experimental observation events as probability spaces $(\mathcal{X},\mathcal{A},\mathbf{Pr})$ (see appendix E for definitions) where the nonempty set $\mathcal{X}$ is understood as the set of elementary events or possible outcomes, the $\sigma$-algebra $\mathcal{A}$ of $\mathcal{X}$ is understood as the observable events and $\mathbf{Pr}$ is the probability measure of $\mathcal{A}$.

A single measurement is modeled as a random variable $X : \mathcal{X} \to \mathbb{R}$. If we repeat the experiment several times the sample of data we get can be expressed as set of independent identical distribution (i.i.d.) random variables. From the set of experiment's measured data we shall try to estimate the unknown parameter.

**Definition 3.1.1.** *Let $X_i : \mathcal{X}' \to \mathcal{X}$, $i \in \{1, \cdots ,n\}$ i.i.d. random variables with probability density $p(x,\theta)$. The **estimator** of the parameter $\theta \in \Theta$ for a sample of size $n$ given by results of $(X_i)_{i \in \{1,\cdots,n\}}$ is a map $\mathcal{E} : \mathcal{X}^n \to \Theta$.*

An estimator is an attempt to approximate the value of the parameter $\theta$. For example, let $X_1, \cdots , X_n$ be drawn i.d.d.. We can use as an example repeated measures of an experiment. Given a previous knowledge about the physical behavior of the system, we assume a normal distribution for the quantity measured with mean value $\theta$ and standard deviation $\sigma$. We can estimate $\theta$ defining a function $\mathcal{E}$ such that $\mathcal{E}(X_1, \cdots , X_n) = \frac{1}{n} \sum_{i=1}^{n} X_i$. The function $\mathcal{E}$ is an estimator of the parameter $\theta$ from the sample.

In order to evaluate how good is the estimator, we define the error of the estimator

**Definition 3.1.2.** *Let $\mathcal{E}$ be an estimator. The function $Er : \Theta \to \mathbb{R}$, $Er(\mathcal{E}) \equiv \mathcal{E} - \theta$ is called the* **error function** *of the estimator $\mathcal{E}$.*

Note that, a priori, we do not know the value $\theta$. We shall call the theoretical "right" value of $\theta$ the *true value* of the parameter. The use of the true value is the development of concepts related to the error of experiment measurements. This will become more clear in the next examples.

**Definition 3.1.3.** *The* **bias** *of an estimator for the parameter $\theta$ is the expected value $E_\theta(\mathcal{E} - \theta)$ of the error of the estimator. The subscript $\theta$ means that the expectation is referent to the density $p(x,\theta)$.*

The estimator is called unbiased when $E_\theta(\mathcal{E} - \theta) = 0 \; \forall \theta \in \Theta$. This implies that $E_\theta(\mathcal{E}) = \theta$ *i.e.* the experiment has no systematic error.

As an example, let $X_1, \cdots , X_n$ be drawn i.d.d. with density function $p(x,\lambda) = \frac{1}{\lambda}e^{-x/\lambda}$, $x \in [0,\infty)$. We define 2 estimators for $\lambda$, $\mathcal{E}_1 = X_1$ and $\mathcal{E}_2 = 1/n \sum_{i=1}^{n} X_i$. We have that $E_\lambda(\mathcal{E}_1) = \int_0^\infty x\frac{e^{-x/\lambda}}{\lambda}dx$. Integrating by parts we have $E_\lambda(\mathcal{E}_1) = \lambda$ and $\mathcal{E}_1$ is, then, unbiased. Also $E_\lambda(\mathcal{E}_2) = \int \frac{1}{n} \left( \sum_{i=1}^{n} x_i \right) \left( \prod_{i=1}^{n} \frac{e^{-x_i/\lambda}}{\lambda}dx_i \right) = \frac{1}{n} \sum_{i=1}^{n} \int \prod_{j=1}^{n} x_i \frac{e^{-x_j/\lambda}}{\lambda}dx_j = n\frac{\lambda}{n} = \lambda$. So, $\mathcal{E}_2 = \lambda$ which is unbiased as well.

The bias is an approach to measure how good is the estimator we choose. However, it's not a good measure of quality since a vanishing bias does not guarantee that the error function has a small result with high probability. For example, if we define a unbiased estimator to a probability density with large dispersion, there will be a high probability of obtaining a large value of error. So, we still need something else to measure and compare estimators. That will be a loss function.

Let $(\mathcal{X}',\mathcal{A}',\mathbf{Pr}_\theta)$ be a probability space where the subscript $\theta$ means that the probability measure depends on the parameter $\theta \in \Theta$ and $(X_i)_{i \in \{1,\cdots,n\}}$ a family of i.d.d. random variables with value in $(\mathcal{X},\mathcal{A})$.

**Definition 3.1.4.** *Let $(\mathcal{D},\mathcal{B})$ be a measurable space called a decision space. A* **decision function** *is a map $\delta : \mathcal{X}^n \to \mathcal{D}$. The cartesian product $\mathcal{X}^n$ depends on how many observations the decision is based on.*

**Definition 3.1.5.** *A* **loss function** *is a map $L : \Theta \times \mathcal{D} \to \mathbb{R}$ that is meant to quantify the cost of a bad decision making.*

**Definition 3.1.6.** *We call a* **risk function** *of a decision $\delta$ the average*

$$R(\theta,\delta) = E_\theta(L(\theta,\delta)) = \int_{\mathcal{X}^n} L(\theta,\delta(x_1,\cdots,x_n))p_{X_1}(x_1,\theta)\cdots p_{X_n}(x_n,\theta)dx_1\cdots dx_n.$$

In the integral above we assume that the observation space is the measure space *i.e.* $\mathcal{X} = X_i(\mathcal{X}')$, $\forall i \in \{1,\cdots,n\}$. In statistical estimation context, the decision is the estimator to choose and the loss function is meant to quantify how bad is the estimator approximation. The

most commonly chosen loss function is the squared error loss $L(\theta,\mathcal{E}) = (\mathcal{E} - \theta)^2$ since it's risk function is exactly the variance of the estimator $R(\theta,\mathcal{E}) = E_\theta[(\mathcal{E} - \theta)^2]$ and the calculations in this case are simpler [71]. We'll choose as loss function the squared loss function and we will understand a good estimator the one that has a low average squared error (low risk function) and which error vanishes when $n \to \infty$.

Since we assign a notion of quality to estimator we shall rank them according to their quality:

**Definition 3.1.7.** *An estimator $\mathcal{E}_1$ is said to dominate another estimator $\mathcal{E}_2$ if, for all $\theta \in \Theta$ we have*

$$E_\theta[(\mathcal{E}_1(X_1,\cdots,X_n) - \theta)^2] \leq E_\theta[(\mathcal{E}_2(X_1,\cdots,X_n) - \theta)^2]$$

If it's now possible to rank estimators a natural question to make is: "Is there a best estimator?" or, to put it in our terms "Is there a estimator that dominates over the others?". To answer this question we shall present the Cramér-Rao bound on the mean squared error of any estimator. In order achieve it, we define the score of the distribution $p(x,\theta)$:

**Definition 3.1.8.** *Let $X : \mathcal{X}' \to \mathcal{X}$ be a random variable of density $p(x,\theta)$. The* **score function** *$V : \mathcal{X}' \to \mathbb{R}$ is a random variable defined by*

$$V = \frac{\partial}{\partial \theta} \ln p(X,\theta) = \frac{\frac{\partial}{\partial \theta} p(X,\theta)}{p(X,\theta)}$$

Note that the mean value of the score is

$$E_\theta(V) = \int_{\mathcal{X}} \frac{\frac{\partial}{\partial \theta} p(x,\theta)}{p(x,\theta)} p(x,\theta) dx = \int_{\mathcal{X}} \frac{\partial}{\partial \theta} p(x,\theta) dx = \frac{\partial}{\partial \theta} \int_{\mathcal{X}} p(x,\theta) dx = \frac{\partial}{\partial \theta} 1 = 0. \quad (3.1)$$

Here we assume $p(x,\theta)$ and $\frac{\partial}{\partial \theta} p(x,\theta)$ are continuous over their domain. The expression above implies that the variance of the score is $E_\theta[(V - E_\theta(V))^2] = E_\theta(V^2) - E_\theta(V)^2 = E_\theta(V^2)$. We can define the score of a family of random variables $(X_i)_{i \in \{1,\cdots,n\}}$ drawn i.i.d.. Since the probability density of a combined observation of the family of random variables satisfies $p(x_1,\cdots,x_n,\theta) = \prod_{i=1}^n p(x_i,\theta)$ we have:

$$V(X_1,\cdots,X_n) = \frac{\partial}{\partial \theta} \ln p(x_1,\cdots,x_n,\theta) = \sum_{i=1}^n \frac{\partial}{\partial \theta} \ln p(x_i,\theta) = \sum_{i=1}^n V(X_i) \quad (3.2)$$

The family of random variables $(V(X_i))_{i \in \{1,\cdots,n\}}$ are i.d.d. and has vanishing mean value. We are finally ready to define the Fisher information (F.I.).

**Definition 3.1.9.** *The* **Fisher Information** *of the parameter $\theta$ is the variance of the score*

$$F(\theta) = E_\theta([(\frac{\partial}{\partial \theta} \ln p(x,\theta))^2]$$

In the case we have multiples observations expressed by the family $(X_i)_{i\in\{1,\cdots,n\}}$, the Fisher information is

$$
\begin{aligned}
F_n(\theta) = E_\theta[V(X_1,\cdots,X_n)^2] &= E_\theta\left[\left(\sum_{i=1}^n V(X_i)\right)^2\right]\\
&= E_\theta\left[\sum_{i=1}^n (V(X_i))^2 + \sum_{\substack{i,j=1\\i\neq j}}^n V(X_i)V(X_j)\right]\\
&= E_\theta\left[\sum_{i=1}^n (V(X_i))^2\right] + E_\theta\left[\sum_{\substack{i,j=1\\i\neq j}}^n V(X_i)V(X_j)\right]\\
&= \sum_{i=1}^n F(\theta) + \sum_{\substack{i,j=1\\i\neq j}}^n E_\theta[V(X_i)]E_\theta[V(X_j)]\\
&= \sum_{i=1}^n F(\theta) = nF(\theta).
\end{aligned}
\tag{3.3}
$$

So, the F.I. of $n$ i.d.d. random variables is $n$ times the F.I. of a single random variable. The significance of Fisher information get clear with Cramér-Rao theorem [33, 34].

**Theorem 3.1.1** (Cramér-Rao Theorem). *The mean squared error of an unbiased estimator of the parameter $\theta$, $\mathcal{E}(X)$, is lower bounded by the inverse of the Fisher information*

$$
Var(\mathcal{E}) \geq \frac{1}{F(\theta)}
$$

The proof is presented in appendix F.

The above bound is valid for any unbiased estimator. It establishes a limit for the quality of the estimator. Estimators that meet the lower bound of Cramér-Rao inequality are called *efficient estimators*. Since F.I. is the mathematical entity that sets the bound for any estimator of $\theta$, F.I. is related to the amount of information that exists about the parameter $\theta$ in the sample of data. The larger F.I. is, smaller the variance bound will be and more accurate may be the estimations of $\theta$. So, the larger F.I. is, the larger the amount of information about $\theta$ in the sample of data.

As an example, let $(X_i)_{i\in\{1,\cdots,n\}}$ i.d.d. random variables with probability density $p(x,\theta) =$

$N(\theta,1)$, where $N(\theta,1)$ denotes the normal distribution with mean value $\theta$ and unit variance.

$$N(\theta,1) = \frac{1}{2\pi}e^{-(x-\theta)^2/2}. \qquad (3.4)$$

Let's calculate the F.I. of this probability distribution. First the score:

$$V = \frac{\frac{\partial}{\partial\theta}\left[\frac{1}{2\pi}e^{-(x-\theta)^2/2}\right]}{\frac{1}{2\pi}e^{-(x-\theta)^2/2}} = x - \theta \qquad (3.5)$$

Then, we calculate the F.I.

$$F(\theta) = E_\theta[(x-\theta)^2] = 1. \qquad (3.6)$$

The expression above holds for a single random variable, since we have a sample of $n$ observations, we have

$$F(\theta) = n. \qquad (3.7)$$

So we claim that, for any estimator $\mathcal{E}$ of the parameter $\theta$, $Var(\mathcal{E}) \geq 1/n$. Now if we calculate the variance of the usual estimator of the mean value $\overline{X} = \frac{X_1+\cdots+X_n}{n}$ we obtain

$$E_\theta[(\overline{X}-\theta)^2] = E_\theta\left[\left(\frac{x_1+\cdots+x_n}{n}-\theta\right)^2\right] = E_\theta\left[\sum_{i=1}^{n}\frac{x_i^2}{n^2} + \sum_{\substack{i,j=1\\i\neq j}}^{n}\frac{x_i x_j}{n^2} - 2\sum_{i=1}^{n}\frac{\theta x_i}{n} + \theta^2\right]$$

$$= \frac{E_\theta(x^2)}{n} + \frac{\theta^2(n^2-n)}{n^2} - 2\theta^2 + \theta^2 = \frac{E_\theta(x^2) - \theta^2}{n} = \frac{1}{n} = \frac{1}{F(\theta)}. \qquad (3.8)$$

So we conclude that $\overline{X}$ meets the Cramér-Rao bound of the variance of the estimator of $\theta$ and, then, it is an efficient estimator.

We presented here the 1-parameter Fisher Information. It is possible to generalize the concept of Fisher Information for the multiparameter case. If we consider a probability density function that depends on various parameters $p(x,\theta^1,\cdots,\theta^n) = p(x,\boldsymbol{\theta})$. The Fisher information of that probability density will be the matrix:

$$F_{ij}(\boldsymbol{\theta}) = \int dx\, p(x,\boldsymbol{\theta})\frac{\partial}{\partial\theta^i}\left(\ln p(x,\boldsymbol{\theta})\right)\frac{\partial}{\partial\theta^j}\left(\ln p(x,\boldsymbol{\theta})\right). \qquad (3.9)$$

This matrix is also called the Fisher information matrix. If we denote by $\mathcal{E}_i$ an unbiased estimator of the parameter $\theta^i$ and the covariance matrix $\Sigma_{ij} = E_{\boldsymbol{\theta}}[(\mathcal{E}_i - \theta^i)(\mathcal{E}_j - \theta^j)]$, the Cramér-Rao inequality gets the form [72]:

$$\Sigma_{ij} \geq F_{ij}^{-1}(\boldsymbol{\theta}). \qquad (3.10)$$

Here, $F_{ij}^{-1}(\boldsymbol{\theta})$ denotes the inverse of $F_{ij}(\boldsymbol{\theta})$ and the inequality means that the difference

$\Sigma_{ij} - F_{ij}^{-1}(\boldsymbol{\theta})$ is a nonnegative matrix.

Finally, F.I. quantifies how sensible the probability distribution is regarding the change in the parameter. If a small variation in the parameter results in a drastic change in the probability distribution, there will be a very distinguishable relation between the parameter value and the probability distribution such that we can estimate the value of the parameter by looking at it's probability distribution precisely. Otherwise, if even a large variation results in a small perturbation of the probability distribution, it will be a much harder task to distinguish which probability distribution is related to each parameter value and we will not be able to realize a precise parameter value estimation. Next we explore what is Fisher information for quantum systems.

## 3.2    Quantum Fisher Information

We saw in the last section that the estimation theory is concerned about the problem of estimation of a parameter from a sample of data. This theory was developed before the raise of quantum mechanics and was thought in classical context. When we are dealing with quantum systems, quantum properties discussed in sections B and C bring new characteristics to estimation problem. In quantum context estimation problem is known as quantum metrology. We shall present in this section an important quantity for quantum metrology accuracy: the quantum Fisher information.

The problem of quantum metrology is the estimation of a parameter $\xi$ from a set of quantum measurements. This is done in following steps:

1. Preparation of the probe state;

2. Interaction of the probe with the system of interest;

3. Measurement of the probe;

4. Decode the information of interest from measured data.

In the first step we prepare a controllable system in a definite, blank state. During step 2 the information about the desired parameter is codified into the state of the probe. This is the codification step. In step 3 we read this information by measuring the probe. By repeating this process several times, we realize the last step, the decoding. We shall estimate the state obtained after the interaction with the system of interest and we can get from the state the value of $\xi$ up to a certain precision.

A key quantity in this process is Fisher information. We already discussed Fisher information in classical context in section 3.1. It bounds the precision at which $\xi$ can be estimated as stated by Cramér-Rao theorem. This same theorem has a version for quantum context as we

will present here [45]. From definition 3.1.9, the classical Fisher information is given by:

$$F(\xi) = \int_{\mathcal{X}} dx \, \frac{[\partial_\xi p(x,\xi)]^2}{p(x,\xi)} \tag{3.11}$$

where we used the notation $\partial_\xi \equiv \partial/\partial\xi$.

In quantum metrology context, as described above, we prepare a state $\rho_0$ and leave it to interact with the system of interest so it evolve to $\rho_\xi$, the state in which the information about the parameter $\xi$ is encoded. As discussed in section B, measurements are described by a POVM $\{\Pi_x\}$ and the probabilities are computed applying Born's rule:

$$p(x,\xi) = Tr(\Pi_x \rho_\xi). \tag{3.12}$$

combining Eq. ( 3.12) and Eq. ( 3.11), we get:

$$F(\xi) = \int_{\mathcal{X}} dx \, \frac{[\partial_\xi Tr(\Pi_x \rho_\xi)]^2}{Tr(\Pi_x \rho_\xi)}. \tag{3.13}$$

There are several possible sets of POVM's that we can choose and different sets will provide a distinct amount of information about the probe, thus resulting in a different value for the Fisher information. We define quantum Fisher information (QFI), $J(\xi)$, as the maximum of $F(\xi)$ over all possible POVM's. In appendix G we present the derivation of QFI in details. The maximization of $F(\xi)$ over all possible POVM's results in:

$$J(\xi) = Tr(\rho_\xi L_\xi^2) = Tr(\partial_\xi \rho_\xi L_\xi), \tag{3.14}$$

where $L_\xi$ is the symmetric logarithm derivative (SLD), defined by the Lyapunov equation:

$$\partial_\xi \rho_\xi = \frac{L_\xi \rho_\xi + \rho_\xi L_\xi}{2}. \tag{3.15}$$

The solution of Eq. ( 3.15) is:

$$L_\xi = 2 \int_0^\infty dt e^{-\rho_\xi t} \partial_\xi \rho_\xi e^{-\rho_\xi t}. \tag{3.16}$$

Such that if we consider the spectral decomposition of the probe system $\rho_\xi = \sum_j p_j |e_j\rangle\langle e_j|$, $0 < p_j \leq 1$, $\sum_j p_j = 1$, as we present in appendix G, we obtain the expression:

$$J(\xi) = 2 \sum_{ij} \frac{|\langle e_i|\partial_\xi \rho_\xi|e_j\rangle|^2}{p_i + p_j}, \tag{3.17}$$

where $i,j$ in summation just includes terms such that $p_i + p_j \neq 0$.

We can then compute the infinitesimal change in the probe state as $\partial_\xi \rho_\xi = \sum_j \partial_\xi p_j |j\rangle\langle j| +$

$p_j|\partial_\xi j\rangle\langle j|+p_j|j\rangle\langle\partial_\xi j|$. Remembering that $\partial_\xi\langle i|j\rangle = \langle\partial_\xi i|j\rangle+\langle i|\partial_\xi j\rangle = 0$, we show in appendix G that the quantum Fisher information can be written in the following form

$$J(\xi) = \sum_i \frac{(\partial_\xi p_i)^2}{p_i} + 2\sum_{i<j} \frac{2(p_i - p_j)^2}{p_i + p_j}|\langle i|\partial_\xi j\rangle|^2. \tag{3.18}$$

We will refer to the first and second terms as the classical and quantum parts of the Fisher information. We note that the Crammér-Rao inequality is also valid for $J(\xi)$ [45]. This fact allow us to interpret Fisher information as a measure of the amount of information about $\xi$ that was codified in the state $\rho_\xi$. Qantum Fisher Information quantifies how sensible the quantum state is regarding the parameter. If a small variation in the parameter results in a large change in the quantum state, parameters values close to each othe will be related to more distinguishable quantum states and we will be able to estimate the value of the parameter through it's quantum state precisely. Otherwise, we will not be able to realize a precise parameter value estimation.

## 3.3 Signal-to-Noise Ratio

We have discussed in last section that, through Crammér-Rao theorem, quantum Fisher information defines a boundary to the parameter estimation precision in quantum context. We chose the variance to quantify the precision of estimation. However, the variance alone is not enough to determine how credible is an estimation. A small variance is needed to claim that an estimated parameter is zero while a larger variance is tolerable if the estimated value is large. So, in order to determine how good is an estimation it is more suitable to look at the signal-to-noise ratio (SNR).

**Definition 3.3.1.** *Let $\xi$ be the parameter of interest. The* **signal-to-noise ratio** *for a single measurement is defined as:*

$$R_\xi = \frac{\xi^2}{Var(\mathcal{E}_\xi)}; \tag{3.19}$$

The definition of SNR is nothing more than the square of the ratio between the estimation mean value and the standard deviation. It is dimensionless and when it results in a large value, it means the standard deviation of the estimation is small compared to it's mean value. In this case estimation is precise.

By Crámmer-Rao inequality, $R_\xi$ is bounded:

$$R_\xi \leq \xi^2 J_\xi \equiv Q_\xi. \tag{3.20}$$

Here we denote as $Q_\xi$ the maximum SNR possible to a estimation scheme. In the next chapter we will analyze the behavior of $Q_\xi$ of our system regarding some parameters we control.

# Chapter 4

# Quantum Metrology and Unruh Effect

We will study the precision in the estimation of Unruh temperature. The choice of the system we will consider is crucial since it has to be simple enough we can handle with but still be able to model a practical measurement scenario to be useful. We use a two-level detector model in flat spacetime to study two distinct cases: *(i)* one detector with constant acceleration interacting with en external field in vacuum state so we can study the influence of coherence in estimation of Unruh temperature. *(ii)* two detectors, one with constant acceleration interacting with an external field in vacuum state and the other one inertial, in order to study the influence of entanglement in estimation precision of Unruh temperature.

In section 4.1 we define our detector as an Unruh-DeWitt detector. In section 4.2 we look for the role of coherence in estimation precision of Unruh temperature. In section 4.3 we look for the role of entanglement in estimation precision of Unruh temperature and find the relation between signal-to-noise ratio and the entanglement sudden death acceleration. Finally in section 4.4 we translate our results to the vicinity of a Schwarzchild black hole event horizon.

## 4.1   Detector Model

In this section we present the Unruh-DeWitt detector model introduced by Unruh and Wald in 1984 [18]. This model was widely used in literature [47–49, 78–80] and the following calculations can be found in the cited papers. The detectors we are considering are two-level atoms with energy gap $\omega$. They are examples of qubits. In our approach, detectors internal degrees of freedom are ruled by quantum mechanics while we assign to them a well defined worldline. We call this approach *semiclassical*. The detector proper Hamiltonian is defined as

$$H_d = \omega d^\dagger d, \tag{4.1}$$

where $\omega > 0$ is the detector energy gap and $d^\dagger$, $d$ are the transition operators for the qubit energy eigenstates. We denote $|0\rangle$ the unexcited and $|1\rangle$ the excited qubit energy eigenstates. We have:

$$d^\dagger |1\rangle = d |0\rangle = 0$$
$$d^\dagger |0\rangle = |1\rangle$$
$$d |1\rangle = |0\rangle . \tag{4.2}$$

Given a timelike isometry followed by the detector, we define the time coordinate $t$ as the proper time of an observer co-moving with the detector and we shall define creation, anihilation and any other operator depending on time coordinate regarding $t$. We coupled the accelerated qubit to a massless free scalar field $\phi(x)$, satisfying Eq. ( 2.42), through the interaction Hamiltonian [18]:

$$H_{int}(t) = \epsilon(t) \int_{\Sigma_t} d^3\mathbf{x}\sqrt{-g}\phi(x) \left[\varphi(\mathbf{x})d + \varphi^*(\mathbf{x})d^\dagger\right] , \tag{4.3}$$

where $g = \det(g_{ab})$, $g_{ab}$ is the spacetime metric, $\mathbf{x}$ are coordinates defined on the Cauchy surface $\Sigma_{t=const}$ associated with the timelike isometry. The real-valued function $\epsilon$ is introduced to keep the detector switched on for a finite amount of proper time $\delta$. Important to say that $\epsilon$ is smooth and compact support, so we avoid any ultraviolet divergence in excitation probability [81]. Finally, $\varphi$ is a smooth, compact support, complex-valued function which shapes the region where the detector interacts with the field in the neighborhood of its worldline.

The total Hamiltonian is:

$$H = H_0 + H_{int}, \tag{4.4}$$

where $H_0 = H_d + H_{KG}$ and $H_{KG}$ is the free scalar field Hamiltonian. In interaction picture (or Dirac picture), if we denote the initial state as $|\mathbf{\Psi}_{-\infty}\rangle$, the evolved state $|\mathbf{\Psi}_t\rangle$ is given by:

$$|\mathbf{\Psi}_t\rangle = T \exp\left[-i \int_{-\infty}^{t} dt' H_{int}^D(t')\right] |\mathbf{\Psi}_{-\infty}\rangle, \tag{4.5}$$

where $T$ is time-ordering operator, $H_{int}^D(t) = \left(U_0^\dagger(t)H_{int}(t)U_0(t)\right)$ and $U_0(t)$ is the unitary time evolution operator related to Hamiltonian $H_0$.

Since after the amount $\delta$ of proper time the detector does not interact with the external field, we have $|\mathbf{\Psi}_\infty\rangle = |\mathbf{\Psi}_{t>\delta}\rangle$. Noting that in the interaction picture the transition operator gets $d \to e^{-i\omega t}d$, from Eq. ( 4.5) and Eq. ( 4.3) we find:

$$|\mathbf{\Psi}_\infty\rangle = T \exp\left[-i \int d^4x\sqrt{-g}\phi(x) \left(fd + f^*d^\dagger\right)\right] |\mathbf{\Psi}_{-\infty}\rangle, \tag{4.6}$$

where $f \equiv \epsilon(t)e^{-i\omega t}\varphi(\boldsymbol{x})$. In first perturbation order in relation to $\epsilon$, Eq.( 4.6) gets:

$$|\mathbf{\Psi}_\infty\rangle = \left[\mathbb{1} - i(\phi(f)d + \phi(f)^\dagger d^\dagger)\right] |\mathbf{\Psi}_{-\infty}\rangle, \tag{4.7}$$

where:

$$\phi(f) \equiv \int d^4x \sqrt{-g}\phi(x)f. \tag{4.8}$$

Here the function $f$ plays the role of a test function in Klein-Gordon equation. The solution is known [1] and we derive it in appendix H:

$$\phi(f) = i\left[a([KEf^*]^*) - a^\dagger(KEf))\right]. \tag{4.9}$$

Here $K$ is the operator that takes positive frequency part of solutions of massless Klein-Gordon equation (Eq. ( 2.42)) with respect to timelike isometry and

$$Ef = \int d^4x'\sqrt{-g(x')}\left[G_A(x,x') - G_R(x,x')\right]f(x'), \tag{4.10}$$

where $G_A$ denotes advanced Green function and $G_R$ the retarded Green function.

We assume $\epsilon$ is a very slow varying function compared to $\omega$, so the turning on/off process of detector is not so fast that compromises the observation of a flip in detectors state. We also assume that $\delta \gg \omega^{-1}$, so the duration of the acceleration is not short enough we can not observe the detector flip. With both assumption $f$ gets to be approximately a positive frequency function [1,49]:

$$KEf \approx Ef, \ KEf^* \approx 0. \tag{4.11}$$

So, denoting $\lambda \equiv -KEf$ we get to:

$$\phi(f) \approx ia^\dagger(\lambda), \tag{4.12}$$

and the final expression for $|\mathbf{\Psi}_\infty\rangle$ is:

$$|\mathbf{\Psi}_\infty\rangle = (\mathbb{1} + a^\dagger(\lambda)d - a(\lambda^*)d^\dagger)|\mathbf{\Psi}_{-\infty}\rangle. \tag{4.13}$$

Note that Eq. ( 4.13) tells us there are three possible results from interaction between detector and the external field: *(i)* nothing happens to the detector nor to the field; *(ii)* the detector loses one quantum and the external field gains one quantum; *(iii)* the detector gains one quantum and the field loses one. This means the excitation/deexcitation qubit process is attached to its absorption/emission process. The detector is absorbing a quantum from the field, emitting a quantum to the field or nothing is happening. Therefore, only processes where the detector flips once or none at all are considered here. The absorbed/emitted quantum is understood as a particle defined by observers co-moving with the detector according to its timelike isometry as explained in section 2.2. Next section we'll apply this model in our specific system.

## 4.2   Estimation Precision and Coherence

We consider now one accelerated detector with constant acceleration $a$ in Minkowski space-time. Our detector moves along the $x$ axis for the finite amount of proper time $\delta$. The worldline of the detector is:

$$t(\tau) = a^{-1} \sinh a\tau, \, x(\tau) = a^{-1} \cosh a\tau,$$
$$y(\tau) = z(\tau) = 0, \tag{4.14}$$

where $\tau$ is the detector proper time and $(t,x,y,z)$ are the usual Cartesian coordinates of Minkowski spacetime. The detector lies in wedge I. We model the region of interaction around the detector worldline with [49]:

$$\varphi(\mathbf{x}) = (\kappa\sqrt{2\pi})^{-3} \exp(-\mathbf{x}^2/2\kappa^2), \tag{4.15}$$

with variance $\kappa = \text{const} \ll \omega^{-1}$.
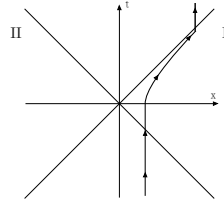


**Figure 4.1** | Detector's worldline representation.

The initial state describing our system is:

$$|\mathbf{\Psi}_{-\infty}\rangle = |\psi^d_{-\infty}\rangle \otimes |0_M\rangle, \tag{4.16}$$

where $|0_M\rangle$ denotes Minkowski vacuum and $|\psi^d_{-\infty}\rangle$ the initial state of the detector which is:

$$|\psi^d_{-\infty}\rangle = c_0|0\rangle + c_1|1\rangle. \tag{4.17}$$

Here $|c_0|^2 + |c_1|^2 = 1$.

The total Hamiltonian of the single detector system interacting with the field is given by Eq. ( 4.4) and we can just follow the development of section 4.1 to apply Eq. ( 4.13) to evolve our initial state Eq. ( 4.16). We find:

$$|\mathbf{\Psi}_{\infty}\rangle = [c_0|0\rangle + c_1|1\rangle] \otimes |0_M\rangle + c_1|0\rangle \otimes [a_{RI}(\lambda^*)|0_M\rangle] - c_0|1\rangle \otimes \left[a^\dagger_{RI}(\lambda)|0_M\rangle\right]. \tag{4.18}$$

Here we denote annihilation and creation operators as $a_{RI}$ and $a^\dagger_{RI}$ to emphasize that these are the operators related to Rindler modes in wedge I according to Rindler time isometry just like presented in section 2.4. We also denote as $a_M$ and $a^\dagger_M$ the operators related to Minkowski

modes. Regarding the relation between annihilation and creation operators in the Rindler and Minkowski spacetimes [8, 18]:

$$a_{RI}(\lambda^*) = \frac{a_M(F_{1\omega}^*) + e^{-\pi\omega/a}a_M^\dagger(F_{2\omega})}{(1 - e^{-2\pi\omega/a})^{1/2}}. \tag{4.19}$$

$$a_{RI}^\dagger(\lambda) = \frac{a_M^\dagger(F_{1\omega}) + e^{-\pi\omega/a}a_M(F_{2\omega}^*)}{(1 - e^{-2\pi\omega/a})^{1/2}}, \tag{4.20}$$

where the positive frequency solutions are given by

$$F_{1\omega} = \frac{\lambda + e^{-\pi\omega/a}\lambda \circ \zeta}{(1 - e^{-2\pi\omega/a})^{1/2}}, \tag{4.21}$$

$$F_{2\omega} = \frac{(\lambda \circ \zeta)^* + e^{-\pi\omega/a}\lambda^*}{(1 - e^{-2\pi\omega/a})^{1/2}}. \tag{4.22}$$

Here, $\zeta$ denotes the Rindler wedge reflection isometry $\zeta(t,x,y,z) = (-t, -x,y,z)$ defined in section 2.4.

Now, let's denote by $\mathcal{H}_{KG}^{RI}$ and $\mathcal{H}_{KG}^{RII}$ the Hilbert space of positive frequencies solutions of Klein-Gordon equation in Rindler wedge I and wedge II respectively, and by $\overline{\mathcal{H}}_{KG}^{RI}$ and $\overline{\mathcal{H}}_{KG}^{RII}$ the related negative frequencies solutions spaces. Since the four cited Hibert spaces are orthogonal in relation to Klein-Gordon inner product, and as $\lambda \in \mathcal{H}_{KG}^{RI}$, and $(\lambda \circ \zeta) \in \overline{\mathcal{H}}_{KG}^{RII}$ [18], it is straightforward that $(F_{1\omega}, F_{2\omega})_{KG} = 0$. We obtain:

$$(F_{i\omega}, F_{j\omega})_{KG} = \| \lambda \|^2 \delta_{ij}, \tag{4.23}$$

where [49, 78]:

$$\| \lambda \|^2 = \frac{\epsilon^2 \omega \delta}{2\pi} e^{-\omega^2 \kappa^2} \equiv \mu. \tag{4.24}$$

Now, if we denote $|1_{\tilde{F}_{i\omega}}\rangle$ as the normalized state corresponding to $|1_{F_{i\omega}}\rangle$, $i \in \{1,2\}$, and remembering the Unruh temperature $T = a/(2\pi)$, we can write the normalized state of the system after the time $\delta$ as

$$|\mathbf{\Psi}_\infty\rangle = [c_0|0\rangle + c_1|1\rangle] \otimes |0_M\rangle + c_1 \frac{\mu^{1/2}}{(1 - e^{-\omega/T})^{1/2}}|0\rangle \otimes |1_{\tilde{F}_{1\Omega}}\rangle - c_0 \frac{\mu^{1/2}e^{-\omega/(2T)}}{(1 - e^{-\omega/T})^{1/2}}|1\rangle \otimes |1_{\tilde{F}_{2\omega}}\rangle. \tag{4.25}$$

Calculating the density matrix of final state $|\mathbf{\Psi}_\infty\rangle$ and tracing out the field degrees of freedom, we obtain the final reduced density matrix:

$$\rho_d = Tr_\phi(|\mathbf{\Psi}_\infty\rangle\langle\mathbf{\Psi}_\infty|) = \frac{1}{N} \begin{bmatrix} (1 - e^{-\omega/T})|c_0|^2 + \mu|c_1|^2 & (1 - e^{-\omega/T})c_0 c_1^* \\ (1 - e^{-\omega/T})c_0^* c_1 & \mu e^{-\omega/T}|c_0|^2 + (1 - e^{-\omega/T})|c_1|^2 \end{bmatrix}, \tag{4.26}$$

where $Tr_\phi$ denotes the partial trace of the field $\phi$ and $N$ is the normalization constant:

$$N = (1 - e^{-\omega/T}) + \mu e^{-\omega/T}|c_0|^2 + \mu|c_1|^2. \tag{4.27}$$

Now we obtained the evolved detector state, we got to the information about Unruh temperature $T$ encoded in $\rho_d$ as discussed in section 3.2. We shall calculate quantum Fisher information of $\rho_d$ related to temperature $T$. By doing so, we can analyze the optimal configurations of parameters of $\rho_d$ we control to get the best estimation precision of $T$. We are supposed to apply Eq. ( 3.18) to $\rho_d$ but to accomplish that, we need first to find spectral form of $\rho_d$.

To obtain the detector's density matrix spectral form we will, first, assume that coefficients $c_0, c_1 \in \mathbb{R}$. Since $|\psi^d_{-\infty}\rangle$ is normalized, we write its coefficients as $c_0 = \sin\eta$ and $c_1 = \cos\eta$. If $\eta = \pi/2$ ($c_0 = 1, c_1 = 0$) or $\eta = 0$ ($c_0 = 0, c_1 = 1$), $\rho_d$ is already in its spectral form and we can calculate quantum Fisher information forthwith. In order to have a clearer notation we will denote here on $J(T) = J_T$. We get:

$$J_T(\eta = \pi/2) = \frac{e^{-\omega/T}\mu\omega^2}{T^4(1 - e^{-\omega/T})(1 - e^{-\omega/T} + \mu e^{-\omega/T})^2}, \tag{4.28}$$

and

$$J_T(\eta = 0) = \frac{e^{-2\omega/T}\mu\omega^2}{T^4(1 - e^{-\omega/T})(1 - e^{-\omega/T} + \mu)^2}. \tag{4.29}$$

Now, for $\eta \in (0, \pi/2)$, we need to diagonalize $\rho_d$. standard diagonalization calculation leeds to:

$$\rho_d = \begin{pmatrix} 1/2 - \gamma & 0 \\ 0 & 1/2 + \gamma \end{pmatrix}. \tag{4.30}$$

The term $\gamma$ is written as:

$$\gamma = \frac{\sqrt{\gamma_1 + \gamma_2 + \gamma_3}}{\gamma_4}, \tag{4.31}$$

where

$$\gamma_1 = e^{-2\omega/T}(8 + \mu(4 + 3\mu)) - 2e^{-\omega/T}(8 + \mu^2) + (8 + \mu(-4 + 3\mu)), \tag{4.32}$$

$$\gamma_2 = 4\mu(1 - e^{-\omega/T})(-2 + \mu + e^{-\omega/T}(2 + \mu))\cos(2\eta), \tag{4.33}$$

$$\gamma_3 = \mu(1 + e^{-\omega/T})(-4 + \mu + e^{-\omega/T}(4 + \mu))\cos(4\eta), \tag{4.34}$$

$$\gamma_4 = 2\sqrt{2}\left[(e^{-\omega/T} + 1)\mu + (1 - e^{-\omega/T})(2 + \mu\cos(2\eta))\right], \tag{4.35}$$

and $f_\omega = (1 - e^{-\omega/T})$.

The eigenbasis of the density operator is $\{V_1, V_2\}$, with

$$V_1 = 1/N_1 \left(\frac{1}{4(1 - e^{-\omega/T})\sin(2\eta)}(\gamma_5 - \sqrt{2}\sqrt{\gamma_1 + \gamma_2 + \gamma_3}), 1\right), \tag{4.36}$$

and

$$V_2 = \frac{1}{N_2} \left( \frac{1}{4(1 - e^{-\omega/T}) \sin(2\eta)} (\gamma_5 + \sqrt{2}\sqrt{\gamma_1 + \gamma_2 + \gamma_3}), 1 \right), \qquad (4.37)$$

where

$$\gamma_5 = -2e^{-\omega/T}(\mu - (2 + \mu)\cos(2\eta)) + 2(\mu + (-2 + \mu)\cos(2\eta)), \qquad (4.38)$$

and $N_1$ and $N_2$ are the normalization constants

$$N_1 = \sqrt{\left( \frac{(\gamma_5 - \sqrt{2}\sqrt{\gamma_1 + \gamma_2 + \gamma_3})}{4(1 - e^{-\omega/T})\sin(2\eta)} \right)^2 + 1},$$

$$N_2 = \sqrt{\left( \frac{(\gamma_5 + \sqrt{2}\sqrt{\gamma_1 + \gamma_2 + \gamma_3})}{4(1 - e^{-\omega/T})\sin(2\eta)} \right)^2 + 1}. \qquad (4.39)$$

Now, with the spectral form Eq. ( 4.30) we can finally apply Eq. (3.18) which lead us to quantum Fisher information $J_T(\eta)$ for $\eta \in (0, \pi/2)$. The expression happens to be too cumbersome, so we present it in appendix I.

Quantum Fisher information $J_T$ depends on the parameters $(\eta, \mu, T)$. Parameter $\eta$ is related to the prepared initial state. Parameter $\mu$ is related to the period in which the acceleration is on, $\delta$, to the energy gap $\omega$ of the detector energy levels and the range of interaction of the detector $\kappa$. Finally $T$ is related to the constant acceleration $a$ by Unruh temperature expression $T = a/(2\pi)$.

In section 4.1 we presented the conditions of validity of the model: $\epsilon \ll 1$, $\delta\omega \gg 1$ and $\omega\kappa \ll 1$. Therefore, we set $\epsilon = 10^{-3}$, $\delta\omega = 2\pi 10^4$ and $\omega\kappa = 10^{-3}$ such that $\mu = 0.01$ here on. We also set $\omega = 1$ such that all the quantities of energy may be presented in units of $\omega$. We present quantum Fisher information as function of acceleration for some different prepared initial states in Fig. ( 4.2).

Note that quantum Fisher information has a maximum for accelerations of the order of the energy gap. This phenomenon is due to the fact that the responsiveness of detector's state to temperature change is low, and the proportional variation in $\rho_d$ is only appreciable when acceleration reaches the order of the detector's energy gap. As the acceleration increases the quantum Fisher information approaches asymptotically to zero. For high acceleration regime, i. e., for acceleration value much higher than $\omega$, the probability of a click in the detector will be already very high and it will be hard to distinguish two high temperature values since the responsiveness of the detector state in this regime will be very low.

This effect adds an additional challenge to Unruh effect experimental detection, since we expect that the higher the temperature, easier it is to detect it, but harder it will be to make a precise measurement. We present in Fig. ( 4.3) the behavior of the maximum value of quantum Fisher information $J_{Tmax}$ and its respective acceleration value $a_{max}$ as a function of the parameter $\eta$.
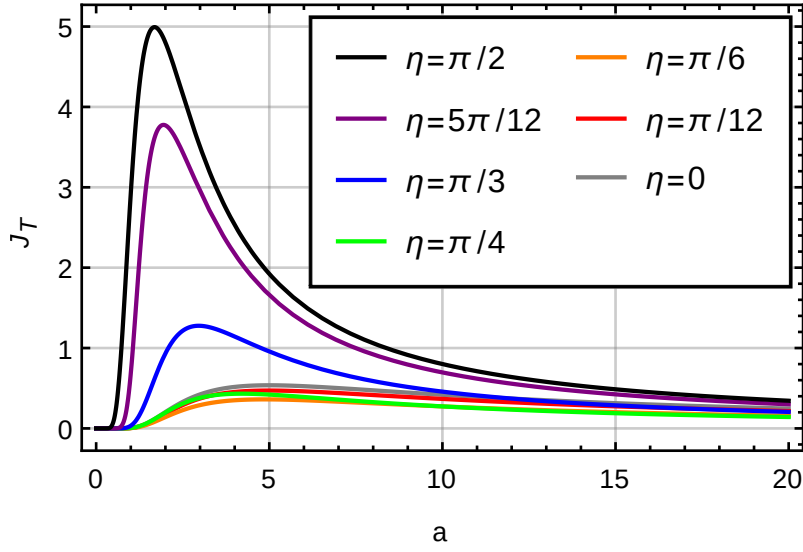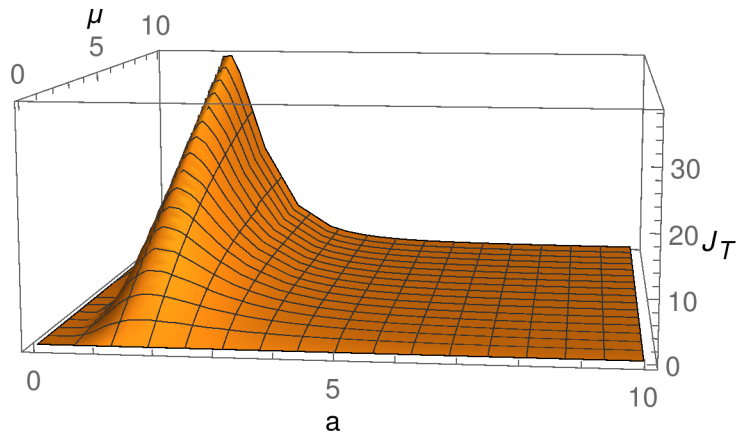
**Figure 4.2** | Quantum Fisher information $J_T$ as function of acceleration for the single detector and $\mu = 0.01$. Each line in the graphic represent a distinct initial state $|\psi_{-\infty}^d\rangle$ with a different value of $\eta$. Acceleration is in units of $\omega$ and $J_T$ in units of $(10^{-2}\omega^{-2})$.
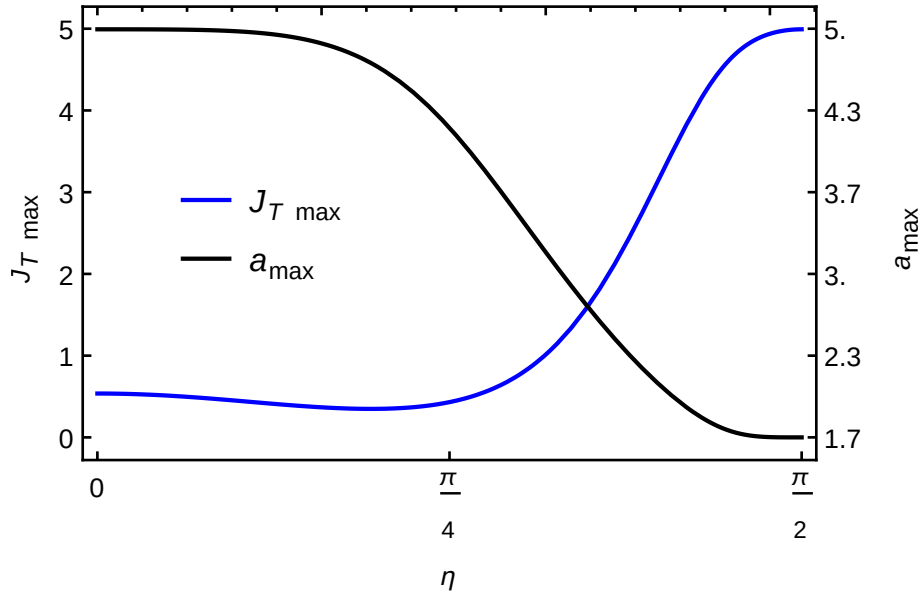


**Figure 4.3** | Maximum value of quantum Fisher information $J_{Tmax}$ and its respective acceleration value $a_{max}$ as function of initial state parameter $\eta$ and $\mu = 0.01$. Acceleration is in units of $\omega$ and $J_{Tmax}$ in units of $(10^{-2}\omega^{-2})$.

We note in the graphic that the highest value of $J_{Tmax}$ has the smallest value of $a_{max}$ (or temperature). This confirms that the higher sensibility of detector to temperature variation will happens for the smaller acceleration. This graphic also shows that the greatest accuracy of temperature detection holds for $\eta = \pi/2$ *i.e.* $|\psi_{-\infty}^d\rangle = |0\rangle$. First conclusion we can take from that is the coherence is not a quantum resource for Unruh temperature estimation. Second, this happens due to the probability of a spontaneous decay of the excited state because of the

interaction with the vacuum. The mixture between the spontaneous decay effect and the decay probability from contact to Unruh thermal bath makes a detector click less informative about Unruh temperature. At last we present the 3D graphic $J_T \times a \times \mu$ for $\eta = \pi/2$ in Fig. ( 4.4).



**Figure 4.4 |** Quantum Fisher information $J_T$ as function of acceleration $a$ and $\mu$ for the optimal estimation initial state parameter $\eta = \pi/2$. Acceleration is in units of $\omega$, $\mu$ has no dimension and $J_T$ in units of $(\omega^{-2})$.

The parameter $\mu$ is directly related to the period of interaction $\delta$ between detector and the external field. When the experimental apparatus is mounted, parameters $\epsilon$, $\omega$, $\kappa$ are fixed and we control only the time of duration of the acceleration of the detector. Figure ( 4.4) shows that for all values of $\eta$, quantum Fisher information increases when $\mu$ increases. This is already expected since greater the duration of interaction, larger probability of detectors transition. The increase of $\mu$ results in a greater rate of probability of detector's click growth which results in a greater proportional variation of $\rho_d$. This means the responsiveness of $\rho_d$ to temperature variation will be greater.

We have presented the results about quantum Fisher information, however, as discussed in section 3.3, QFI does not tell us about the estimation credibility since it depends on the relation between the variance of the estimation and the estimation mean value, for which QFI gives no information about. Therefore, we should look at the signal-to-noise ratio. Bellow, we will analyze the $Q_T$, the upper limit of the SNR, defined by Eq. ( 3.20). In Fig. ( 4.5) we plot the SNR for different initial states.

Note that the maximum of $Q_T$ is reached for accelerations two orders of magnitude greater than $J_T$. Despite the fact that $J_T$ reaches its maximum for acceleration of order of $\omega$, just after its maximum, $J_T$ approaches zero in a rate such that $Q_T \equiv T^2 J_T$ still increases. We observe $Q_T$ increasing until its maximum, around accelerations values between $a = 400\omega$ and $a = 630\omega$, depending on $\eta$, and from this point on $Q_T$ tends to zero.
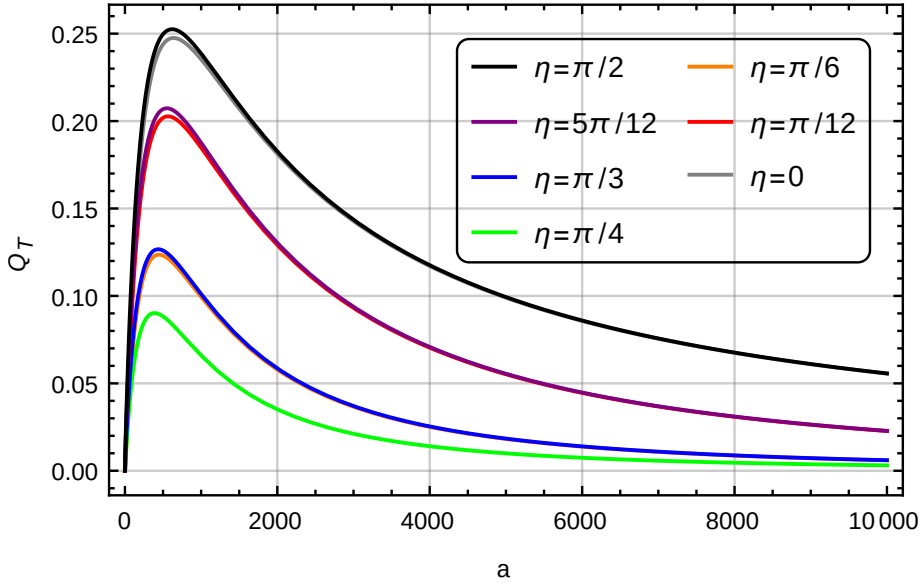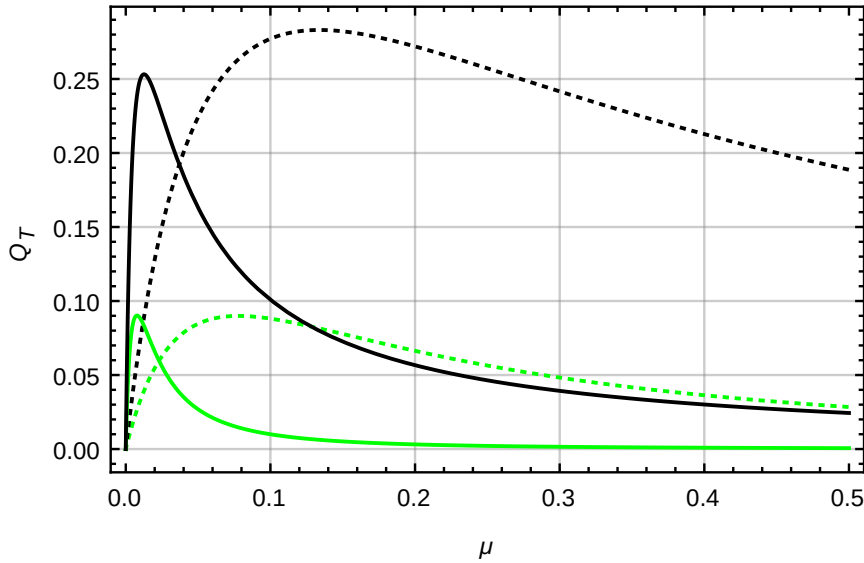
**Figure 4.5** $Q_T$ as function of acceleration for the single detector and $\mu = 0.01$. Each line in the graphic represent a distinct initial state $|\psi_{-\infty}^d\rangle$ with a different value of $\eta$. Acceleration is in units of $\omega$ and $Q_T$ has no dimension.
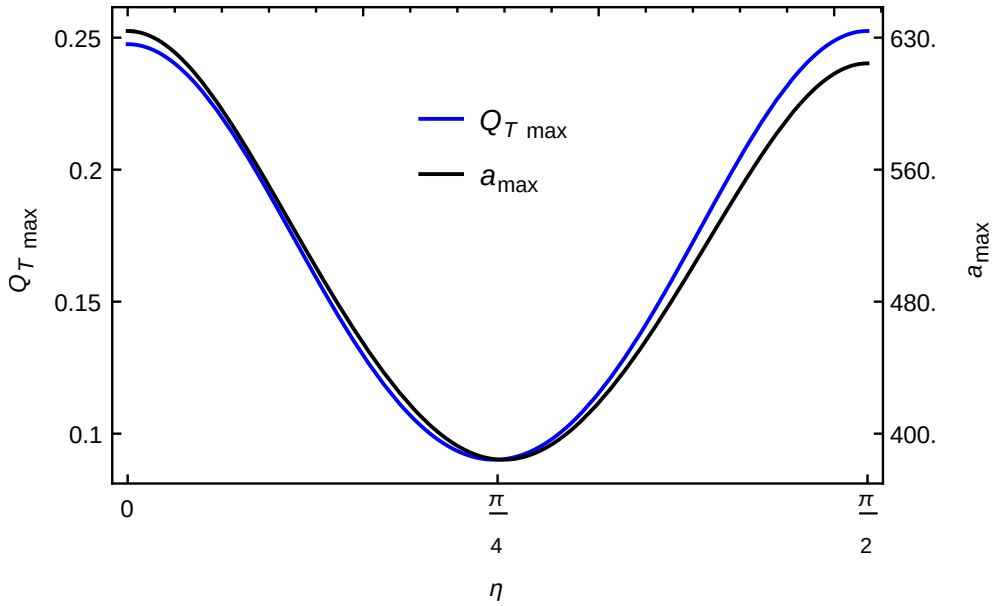


**Figure 4.6** Maximum value of $Q_T$, denoted by $Q_{Tmax}$, and its respective acceleration value $a_{max}$ as function of initial state parameter $\eta$ and $\mu = 0.01$.

We plot in Fig. ( 4.6) the maximum value of $Q_T$ denoted $Q_{Tmax}$ by its respective acceleration value denoted $a_{max}$. Here we see a remarkable negative correlation between the coherence of the initial state and $Q_{Tmax}$. We conclude that quantum coherence spoils the precision of estimation of Unruh temperature. We can also note the higher value of $Q_{Tmax}$ for $\eta = \pi/2$. This is also explained by the probability of spontaneous decay of the excited initial state which makes less informative the detector click, as discussed before.

**Figure 4.7** | $Q_T$ as function of acceleration $\mu$ for different initial state parameters $\eta$ and acceleration value. Green lines are $\eta = \pi/4$, black lines have $\eta = \pi/2$. Dotted lines have $a = 50\omega$ and solid lines have $a = 500\omega$.

Now we observe in Fig. ( 4.7) that, besides we showed that $J_T$ increases with $\mu$, we show here that $Q_T$ has a maximum value when seen as function of $\mu$. It means that after an optimal time of exposure of the detector $\delta$, the responsiveness of the final state to changes in $T$ decreases. We can observe that higher the acceleration value, smaller will be the optimal $\delta$ value. What is expected since large acceleration values results in high probability to detect a click of the probe sooner, which take us to the low responsiveness regime for smaller $\delta$.

We finish here our study of the role of coherence in Unruh temperature estimation precision. Next section we will present our results regarding the role of entanglement in measurement precision.

## 4.3   Estimation Precision and Entanglement

We consider now the case of two non-interacting detectors in Minkowski spacetime. Detector $A$ is kept inertial while detector $B$ has constant proper acceleration $a$ along the $x$ axis for the finite amount of proper time $\delta$. This configuration is refered, in metrology literature, by an ancilla test instead of an entanglement test. It is however, in relativistic quantum metrology literature refered most commonly as an entanglement test and that is how we call it in this study.

The worldline of detector $B$ is given by same expressions as Eq. ( 4.14):

$$t_B(\tau_B) = a^{-1} \sinh a\tau_B, \ x_B(\tau_B) = a^{-1} \cosh a\tau_B,$$
$$y_B(\tau_B) = z_B(\tau_B) = 0, \tag{4.40}$$

where $\tau_B$ is detector $B$ proper time, $(t_B, x_B, y_B, z_B)$ are the points of detector $B$ worldline and

$(t,x,y,z)$ are the usual Cartesian coordinates of Minkowski spacetime. We model interaction region with the same expression of the single detector case

$$\varphi_B(\mathbf{x}) = (\kappa\sqrt{2\pi})^{-3}\exp(-(\mathbf{x} - \mathbf{x}_B)^2/2\kappa^2), \tag{4.41}$$

with variance $\kappa = \text{const} \ll \omega^{-1}$.
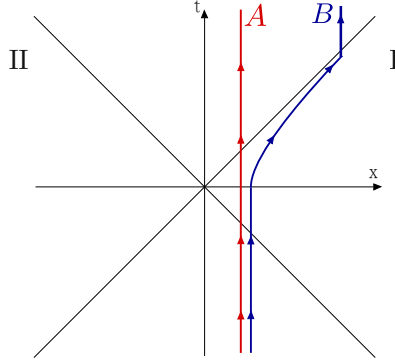


**Figure 4.8 |** Detectors' worldlines representation.

The initial state describing the system is:

$$|\mathbf{\Psi}_{-\infty}\rangle = |\psi^{dd}_{-\infty}\rangle \otimes |0_M\rangle, \tag{4.42}$$

where

$$|\psi^{dd}_{-\infty}\rangle = \sin\theta|01\rangle + \cos\theta|10\rangle. \tag{4.43}$$

Here $|ab\rangle \equiv |a\rangle \otimes |b\rangle$ where $|a\rangle \in \mathcal{H}_A$ and $|a\rangle \in \mathcal{H}_B$, $a,b \in \{0,1\}$, $\mathcal{H}_A$ and $\mathcal{H}_B$ are the Hilbert space associated to detector $A$ and $B$ respectively.

The free Hamiltonian for detectors is given by Eq. (4.1):

$$H_A = \omega d_A^\dagger d_A, \tag{4.44}$$

$$H_B = \omega d_B^\dagger d_B. \tag{4.45}$$

Here $d_A$ and $d_B$ denotes transitions operators of detectors $A$ and $B$. Detector $A$ is kept switched off during the entire process. It means it does not interact with external field, except indirectly. Detector $B$ is switched on during the proper time interval $\delta$ and interacts with the field through the rule Eq. (4.3). The total Hamiltonian of the system is given by:

$$H = H_A + H_R + H_{KG} + H_{int}. \tag{4.46}$$

Therefore we have the time independent Hamiltonian $H_0 = H_A + H_R + H_{KG}$ and $H_{int}$ analogous to Eq. ( 4.3) such that the calculations from section 4.1 holds and we can evolve the state $|\mathbf{\Psi}_{-\infty}\rangle$

up to first order with Eq. (4.13). It results:

$$|\mathbf{\Psi}_\infty\rangle = (\sin\theta|01\rangle + \cos\theta|10\rangle) \otimes |0_M\rangle - \cos\theta|11\rangle \otimes (a_{RI}(\lambda^*)|0_M\rangle)$$
$$+ \sin\theta|00\rangle \otimes \left(a_{RI}^\dagger(\lambda)|0_M\rangle\right) \tag{4.47}$$

Again we denote annihilation and creation operators as $a_{RI}$ and $a_{RI}^\dagger$ to emphasize that these are the operators related to Rindler modes in wedge I according to Rindler time isometry. We also denote as $a_M$ and $a_M^\dagger$ the operators related to Minkowski modes. Applying Eq. ( 4.19) and Eq. ( 4.20) we find:

$$|\mathbf{\Psi}_\infty\rangle = (\sin\theta|01\rangle + \cos\theta|10\rangle) \otimes |0_M\rangle - \frac{\cos\theta\mu^{1/2}e^{-\omega/(2T)}}{(1-e^{-\omega/T})^{1/2}}|11\rangle \otimes |1_{\tilde{F}_{2\omega}}\rangle$$
$$+ \frac{\sin\theta\mu^{1/2}}{(1-e^{-\omega/T})^{1/2}}|00\rangle \otimes |1_{\tilde{F}_{1\omega}}\rangle, \tag{4.48}$$

where $F_{1\omega}$ and $F_{2\omega}$ are defined by Eq. ( 4.21) and Eq. ( 4.22) respectively and $|1_{\tilde{F}_{1\omega}}\rangle$ and $|1_{\tilde{F}_{2\omega}}\rangle$ are normalized states associated to one quantum excitation of external field in modes $F_{1\omega}$ and $F_{2\omega}$.

Calculating the density matrix of final state $|\mathbf{\Psi}_\infty\rangle$ and tracing out the field degrees of freedom, we obtain the final reduced density matrix:

$$\rho_{dd} = Tr_\phi\left(|\mathbf{\Psi}_\infty\rangle\langle\mathbf{\Psi}_\infty|\right) = \begin{bmatrix} P_1 & 0 & 0 & 0 \\ 0 & P_0\sin^2\theta & P_0\sin\theta\cos\theta & 0 \\ 0 & P_0\sin\theta\cos\theta & P_0\cos^2\theta & 0 \\ 0 & 0 & 0 & P_2 \end{bmatrix}. \tag{4.49}$$

Here we use the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

$$P_0 = \frac{1-e^{-\omega/T}}{(1-e^{-\omega/T}) + \mu(\sin^2\theta + e^{-\omega/T}\cos^2\theta)}, \tag{4.50}$$

$$P_1 = \frac{\mu\sin^2\theta}{(1-e^{-\omega/T}) + \mu(\sin^2\theta + e^{-\omega/T}\cos^2\theta)}, \tag{4.51}$$

$$P_2 = \frac{\mu e^{-\omega/T}\cos^2\theta}{(1-e^{-\omega/T}) + \mu(\sin^2\theta + e^{-\omega/T}\cos^2\theta)}. \tag{4.52}$$

Now we have the evolved state where the information about Unruh temperature is encoded, we shall obtain the spectral form of $\rho_{dd}$ so we can apply Eq. ( 3.18) to obtain the quantum Fisher information $J_T$. First we should note that for $\theta = 0$ and $\theta = \pi/2$ (*i.e.* $|\psi_{-\infty}^{dd}\rangle = |10\rangle$ and $|\psi_{-\infty}^{dd}\rangle = |01\rangle$ respectively) Eq. ( 4.49) is already in its spectral form and we can apply Eq.

( 3.18) straightforwardly to obtain:

$$J_T(\theta = 0) = \frac{e^{-\omega/T}\mu\omega^2}{T^4(1 - e^{-\omega/T})((1 - e^{-\omega/T}) + e^{-\omega/T}\mu)^2}, \tag{4.53}$$

and

$$J_T(\theta = \pi/2) = \frac{e^{-2\omega/T}\mu\omega^2}{T^4(1 - e^{-\omega/T})((1 - e^{-\omega/T}) + \mu)^2}. \tag{4.54}$$

However, for $\theta \in (0,\pi/2)$, $\rho_{dd}$ is not diagonalized and we need to make standard diagonalization calculations to obtain:

$$\rho_{dd} = \begin{bmatrix} P_0 & 0 & 0 & 0 \\ 0 & P_1 & 0 & 0 \\ 0 & 0 & P_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.55}$$

in the basis

$$\{\sin\theta|01\rangle + \cos\theta|10\rangle, |00\rangle, |11\rangle, \cos\theta|01\rangle - \sin\theta|10\rangle\}. \tag{4.56}$$

Note that the basis above does not depend on $T$. That implies that the quantum term of Eq. ( 3.18) vanishes and the expression simplifies to:

$$J_T = \frac{(\partial_T P_0)^2}{P_0} + \frac{(\partial_T P_1)^2}{P_1} + \frac{(\partial_T P_2)^2}{P_2}. \tag{4.57}$$

Substituting Eq. ( 4.50), Eq. ( 4.51) and Eq. ( 4.52) in Eq. ( 4.57) we obtain:

$$J_T = \frac{e^{-\omega/T}\mu\omega^2(e^{-\omega/T}(4(1 - \cos(2\theta)) + \mu(-1 + \cos(4\theta))) + 4(1 + \cos(2\theta)) + \mu(1 - \cos(4\theta)))}{2T^4(1 - e^{-\omega/T})(e^{-\omega/T}(-2 + \mu(1 + \cos(2\theta))) + 2 + \mu(1 - \cos(2\theta)))^2}. \tag{4.58}$$

Behavior of $J_T$ is presented in the graphic of Fig. ( 4.9).

We can see that, as in the single detector case, the maximum value of quantum Fisher information holds for acceleration values of the order of the detector's energy gap $\omega$. The explanation is analogous to single detector case, the state $\rho_{dd}$ has low responsiveness to Unruh temperature variation, so the proportional variation of $\rho_{dd}$ with $T$ is just appreciable for small transition probability which happens for such values of acceleration. The graphic of Fig. ( 4.10) corroborates this analysis. Note that the greater value of $J_{Tmax}$ has the lowest value of $a_{max}$.

From Fig. ( 4.9), Fig. ( 4.10) and Fig. ( 4.11) we can conclude that the entanglement is not a quantum resource for Unruh temperature estimation. This result contradicts what was suggested in literature [48]. The best initial state for temperature estimation is the separable state $|\psi_{-\infty}^{dd}\rangle = |10\rangle$ ($\theta = 0$). Entanglement seems to not affect QFI. The graphic of Fig. ( 4.11) present the concurrence as a measure of entanglement (sec. D) overlaid with $J_T$ as a function of $\theta$ for distinct values of acceleration. We see no relation between entanglement and $J_T$.
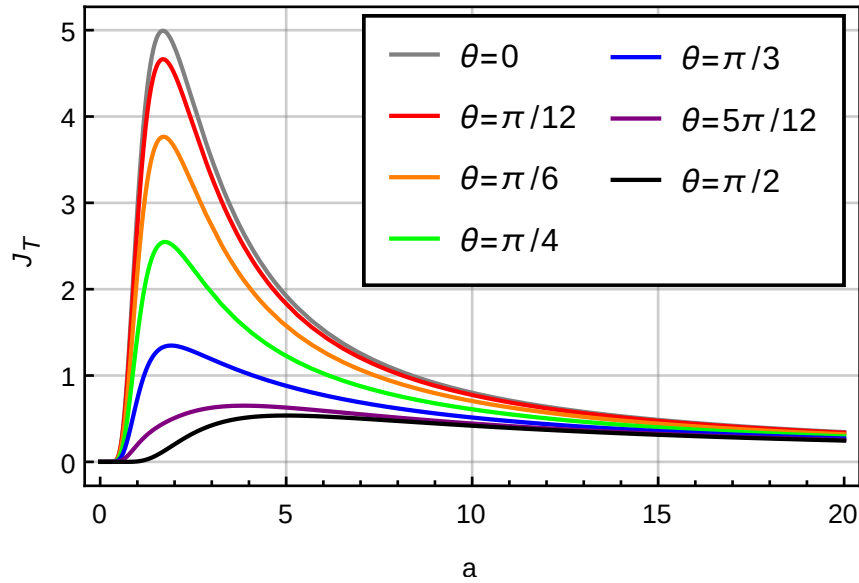
**Figure 4.9** | Quantum Fisher information $J_T$ as function of acceleration $a$ and $\mu = 0.01$. Each line represents a different initial state related to a different value of parameter $\theta$. Acceleration is in units of $\omega$ and $J_T$ in units of $(10^{-2}\omega^{-2})$.
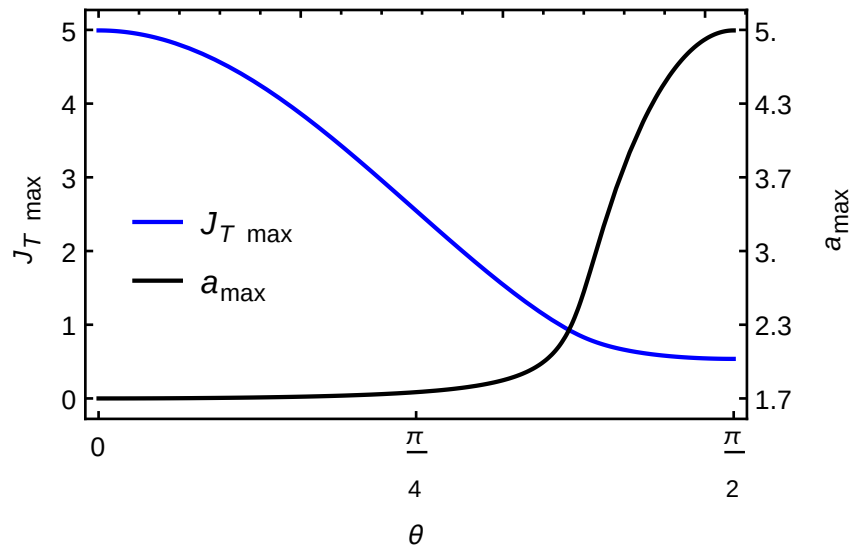


**Figure 4.10** | Maximum value of Quantum Fisher information $J_{Tmax}$ and its respective acceleration $a_{max}$ as function of initial state parameter $\theta$ and $\mu = 0.01$. Acceleration is in units of $\omega$ and $J_T$ in units of $(10^{-2}\omega^{-2})$.

We can also note from Eq. ( 4.53), Eq. ( 4.54), Eq. ( 4.58) and Fig. ( 4.9) that $J_T \rightarrow 0$ as $a \rightarrow \infty$. This can be explained by the low responsiveness of the state $\rho_{dd}$ to variation in temperature in high acceleration regimes. For great acceleration values the final state will change so subtly that it is a hard task to distinguish two distinct values of Unruh temperature in this regime. As in single detector case we face an additional challenge for Unruh effect experimental verification, since detection of small temperatures is a much harder task than

high temperature detections, but in high Unruh temperature regimes we can not have a precise estimation of temperature.

The initial state preparation which results in the worst Unruh temperature estimation precision is $\theta = \pi/2$ ($|\psi_{-\infty}^{dd}\rangle = |01\rangle$). As in the single detector case this happens because, as detector $A$ does not interact with external field, the state $|10\rangle$ can only flip to $|11\rangle$ if accelerated, instead, the state $|01\rangle$ can flip to $|00\rangle$ even without acceleration because of spontaneous decay probability. The mixture between spontaneous decay phenomenon and the transition probability due to Unruh effect results in a less amount of information about temperature.
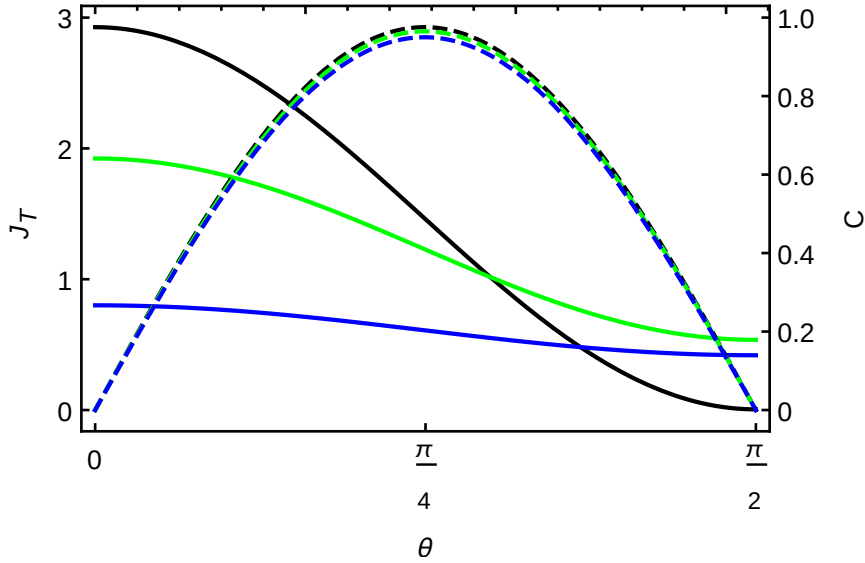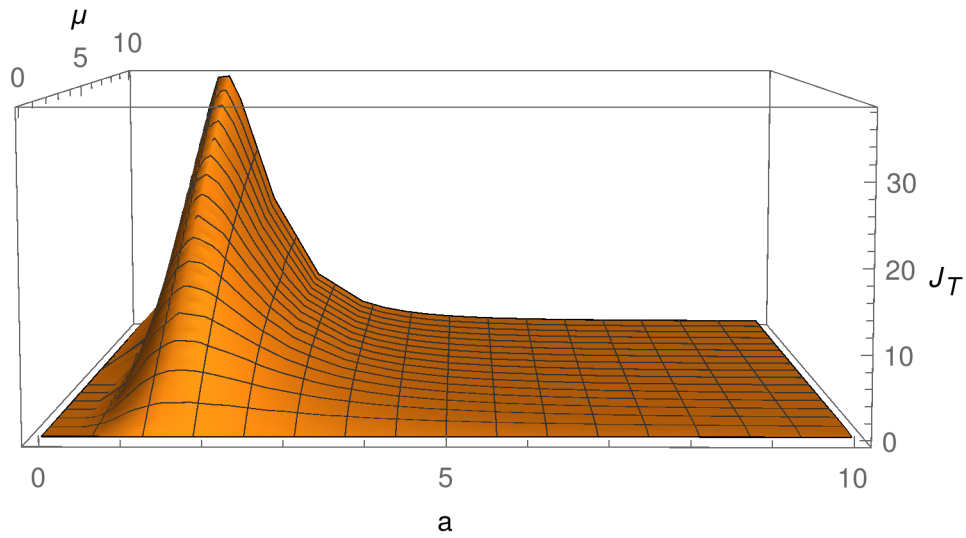


**Figure 4.11 |** Quantum Fisher information $J_T$ in solid line overlaid with concurrence $C$ in dashed line as function of parameter $\theta$ and $\mu = 0.01$. Each line represents a different acceleration value. For black line $a = 1\omega$, green line $a = 5\omega$ and blue line $a = 10\omega$. Concurrence has no dimension and $J_T$ in units of $(10^{-2}\omega^{-2})$.

We point out that $J_T$ increases with $\mu$, as shown in Fig ( 4.12). This is expected since a larger $\mu$ is related with a longer interaction time $\delta$ which increases transition probability slope and consequently the proportional variation of $\rho_{dd}$ with acceleration.

Now on we shall analyze the signal-to-noise ratio. As we did in single detector case, we shall look at SNR upper limit $Q_T$. In Fig. ( 4.13) we plot $Q_T$ as function of acceleration for some different values of $\theta$.

We observe that the initial state has modest influence on $Q_T$. This image corroborates that entanglement does not play a role in our Unruh temperature estimation scheme. As in single detector case, $Q_T$ reaches its maximum value for accelerations two orders of magnitude greater than the acceleration $J_T$ reaches its maximum.

In Fig. ( 4.14) we plot the maximum value of $Q_T$, denoted by $Q_{Tmax}$, overlapped by its acceleration value, denoted by $a_{max}$, as function of $\theta$. The greatest value of $Q_{Tmax}$ is obtained for $\theta = 0$ ($|\psi_{-\infty}^{dd}\rangle = |10\rangle$). This can be explained by the vanishing spontaneous decay probability of this state.

**Figure 4.12** │ A 3D graphic of Quantum Fisher information $J_T$ as function of acceleration $a$ and parameter $\mu$ for the optimal temperature estimation value $\theta = 0$. Acceleration is in units of $\omega$, parameter $\mu$ has no dimension and $J_T$ in units of $(\omega^{-2})$.
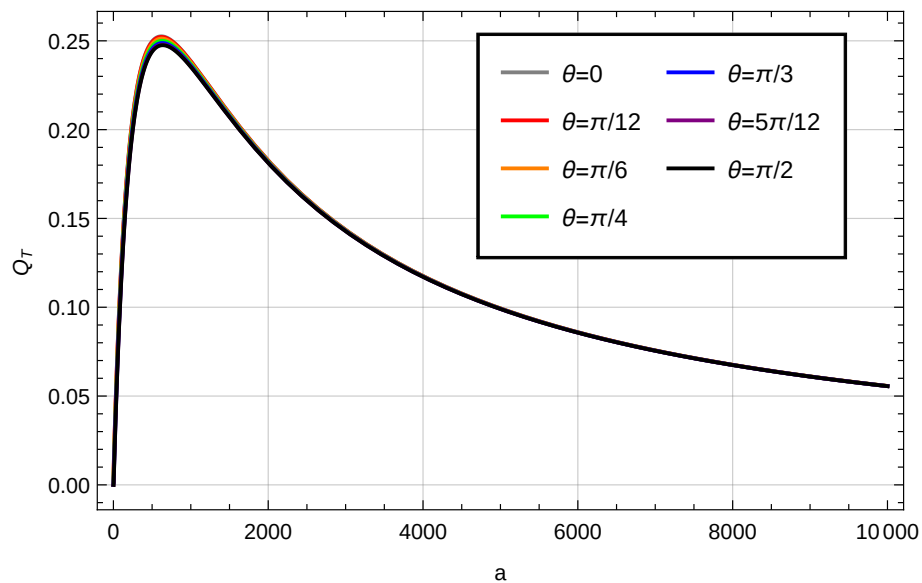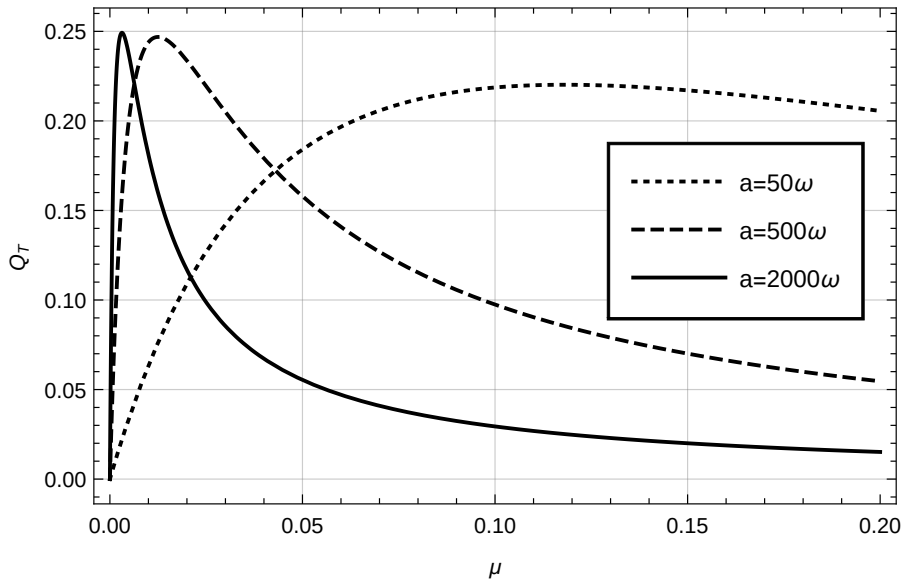


**Figure 4.13** │ $Q_T$ as function of acceleration for the two detector case with $\mu = 0.01$. Acceleration is in units of $\omega$ and $Q_T$ has no dimension.

Just like in single detector case, in the current two detector case, while $J_T$ increases with $\mu$, we can see by Fig. ( 4.15) that $Q_T$ has a maximum value when seen as function of $\mu$. This maximum is reached for $\mu < 1$. We also observe that for higher acceleration values, the value of
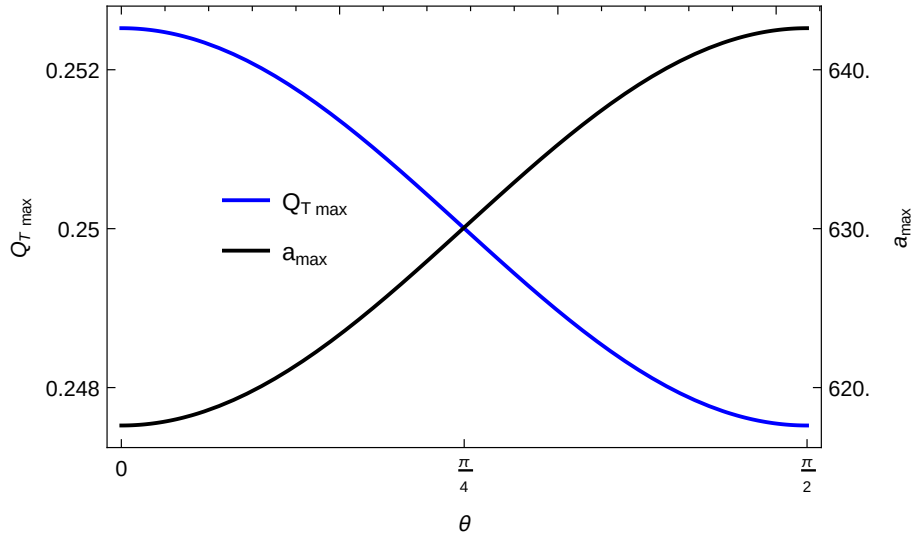
**Figure 4.14** | Maximum value of $Q_T$, denoted by $Q_{Tmax}$, and its respective acceleration value $a_{max}$ as function of initial state parameter $\theta$ and $\mu = 0.01$.
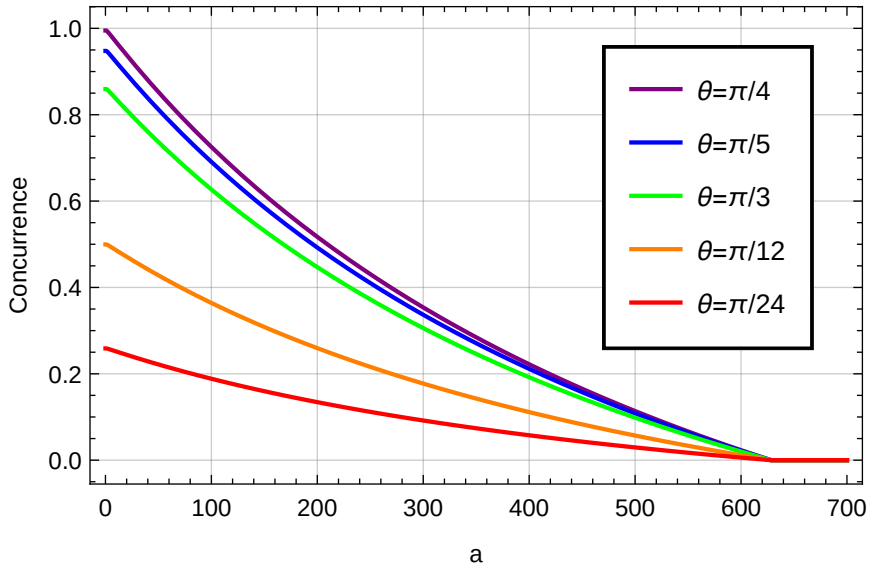


**Figure 4.15** | $Q_T$ as function of acceleration $\mu$ for $\theta = 0$ and different acceleration values. We observe $Q_T$ decreasing for long periods of time $\delta$.

$\mu$ in which $Q_T$ reaches its maximum is smaller. This is due to the fact that for large acceleration values, sooner the detector reaches the high probability of clicking regime when the final state loses responsiveness regarding Unruh temperature variation.

Now, we shall present an interesting relation between $Q_{Tmax}$ and the entanglement sudden death acceleration value. In 2009, A. Landulfo and G. Matsas [49] observed the entanglement sudden death (ESD) in the same two detector system we are studying. If we prepare an entangled initial state, the entanglement of the system vanishes for a finite value of acceleration. Interestingly, if we quantify entanglement by concurrence, ESD acceleration value does

not depend on initial state concurrence, since, of course, we do not have as initial state a separable state. This is shown by Fig. ( 4.16). Note that for different initial states with different concurrence values, the acceleration at which concurrence vanishes is the same. This is the entanglement sudden death acceleration value.



**Figure 4.16** | Concurrence as function of acceleration for different $\theta$, here $\mu = 0.01$. Notice that etanglement sudden death acceleration value is the same for all different $\theta$.

It happens that $Q_{Tmax}$ is related to entanglement sudden death acceleration value. In Fig. ( 4.17) we plot the value of acceleration $a_{max}$ for different values of $\theta$ as a function of $\mu$ overlapped by ESD acceleration value. Observe that $a_{max}$ for $\theta = \pi/4$ coincides with ESD acceleration value.

The graphic of Fig. ( 4.17) interestingly suggests that, despite the fact that entanglement plays no role in Unruh temperature estimation precision or reliability, the decoherence introduced by Unruh Effect that affects the entanglement between both detectors affects the precision of Unruh temperature estimation the same way. The difference observed from ESD acceleration value and $a_{max}$ for $\theta = 0$ and $\theta = \pi/2$ can be explained by the role of spontaneous decay effect in estimation credibility.

We conclude here the presentation and the discussion of our original results. In next section we translate our previous results for Hawking radiation estimation precision.

## 4.4   Estimation Precision in Schwarzschild Black Hole

Although Unruh effect occurs in flat spacetime, we can consider an extension of our results to the estimation of the Hawking temperature in Schwarzschild spacetime. The metric, in
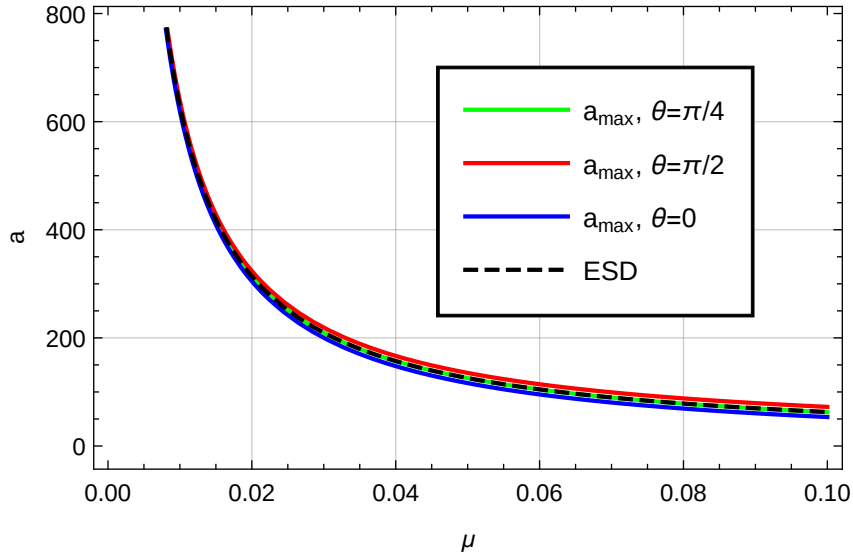
**Figure 4.17** | Acceleration value where $Q_T$ finds its maximum $a_{max}$ as function of $\mu$ for different $\theta$ overlapped with Entanglement Sudden Death (ESD) acceleration value as function of $\mu$. Notice that ESD coincides with $a_{max}$ for $\theta = \pi/4$.

Schwarzschild coordinates and making gravitational constant $G = 1$ is:

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2, \qquad (4.59)$$

where $t$ and $r$ denote the time and radial coordinates respectively, $d\Omega$ denotes the $S^2$ differential element and $m$ the mass in the spacetime.

Schwarzschild coordinates does not cover all the manifold properly since we got a coordinate singularity at $r = 2m$ in Eq. (4.59) [4, 50, 82]. We solve that problem by using a more suitable coordinate which covers all the manifold. We denote:

$$\bar{r} = r + 2m \ln |r/2m - 1|. \qquad (4.60)$$

and define the null coordinates

$$U = -4m e^{(t+\bar{r})/4m}, \qquad (4.61)$$

$$V = 4m e^{(t-\bar{r})/4m}. \qquad (4.62)$$

Null coordinates leads to the following metric:

$$ds^2 = -\frac{2m}{r} e^{-r/2m} dU dV + r^2 d\Omega. \qquad (4.63)$$

Note that metric Eq.(4.63) does not present a singularity at $r = 2m$ and defines $r$ implicitly by

$$- UV = 16m^2 \left( \frac{r}{2m} - 1 \right) e^{r/2m}. \tag{4.64}$$

Coordintes $U$ and $V$ are defined only for $U < 0$ and $V > 0$. However, It's possible to make an analytic continuation to extend the domain of $U$ and $V$ to $(-\infty, \infty)$ [82]. This way we got $UV = 0$ if $r = 2m$ and $UV = 16m^2$ for $r = 0$. Therefore the quadrant $U > 0, V < 0$ denotes a copy of $U < 0, V > 0$, a space identical to the region out of event horizon ($r \geq 2m$). If we define new compactified coordinates

$$u = 2 \arctan(U/4m), \tag{4.65}$$
$$v = 2 \arctan(V/4m), \tag{4.66}$$

we can ilustrate the extended Schwarzschild solution in Figure (4.18).
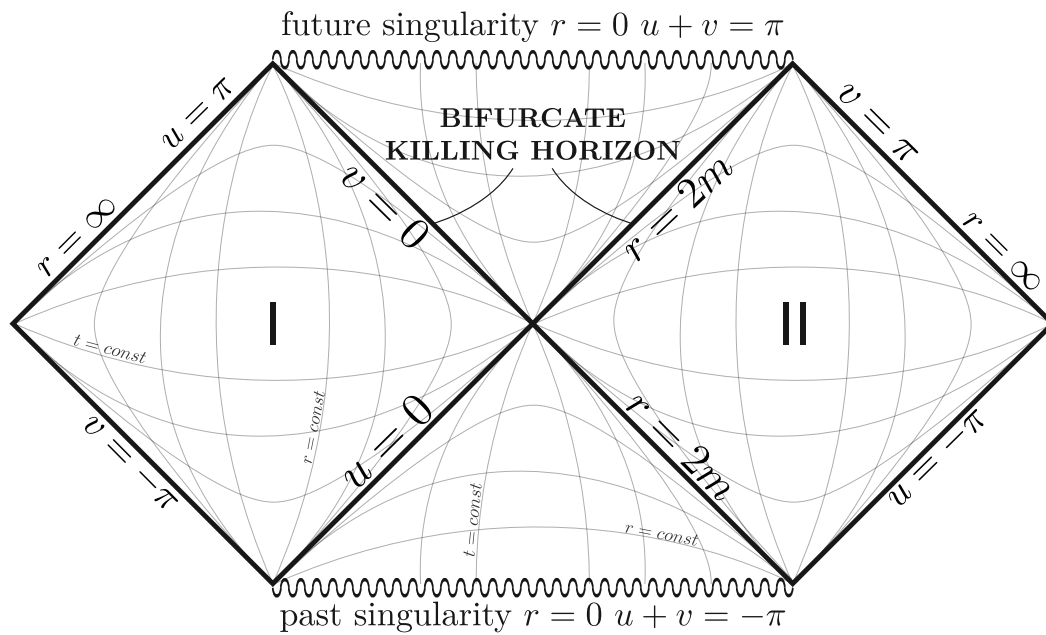


**Figure 4.18** | Extended Schwarzschild diagram. Angular coordinates are suppressed, so each point of the diagram represent a $S^2$. Wedges I and II are two identical copies of $r > 2m$ region.

Since Schwarzschild coordinate does not properly cover $r = 2m$, in order to make analysis on the event horizon we shall introduce other coordinates

$$\bar{t} = t + 2m \ln |r/2m - 1|. \tag{4.67}$$

In this new coordinate the spacetime metric becomes:

$$ds^2 = - \left( 1 - \frac{2m}{r} \right) d\bar{t}^2 + \frac{4m}{r} d\bar{t}dr + \left( 1 + \frac{2m}{r} \right) dr^2 + r^2 d\Omega^2. \tag{4.68}$$

The above equation is well behaved at $r = 2m$. We can proceed an analytical continuation so that $r$ extends its domain to $r > 0$ (or verify that Eq. (4.68) solves Einstein's equation for $r > 0$).

With $(\bar{t}, r, \theta, \phi)$ coordinates we can define the static Killing field $\chi = \partial_{\bar{t}}$ at $r = 2m$. As

$$\langle \partial_{\bar{t}}|_{r=2m}, \partial_{\bar{t}}|_{r=2m} \rangle = 0, \tag{4.69}$$

The Killing field $\partial_{\bar{t}}$ is a null vector at the hypersurface $r = 2m$. It's also normal to this hypersurface since for any vector of the form $v = v_{\bar{t}} \partial_{\bar{t}}|_{r=2m} + v_\theta \partial_\theta|_{r=2m} + v_\phi \partial_\phi|_{r=2m}$, where $\theta, \phi$ denotes usual angular coordinates of $S^2$, we have $\langle \partial_{\bar{t}}|_{r=2m}, v \rangle = 0$. Therefore we call $r = 2m$ a Killing horizon [1]. The event horizon from the extended Schwarzchild spacetime (Fig. ( 4.18)) can be generated by null geodesics orthogonal to surface of $\bar{t} = 0, r = 2m$. Such event horizons are called *bifurcate Killing horizon* [1]. It divides spacetime in four wedges.

If there is a nonsingular vacuum state (*i.e.* Hadamard form) which is invariant under isometries generated by the Killing field $\partial_{\bar{t}}$, then the restriction of such vacuum to one wedge will be a thermal state [83]. Restricting ourselves to the right wedge, $I$, this state can be written as

$$\rho = \prod_i \left( (1 - e^{-2\pi\omega_i/\kappa}) \sum_{n_{iI}} e^{-2\pi\omega_i/\kappa} |n_{iI}\rangle\langle n_{iI}| \right). \tag{4.70}$$

The index $i$ runs over the filed modes. Here, $|n_{iI}\rangle$ is the $n_i$'th excited state of mode $i$ in wedge I. Hawking temperature is

$$T = \kappa/(2\pi), \tag{4.71}$$

where $\kappa$ denotes the surface gravity.

$$\kappa = \lim(a(-\langle \chi, \chi \rangle)). \tag{4.72}$$

The term $-\langle \chi, \chi \rangle$ is the redshift factor and $a$ denotes the proper acceleration of orbits of $\chi$ just outside the horizon *i.e.* $a = (a^c a_c)^{1/2}$ where $a^c = \chi^b \nabla_b \chi^c$. In the vicinity of the horizon $\chi$ will be timelike. In the limit where we approach the horizon $a \to \infty$ and $\langle \chi, \chi \rangle \to 0$.

Note that the definition of a thermal state in Schwarzschild spacetime in the terms we presented depends on the existence of a Hadamard form vacuum. To prove the existence of such a vacuum in a general spacetime is quite a hard task. However it has been proved such existence in Schwarzschild spacetime by Hartle and Hawking [84, 85]. This vacuum state is named after them, it is called the Hartle-Hawking vacuum $|0_{HH}\rangle$. It is invariant under the isometries generated by the static killing field $\chi$. Therefore, when restricted to region $I$ of Fig. (4.18) it assumes the form of Eq. (4.70).

The Hawking temperature, as measured by an observer following a worldline in the direction

of $\partial_{\bar{t}}$, is given by

$$T_H = \frac{\kappa}{2\pi(-\langle\chi,\chi\rangle)}. \tag{4.73}$$

From Eqs. (4.72) and (4.73) we can see that, in the neighborhood of the horizon, Hawking temperature is approximately given by Unruh temperature $T = a/(2\pi)$. So, static observers near the horizon of a Schwarzschild black hole experience the same temperature as the observers with proper acceleration $a$ in Minkowski spacetime.

Therefore, our conclusions regarding the behavior of Fisher information and signal-to-noise ratio bound $Q_T$ under constant acceleration in flat spacetime also hold for a static observer in the vicinity of a Schwarzschild black hole. From the above calculations, the approximation $T_H \approx a/(2\pi)$ holds just if we are very close to the horizon, where $a$ takes a great value, implying bad precision of the estimation of Hawking temperature according to our results expressed in section 4.2 and 4.3. Remember we observed the greatest temperature estimation precision for small values of acceleration regarding $\omega$ scale. However, this problem can be solved if we can employ an Unruh-DeWitt detector with a very large energy gap.

But, how large should be detector's energy gap? From standard calculations of general relativity we conclude that, in the worldline which follows $\partial_{\bar{t}}$, *i.e.*, a static observer, we have

$$a = \frac{m/r^2}{(1 - 2m/r)^{1/2}}. \tag{4.74}$$

Therefore, near the horizon we have

$$T \approx \frac{\frac{m}{2\pi r^2}}{(1 - 2m/r)^{1/2}}. \tag{4.75}$$

For both cases, the single detector and two detectors, the acceleration value of maximum estimation precision was the same $1.6863\omega$. So, in order to obtain the best estimation precision according to our first order approximation of the probe state (one-click experiment), we need a detector with energy gap near $\omega_{max_{QFI}}$:

$$\omega_{max_{QFI}} \approx \frac{\frac{m}{r^2}}{1.6863\left(1 - \frac{2m}{r}\right)^{1/2}}, \; as \; r \to 2m. \tag{4.76}$$

Now, in order to determine the best estimation reliability we need to look at $Q_T$. The single detector and two detectors results in similar maximum values of $Q_T$, but the two detector set up reaches $Q_{Tmax}$ for smaller acceleration, than we will present only the detector with energy gap regarding this case. The acceleration value of maximum estimation reliability was approximately $618\omega$. So, we need a detector with energy gap near $\omega_{max}$:

$$\omega_{max} \approx \frac{\frac{m}{r^2}}{618\left(1 - \frac{2m}{r}\right)^{1/2}}, \; as \; r \to 2m. \tag{4.77}$$

We should point here, however, that a too large $\omega_{max}$ may not be useful as a thermality probe [86]. We finish here the extension of estimation precision results for Unruh temperature to estimation precision of Hawking temperature in Schwarzschild blackhole. We concluded that, since the analogy between Hawking temperature and Unruh temperature holds in the vicinity of event horizon, where $a \rightarrow \infty$, Hawking temperature estimation precision will be poor unless we can afford a detector with a very high energy gap. We presented an estimation of the value $\omega_{max}$ of the needed detector's energy gap which provides an optimal Hawking temperature estimation precision.

# Chapter 5

# Final Discussion

Unruh effect was discovered by Bill Unruh in 1976 in an attempt to gain insight about Hawking radiation. It is one of the most important results from quantum field theory in curved spacetime. It states that in Minkowski spacetime where an inertial observer can see only vacuum, an observer with constant acceleration will see himself immersed in a thermal bath. Unruh effect shows that the concept of particle depends of the observer. However, this phenomenon was not experimentally verified since it predicts that it's necessary an acceleration of order of $10^{20}m/s^2$ so we can observe a temperature of order $1K$. The engineering required to obtain linear acceleration to reach a detectable temperature was not achieved yet. To experimentally verify this physics fundamentals effect would be an important milestone.

Quantum metrology is a field of physics dedicated to achieve estimation of parameters from quantum systems with optimal precision. The estimation protocol has four steps: $(1)$ preparation of probe state; $(2)$ interaction of probe state with the system of interest; $(3)$ measurement of the probe; $(4)$ decode information about the parameter of interest from measured data. Quantum metrology looks for employments of quantum resources like quantum coherence and entanglement in order to improve parameter estimation. We can and should apply quantum metrology protocols and techniques to investigate Unruh temperature estimation to provide theoretical support to experimental attempts of Unruh effect observations. In this context we developed the studies which based this PhD thesis.

In chapter 2 we presented Unruh effect in further details. We introduced quantum scalar fields in curved static spacetimes formalism. We introduced Bogoliubov transformations. We presented Rindler coordinates and discussed how trajectories of constant spatial coordinates in Rindler wedges are related to accelerated trajectories in Minkowski spacetime. We showed Unruh effect as the Minkowski vacuum written in Rindler modes by Bogoliubov transformations results in a thermal state.

In chapter 3 we introduced we talked about quantum Fisher information in its classical and quantum contexts, discussing the main role this quantity plays in parameter estimation precision. We also defined the signal-to-noise ration and its upper boundary $Q_T$.

We present our original result in chapter 4. We prepare a Unruh-DeWitt detector coupled to an external scalar massless field. We accelerate the detector for a finite amount of proper time $\delta$ or until it clicks and measure the detector's state. We observed that maximum quantum Fisher Information is obtained for acceleration values of order of the detector energy gap $\omega$. We concluded this phenomenon is due to the fact that the responsiveness of detector's state to temperature change is low, and the proportional variation in $\rho_d$ is only appreciable for low values of acceleration. We also observed that as the acceleration increases the quantum Fisher information goes asymptotically to zero. This effect adds an additional challenge to Unruh effect experimental detection, since we expect that the higher the temperature, easier it is to detect it, but harder it will be to make a precise measurement.

We also concluded that the greatest precision of temperature detection holds for initial prepared probe in unexcited state. This imply the coherence is not a quantum resource for Unruh temperature estimation. This phenomenon happens due to the probability of a spontaneous decay of the excited state. The mixture between the spontaneous decay effect and the decay probability from contact to Unruh thermal bath makes a detector click less informative about Unruh temperature.

We also concluded that for initial prepared state of the probe, quantum Fisher information increases with time exposition $\delta$. This is already expected since greater the duration of interaction, larger probability of detectors transition and the responsiveness of $\rho_d$ to changes in temperature will be greater.

In order to determine the Unruh temperature estimation reliability, we analyzed the signal-to-noise ratio bound $Q_T$. We observed that that maximum of $Q_T$ is reached for accelerations two orders of magnitude greater than $J_T$.

We found a remarkable negative correlation between the coherence of the detector state and the maximum estimation SNR. Quantum coherence spoils the credibility of Unruh temperature estimation. We also noted through $Q_T$ analysis the consequence of spontaneous decay probability of the excited initial state. It results to harm Unruh temperature estimation credibility.

We observed that reliability of Unruh temperature estimation has an optimal period of acceleration $\delta$. The responsiveness of the detector to changes in $T$ decreases after long period of accelerations and the higher the acceleration value is, smaller will be the optimal $\delta$ value.

We also studied the case of two detectors, one inertial and one uniformly accelerated. The accelerated detector coupled to an external massless scalar field and the inertial detector switched off. We also observed the maximum precision estimation for values of acceleration of order of the energy gap $\omega$. The explanation is analogous to single detector case, detector's state has low responsiveness to changes in Unruh temperature, so the proportional variation of $\rho_{dd}$ with $T$ is just appreciable for small transition probability which happens for acceleration values of order of $\omega$.

We also observed quantum Fisher information going asymptotically to zero as acceleration increases, what is explained by final state responsiveness regarding changes in $T$ lost at high

acceleration regimes. About our initial objective with this study, to look for enhancement of estimation precision on Unruh temperature with quantum resources, we concluded that entanglement is not a quantum resource to enhance Unruh temperature measurement. This result contradicts what was suggested in literature [48].

We found that the best probe initial state for temperature estimation precision is $|\psi_{-\infty}^{dd}\rangle = |10\rangle$. This consequence of spontaneous decay phenomenon impact on Unruh temperature information encoded in detectors after interaction.

We concluded that temperature QFI increases with $\delta$. This is expected since a larger exposition time increases transition probability slope and consequently the proportional variation of detector's state with acceleration.

Analyzing signal-to-noise ratio bound $Q_T$, we corroborated our previous conclusion that entanglement is not a quantum resource to our Unruh temperature estimation scheme. As in single detector case, $Q_T$ reaches its maximum value for accelerations two orders of magnitude greater than the acceleration $J_T$ reaches its maximum.

The maximum estimation credibility was found for the initial state $|\rho_{-\infty}^{dd}\rangle = |10\rangle$. This can be explained by the vanishing spontaneous decay probability for this state.

We observed that $Q_T$ has an optimal value of $\delta$ and that higher the acceleration smaller the optimal $\delta$ value of maximum $Q_T$. This phenomenon happens also in the single detector case for the same reasons.

We also observed a remarkable relation between the maximum estimation reliability and the entanglement sudden death acceleration value. We found that $a_{max}$ for $\theta = \pi/4$ coincides with ESD acceleration value. Interestingly, this suggests that, despite the fact that entanglement plays no role in Unruh temperature estimation reliability, the decoherence introduced by Unruh Effect that affects the entanglement between both detectors affects the credibility of Unruh temperature estimation the same way. The difference observed from ESD acceleration value and $a_{max}$ for $\theta = 0$ and $\theta = \pi/2$ can be explained by the role of spontaneous decay effect in estimation credibility.

Finally, we translated our estimation precision results for Unruh temperature to estimation precision of Hawking temperature in Schwarzschild blackhole. We concluded that, since the analogy between Hawking temperature and Unruh temperature holds in the vicinity of event horizon, where the proper acceleration felt by an static observer diverges, temperature estimation precision will be poor unless we afford a detector with a very high energy gap. We obtained an estimation of the value of detector's energy gap needed to an optimal Hawking temperature estimation precision.

The present study provides perspective of some interesting future studies. We are currently investigating if vacuum entanglement studied by Resnik [79] has any relation with Unruh temperature estimation precision. It would also be interesting to study quantum Fisher information for Unruh temperature in systems of electrons in circular and oscillatory motion. This way we could investigate the efficacy of Unruh temperature estimation precision of several experimental

schemes proposed in literature.

# Appendix  A

# Classical Information

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Information is a widely used word in everyday's vocabulary. It is, however, a vague concept in common sense. We associate information to so many events of different nature that is hard to find a way to define it in all its generality. We can point someone who explored information concept in a specially productive way. His name was Claude Shannon, a former IBM engineer who searched a mathematical approach to communication problem. Communication problem can be expressed as the problem of reproducing in one point exactly or approximately a message selected in other point. Shannon's conclusions are expressed in his major paper [52].

Communication phenomenon gets possible with instruments that compose a communication system. It consists of five essential parts.

1. The *information source*, which produces a sequence of messages to be communicated. The message may be of many sorts, a sequence of letters in a telegraph, a time function $f(t)$ of air pressure as in telephone or three functions $r(x,y,t)$, $g(x,y,t)$ and $b(x,y,t)$ of intensity of each color of RGB color system for each pixel at each time as in a color tv transmission.

2. The *transmitter* or *codifier* translates the message to signal code suitable for transmission over the channel. In telegraphy we use an encoding operation which relates letters to a sequence of dots, dashes and spaces which will be translated again to long, short or non electrical pulses in a electrical cable. In telephony sound pressure is encoded into proportional electrical current and in digital television transmission it's about transform $r(x,y,t)$, $g(x,y,t)$ and $b(x,y,t)$ in a sequence of $0$ and $1$.

3. The *channel* is the physical medium used to transmit the signal. For telegraphs it used to be a copper wire, for telephony it used to be a coaxial cable and for digital television transmission it's a band of frequency of electromagnetic waves.

4. The *receiver* or *decodifier* makes the inverse operation of the codifier and reconstruct the message from the signal.

5. The *destination* is the one for whom the message is intended.

When there is noise (that's true for any real channel) the received message is partially different from the emitted one. We shall not worry with systems with noise in this study.
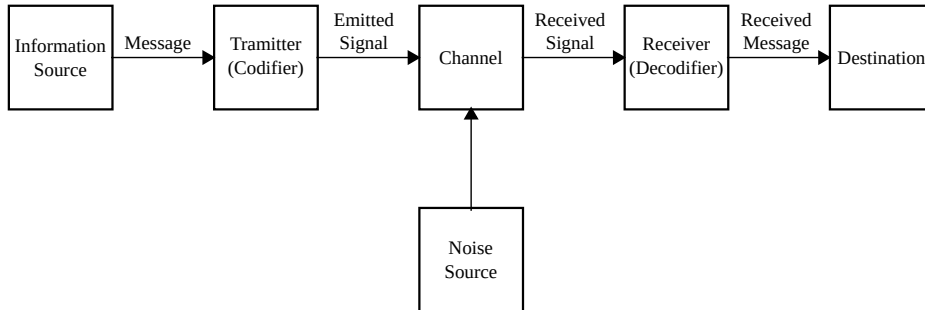


**Figure A.1** | Scheme of a communication system.

Now, going back to communication problem, we saw that the message may be entities of very different nature. If you look in our communication system for a role of the content of the message, you will find none. The message itself is not important for the communication problem. The message can be chosen from a list of possible messages. In the telegraph example it is the alphabet plus space (pause), in the telephone example it is the possible variations of sound pressure, in television example it is the combinations of all values of $r(x,y,t)$, $g(x,y,t)$ and $b(x,y,t)$ in a scale from 0 to 255, at each time interval. Since we are able to distinguish between all possible messages the selected one, we do not have to worry about the meaning of it. The ability to make a distinction between different messages is the important issue for communication problem. Shannon got to this conclusion and called the ability to distinguish different messages information.

The fundamental issue of communication is not, then, about transmit messages, but about transmit information. A natural question that arises, then, is "what is the smallest possible amount of information?". Naturally the simplest distinction is made between two different things. A yes-no answer or a true-false message. There is a very convenient way to express mathematically such sort of message. If we use binary numerical base, the available symbols are 0 and 1, which are called in this context binary digits, or briefly *bits*. One bit is one blank space that can be filled with one 0 or one 1. Mathematically, it is a variable that can assume 0 or 1. If we endow a probability to each digit, we can understand a bit as a discrete random variable defined over $\{0,1\}$ (see definitions in appendix E). A bit is a unit of information.

Understanding what is a unit of information provides a first step to quantify information. To associate a number to information amount, we need to model mathematically communication. In order to do so we will interpret the message as a sequence of identical independent distributed (i.i.d., see appendix E for definition) random variables $X_1, \cdots, X_n$ (definitions in appendix E). Each random variable will be defined over a probability space $(\mathcal{X},\mathcal{A},\mathbf{Pr})$ where $\mathcal{X}$ is the elemental events set, in our examples, the alphabet, the possible sounds or the $r(x,y,t)$,

$g(x,y,t)$ and $b(x,y,t)$ functions. The $\sigma$-algebra $\mathcal{A}$ is the possible events that consists of the possible combinations of elemental events: the possible combinations of letters, voices speaking or images of tv in $r$, $g$ and $b$ function forms. Finally, the probability measure $\mathbf{Pr}$ will express the probability of the information source emitting the elemental event. This approach provides the generality needed to model communication.

Now we have a mathematical model to communication, we can go on pursuing the task to quantify information. One possible way to know how much we have of something is knowing how much we do not have of it. That's the case for information. To be able to distinguish a message between all possible messages you should be able to determine how much "choice" there is in the signal we are receiving, this is equivalent to determine how random the received signal is. We quantify information by quantifying uncertainty. That is how much information we do not have about a process.

If we are going to define a function of uncertainty $H$, we demand some properties from it. The properties we choose are a matter of taste, but there is an agreement that the choices reproduced in this study are very reasonable assumptions.

1. *H depends on events' probabilities*: $\mathcal{X}$ is an abstract set. We can think $\mathcal{X}$ is finite $\{x_1,\cdots,x_n\}$ with probabilities $p_1,\cdots,p_n$, $p_i \geq 0 \forall i \in \{1,\cdots,n\}$, $\sum_i p_i = 1$, so arguments are clearer. If we are going to quantify how random is a message with elements taken from $\mathcal{X}$, we should look for the probabilities values $p_i$. A random message will have an approximately equal probabilities $p_i \approx p_j \forall i,j \in \{1,\cdots,n\}$, however, a chosen message will have one or some values of $i \in I \subset \{1,\cdots,n\}$ for which $p_i \gg p_j$, $i \in I, j \notin I$. So, we expect that $H = H(p_1,\cdots,p_n)$.

2. *H is continuous in* $p_1,\cdots,p_n$: It is natural to expect that small changes in the probabilities to measure outcomes results in small changes in uncertainty of the message.

3. *If* $\mathcal{X} = \{x_1,\cdots,x_n\}$ *and* $\mathcal{Y} = \{y_1,\cdots,y_m\}$ *such that all* $x_i \in \mathcal{X}$ *have equal probability* $1/n$ *and all* $y_j \in \mathcal{Y}$ *have equal probability* $1/m$, $m > n$, $H(1/m,\cdots,1/m) > H(1/n,\cdots,1/n)$: if we have a completely random message with equally probable outcomes, the message will be more uncertain the larger the set of possible choices.

4. $H(p_1,\cdots,p_n) = H(\sum_{i=1}^s p_i, p_{s+1},\cdots,p_n) + (\sum_{i=1}^s p_i) H(p_1/(\sum_{i=1}^s p_i),\cdots,p_s/(\sum_{i=1}^s p_i), p_{s+1},\cdots,p_n)$: if a choice is broken into two successive choices, original $H$ should be the uncertainty of the last choice weighted by the probabilities of the first choice. As an example, we discuss the situation illustrated in Fig. ( A.2). If we decompose a three-options, one-step choice, options with probabilities 1/4, 1/4, 1/2, into a two-step choice: first, a two-options choice with probabilities 1/2, 1/2. Then, if first option is chosen, there is another choice step with two-options of probabilities 1/2, 1/2, then we should have $H(1/4,1/4,1/2) = H(1/2,1/2) + 1/2H(1/2,1/2)$.
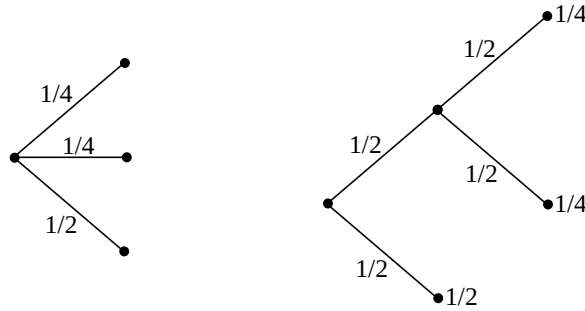
**Figure A.2 |** Example of decomposition of choices.

Now we made our assumptions we naturally ask ourselves if there is such a function that satisfies above properties. It happens there is and is unique up to a constant [52].

**Theorem A.0.1.** *The $H$ satisfying specified assumptions is of the form:*

$$H = -K \sum_{i=1}^{n} p_i \log_b p_i, \tag{A.1}$$

*where K and b are positive constants.*

**Definition A.0.1.** *When $K = 1$ we call $H$ Shannon entropy.*

The constant $b$ denotes the basis of the logarithm. Changing $b$ means we are changing the unit of information (uncertainty) we adopt. If $b = 2$ we are quantifying information in units of bits. We assume from here on $b = 2$ unless specified otherwise. Note that, if we have $\mathcal{X} = \{x_1, x_2\}$ with probabilities $p$ and $1 - p$, respectively, as shown in Fig.( A.3), the maximum of the entropy is reached in the maximum of randomness of choice ($p = 0.5$), and the minimum when the message is completely certain ($p = 1$ or $p = 0$), as shown in Fig. ( A.3).
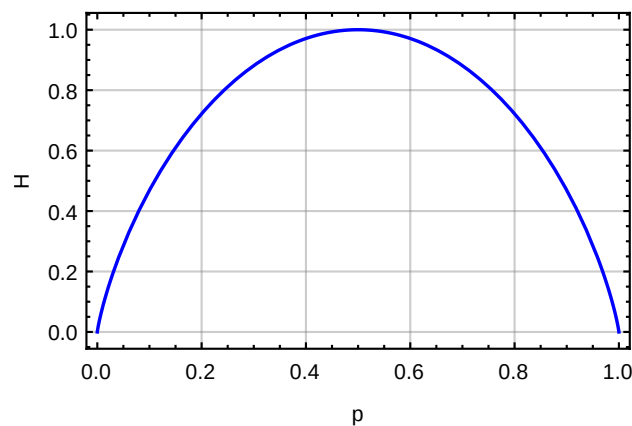


**Figure A.3 |** Shannon entropy with $\mathcal{X} = \{x_1, x_2\}$ with probabilities $p$ and $1 - p$ respectively.

Now that we have already introduced Shannon entropy, there are other relevant information measures that we should present. In appendix E we define a *probability density* of a continuous

random variable $X$. The analogue of the probability density for a discrete random variable we call a *probability mass function*. We present in this study only the treatment for discrete random variables because this is enough for our purposes. The reader must assume random variables are discrete unless specified otherwise. We denote $H(X)$ the Shannon entropy relative to a probability distribution of a random variable $X$. Let $X$ and $Y$ be two random variables. We difine the duplet $(X,Y)$, we are making a measurement of two different quantities that we consider together. We can define in this scenario a join probability $p_{XY}$ which gives us the probability of the possibles outcomes $(x_i,y_j)$, $x_i \in \mathcal{X}$, $y_j \in \mathcal{Y}$. We define:

**Definition A.0.2.** *The* **join entropy** $H(X,Y)$ *is defined by:*

$$H(X,Y) = \sum_{ij} p_{XY}(x_i,y_j) \log_2 p_{XY}(x_i,y_j).$$

Join probability is a measure of the uncertainty of the join distribution $p_{XY}$.

We also define the *relative entropy* or *Kullback Leibler distance*:

**Definition A.0.3.** *Let $X$ and $Y$ be random variables over the same probability space and $p_X$ and $q_Y$ it's respective probability mass function. The* **relative entropy** *or* **Kullback Leibler distance** *is defined by:*

$$D(p_X \parallel q_Y) = \sum_{i} p_X(x_i) \log_2 \frac{p_X(x_i)}{q_Y(x_i)}.$$

Relative entropy is an analogue of the distance between probability distributions, and here is important to emphasize that it's an analogy because it's not a distance in the mathematical sense. It's a measure of how different are two probability mass functions.

The relative entropy satisfies the property [53]

$$D(p_X \parallel q_Y) \geq 0. \tag{A.2}$$

The equality holds if and only if $p_X = q_Y$. The reason relative entropy is not a distance in mathematical sense is that it fails to satisfy triangle inequality and also is not symmetric *i.e.* $D(p_X \parallel q_Y) \neq D(q_Y \parallel p_X)$.

From Eq. ( A.2) we can show that the constant probability mass function is the greater uncertainty there is. Let $\mathcal{X} = \{x_i, \cdots ,x_n\}$ and $q_Y(x_i) = 1/n \forall x_i \in \mathcal{X}$. Then

$$D(p_X \parallel q_Y) = -H(X) - \sum_{i} p_X(x_i) \log_2 \left(\frac{1}{n}\right) = -H(X) + \log_2 n \geq 0. \tag{A.3}$$

This implies that

$$0 \leq H(X) \leq \log_2 n. \tag{A.4}$$

Other important property we find through Eq. ( A.2) is $H(X,Y) \leq H(X) + H(Y)$, where $X$ and $Y$ are any discrete random variables. To prove that we should remember $p_X(x_i) = \sum_j p_{XY}(x_i,y_j)$ and $p_Y(y_j) = \sum_i p_{XY}(x_i,y_j)$. We have:

$$H(X) + H(Y) = \sum_i -p_X(x_i) \log_2 p_X(x_i) + \sum_j -p_Y(y_j) \log_2 p_Y(y_j) =$$
$$= \sum_{ij} -p_{XY}(x_i,y_j) \log_2 p_X(x_i) + \sum_{ij} -p_{XY}(x_i,y_j) \log_2 p_Y(y_j) =$$
$$= \sum_{ij} -p_{XY}(x_i,y_j) \log_2 p_X(x_i)p_Y(y_j). \qquad \text{(A.5)}$$

Since

$$D(p_{XY} \parallel p_X p_Y) = \sum_{ij} [p_{XY}(x_i,y_j) \log_2 p_{XY}(x_i,y_j) - p_{XY}(x_i,y_j) \log_2 p_X(x_i)p_Y(y_j)] \geq 0,$$
$$\text{(A.6)}$$

we find

$$\sum_{ij} p_{XY}(x_i,y_j) \log_2 p_{XY}(x_i,y_j) \geq p_{XY}(x_i,y_j) \log_2 p_X(x_i)p_Y(y_j). \qquad \text{(A.7)}$$

From Eq. ( A.5) and definition A.0.2, the above equation is equivalent to:

$$H(X,Y) \leq H(X) + H(Y). \qquad \text{(A.8)}$$

The equality holds if and only if $p_{XY}(x_i,y_j) = p_X(x_i)p_Y(y_j) \forall x_i \in \mathcal{X}, y_j \in \mathcal{Y}$, *i.e.* if random variables are independent. Equation ( A.8) has an important meaning. The uncertainty of the whole system is smaller than the sum of the uncertainty of the parts if the variables are not independent *i.e.* if there are correlations. In order to explore correlations we will define conditional entropy to get to mutual information.

**Definition A.0.4.** *Let $X$ and $Y$ discrete random variables. The* **conditional entropy** $H(X|Y)$ *is defined:*

$$H(X|Y) = \sum_j p_Y(y_j)H(X|y_j) = \sum_j p_Y(y_j) \sum_i -p_X(x_i|y_j) \log_2 p_X(x_i|y_j) =$$
$$= \sum_{ij} -p_{XY}(x_i,y_j) \log_2 p_X(x_i|y_j).$$

Here $p_X(x_i|y_j)$ is the conditional probability to observe $x_i$ given that we already observed $y_j$. The conditional entropy $H(X|Y)$ is the uncertainty that remains over $X$ after the measurement of $Y$ is performed. From straightforward calculations we obtain $H(X|Y) = H(X,Y) - H(Y)$. Now we have all we need to define *mutual information*

**Definition A.0.5.** *Let $x$ and $Y$ be discrete random variables. The* **mutual information** $I(X :$

*Y) between X and Y is given by:*

$$I(X:Y) = H(X) - H(X|Y).$$

Since The conditional entropy $H(X|Y)$ is the uncertainty that remains over $X$ after the measurement of $Y$, the mutual information is the uncertainty common to $X$ and $Y$, or equivalently, the information we gain of $X$ by measuring $Y$. There is a pictorial representation of the concepts above we call the Venn diagram we present in Fig. ( A.4).
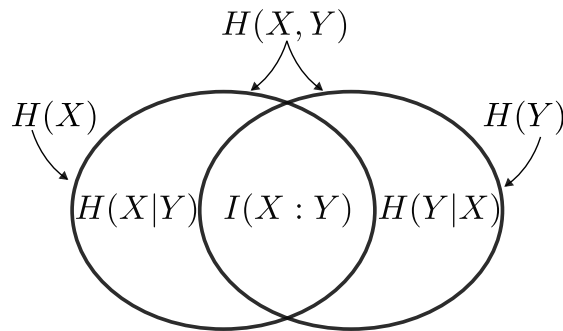


**Figure A.4** | Venn Diagram.

An even more useful and enlightening expression for mutual information if we combine definition A.0.5 with the expression obtained above $H(X|Y) = H(X,Y) - H(Y)$:

$$I(X:Y) = H(X) + H(Y) - H(X,Y). \tag{A.9}$$

Note here that mutual information is symmetric. Now, Eq. ( A.8) with Eq. ( A.9) implies that

$$I(X:Y) \geq 0, \tag{A.10}$$

the equality holds when $X$ and $Y$ are independent. This shows that mutual information can be associated to a measure of correlation between random variables $X$ and $Y$. This fact is going to be of central importance in quantum context, when we are going to define entanglement.

# Appendix B

# Quantum mechanics

We presented the basic concept of information in its classical context. We defined channel as a physical medium without specifying what physics theory rules it. It's natural to assume we are talking about classical physics. However, it is possible to employ physical media ruled by quantum mechanics. We need a transmitter that codifies messages to quantum states and the quantum medium to transmit it. In this case we have a whole new set of phenomenons associated to information. Such that we call information in quantum context *quantum information*.

We are interested in analyzing precision in measurements of quantum accelerated systems. So, we are interested in information in quantum context. We shall present and discuss some points of quantum mechanics which are going to be important for us. For a complete discussion about the subject see [54–58]. Quantum mechanics can be derived from a set of postulates. How many postulates or which postulates are them is an open debate [59,60] and even a matter of taste [45,58,61–63], but the postulates presented bellow are good enough for our purposes.

**Postulate 1**: *A quantum system is described as a vector of projective space of Hilbert space $\mathcal{H}$.*

The space $\mathcal{H}$ is called *state space* and the vector that describes the quantum system is called *state vector* or, simply, state of the system. We adopt Dirac's notation and denote a state vector as a ket $|\psi\rangle$. Note that the above postulate does not tell us what is a quantum system, but how it is described. The way to describe a particular system is a difficult task and, for each system there is a specific theory ( QED for interaction light-atoms, quantum optics for light, molecular physics for molecules...).

**Postulate 2**: *Evolution of a closed quantum system $|\psi(t_0)\rangle$ is described by a unitary operator $U(t)$ (or an one parameter family of unitary transformations) such that $|\psi(t)\rangle = U(t - t_0)|\psi(t_0)\rangle$ and $lim_{t \to t_0} U(t - t_0)|\psi(t_0)\rangle = |\psi(t_0)\rangle$.*

Note that, again, the postulate just tells us that dynamics is described by unitary operators but do not tells us what operator describes evolution. The operator depends on each system

and calculate it can be challenging. Postulate 2 refers to relation between a quantum system state at two different times, the generator of $U(t)$ is given by the differential equation that rules evolution called Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle, \tag{B.1}$$

where $\hbar$ is a fundamental constant which, in natural units, we set $\hbar = 1$ and $H$ is the Hamiltonian operator which will be a different operator to each system [56]. Eq. ( B.1) leads to $U(t) = e^{-iHt/\hbar}$ [56–58]. Note that Postulate 2 refers to closed systems. However, it's possible to define a time dependent Hamiltonian which varies to parameters controlled by an external agent, as a physicist running an experiment. Such a system is therefore, an open system. This will still, in good approximation, evolve according to Schrödinger equation and satisfy postulate 2 [56, 63]. This will be the case of our study.

**Postulate 3**: *Quantum measurements are described by* **measurements operators** $\{M_\lambda\}$, $M_\lambda : \mathcal{H} \rightarrow \mathcal{H}$, $\sum_\lambda M_\lambda^\dagger M_\lambda = \mathbb{1}$. *The index $\lambda$ refers to possible measurements outcomes, $\lambda \in I \subset \mathbb{R}$. Given the state vector $|\psi\rangle$ that describes the system the probability to get a result $\lambda$ is given by*

$$p(\lambda) = \langle\psi|M_\lambda^\dagger M_\lambda|\psi\rangle,$$

*and the state vector after the event of measurement $|\psi'\rangle$ is given by:*

$$|\psi'\rangle = \frac{M_\lambda|\psi\rangle}{\sqrt{\langle\psi|M_\lambda^\dagger M_\lambda|\psi\rangle}}.$$

There are two fundamental aspects about quantum measurement that distinguish classical from quantum physics. First, the outcomes has a probabilistic nature. Quantum mechanics give us probabilities of the possible outcomes. Second, the measurement alters the state of the system. This characteristics are in the core of the nature of quantum world.

If measurement operators are also Hermitian and satisfy additional condition $M_\lambda M_{\lambda'} = \delta_{\lambda\lambda'} M_\lambda$ *i.e.* if measurements operators are orthogonal projectors, then the measurement is called *projective*. In this case we denote $M_\lambda = P_\lambda$. This class of measurements is associated to observables measurements as energy and it's straightforward to calculate average value of measurements. If an observable is expressed as $O = \sum_\lambda \lambda P_\lambda$, then:

$$\langle O\rangle = \sum_\lambda p(\lambda)\lambda = \sum_\lambda \langle\psi|P_\lambda|\psi\rangle\lambda = \langle\psi|\left(\sum_\lambda \lambda P_\lambda\right)|\psi\rangle = \langle\psi|O|\psi\rangle. \tag{B.2}$$

However, sometimes we are interested in performing a measurement just once, or we simply can not perform measurement more than once, as in case we use a silvered screen to measure

the position of a photon. In order to perform the measurement, we destroy the state. We can just use the statistics of outcomes. To handle this cases there is a very suitable formalism called Positive Operator-Value Measurement:

**Definition B.0.1.** *Let $\{M_\lambda\}$ be a measurement operator set. The set $\{\Pi_\lambda\}$ where $\Pi_\lambda = M_\lambda^\dagger M_\lambda$ is called a* **Positive Operator-Valued Measure** *(POVM).*

Alternatively, given a set of positive operators $\{\Pi_\lambda\}$ such that $\sum_\lambda \Pi_\lambda = \mathbb{1}$, the operators $M_\lambda \equiv \sqrt{\Pi_\lambda}$ satisfy $\Pi_\lambda = M_\lambda^\dagger M_\lambda$ and $\sum_\lambda M_\lambda^\dagger M_\lambda = \mathbb{1}$ such that $M_\lambda \equiv \sqrt{\Pi_\lambda}$ define a set of measurement operators. Therefore any set of positive operators which satisfy completeness relation is called POVM.

POVM formalism is also used to distinguish quantum states. A set of quantum states $\{|\psi_1\rangle, \cdots, |\psi_n\rangle\}$ is distinguishable if there is a measurement which aloud us to determine with probability 1 the measured state. This is equivalent to define a POVM $\{\Pi_1, \cdots, \Pi_m\}$, $m \geq n$, such that $\langle\psi_i|\Pi_j|\psi_i\rangle = \delta_{ij}$ $\forall i \in \{i, \cdots, n\}$, $j \in \{1, \cdots, m\}$. That's only possible when the states are orthogonal to each other. When two states are not orthogonal, there is no way to determine for sure which one we are measuring [45, 63].

It is also common to have an ensemble where we can just tell the probability $p_i$ of the state being $|\psi_i\rangle$. This happens for example if we measure electrons spin in a given direction coming from a hot oven. We can not say for sure if the state is up or down, we just assume a random distribution of electron spin coming from a thermal source and describe the state as up with probability $0.5$ and down with same probability. The set $\{p_i, |\psi_i\rangle\}$ is called a mixture. When for a given $j$, $p_i = \delta_{ij}$ *i.e.* we are certain that the state is $|\psi_j\rangle$, we call the ensemble a pure state. It's not possible to describe a mixture state with the formalism we have seen so far. From postulate 3 we can obtain probabilities of results of a measurement in superposition states. But in this case we are certain about the state of the system and we get different possible outcomes from the same state. It's a different situation from a mixture state, which we don't know for sure the state of the system. In order to describe mixture states we need the formalism of density operators:

**Definition B.0.2.** *A* **Density Operator** $\rho : \mathcal{H} \to \mathcal{H}$ *is a positive operator such that* $Tr(\rho) = 1$.

Here $Tr$ means the trace operation over a representation of the operator $\rho$ in matrix space. There are infinite matrices which represent such an operator (for each basis of Hilbert space will get a possibly different matrix) and all of them are physically equivalent. However there is a very suitable basis to work with which defines uniquely the density matrix $\rho$. The spectral theorem [45] states that any self-adjoint operator admits a spectral decomposition $\sum_i \lambda_i P_{\lambda_i}$, where $\lambda_i$ are operator's eigenvalues and $P_{\lambda_i}$ are projectors in subspace associated to $\lambda_i$. Density matrix is positive (so, also self-adjoint) and $Tr(\rho) = 1$ which implies it can be written in the form

$$\rho = \sum_i \lambda_i |e_i\rangle\langle e_i|. \tag{B.3}$$

where $\lambda_i \geq 0$, $\sum_i \lambda_i = 1$ and $\{|e_i\rangle\}$ are orthonormal. Eq. ( B.3) is called the *spectral decomposition* of the mixture state $\{p_i, |\psi_i\rangle\}$. Of course if $\{|\psi_i\rangle\}$ forms a orthonormal basis of the Hilbert space associated to the system of interest then $p_i = \lambda_i$ and $|e_i\rangle = |\psi_i\rangle$, but this is not always the case.

Alternatively, it's possible to define density operator as

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|. \tag{B.4}$$

Note that, given an orthonormal basis $\{|e_i\rangle\}$:

$$Tr\rho = \sum_j \langle e_j|\rho|e_j\rangle = \sum_{ji} p_i|\langle e_j|\psi_i\rangle|^2 = \sum_i p_i = 1. \tag{B.5}$$

Also note that, for any state vector $|\psi\rangle$:

$$\langle\psi|\rho|\psi\rangle = \sum_i p_i|\langle\psi_i|\psi\rangle|^2 \geq 0. \tag{B.6}$$

Therefore, $\rho$ is a positive operator. The density operator can describe pure or mixture states, so it's a more general representation of a quantum system. We will sometimes refer to density operator simply as the state of the system. We denote the space of density operators over a Hilbert space $\mathcal{H}$ as $\mathcal{D}(\mathcal{H})$.

Given a state $\rho$, the probability to obtain a result $\lambda$ from a measurement described by the POVM $\{\Pi_\lambda\}$ is:

$$p(\lambda) = Tr(\rho\Pi_\lambda). \tag{B.7}$$

The evolution of the density equation will be given by a unitary transformation as described in Postulate 2:

$$\rho(t) = \sum_i p_i U(t - t_0)|\psi_i\rangle\langle\psi_i|U^\dagger(t - t_0) = U(t - t_0)\rho(t_0)U^\dagger(t - t_0). \tag{B.8}$$

Now we shall enunciate the last postulate of quantum mechanics.

**Postulate 4**: *If a quantum system with associated Hilbert space $\mathcal{H}$ consists of $n$ component physical systems, each subsystem with associated Hilbert space $\mathcal{H}_i$, $i \in \{1, \cdots, n\}$, then the composite total system is described by the tensor product space of the parts $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$.*

Postulate 4 tells us the way to describe quantum composite systems. This last postulate is the origin of one of the most remarking and puzzling phenomenons of quantum mechanics which we will explain in next sections: Entanglement. Next sections we will present some concepts that arise in information theory in quantum context.

# Appendix C

# Quantum Information

Now we have pointed some quantum mechanic's concepts we are going to use, we shall go on and apply such concepts in information theory to present what is new from information theory in quantum context. First entity we present is the information unit of a quantum system, the *qubit*.

As we have discussed in section A, the bit is the unit of classical information. It can admit two values which correspond to two states of a physical system (as it is transmitted by a channel). We denote it mathematically by 0 and 1. Now, if we are in a quantum context, the equivalent to a bit is a quantum bit (qubit). Formally, the qubit is defined as:

**Definition C.0.1.** *A **qubit** is a quantum system represented by the projective space of a two dimensional Hilbert space.*

As a classical bit can admit the states 0 or 1, the qubit can take the orthogonal states $|0\rangle$ or $|1\rangle$ usually called unexcited and excited state. However, as a quantum system, the qubit can take any superposition state $\alpha|0\rangle + \beta|1\rangle$, $\alpha,\beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$. So, a qubit can admit an infinite number of states. While a bit carries one unit of information, we could use a qubit to encode, in principle, an infinite amount of information, however not all this information would be accessible since we have to measure the state and not all the qubit states would be distinguishable. There is an upper limit to the amount of information one can extract from a quantum system, it is called Holevo limit [63].

There is a nice geometrical representation of the qubit as a *Bloch sphere*. If we denote our qubit $|\psi\rangle$, we can represent it as

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)|1\rangle. \tag{C.1}$$

Here, $\theta$ is usually called the weight parameter and $\varphi$ the phase parameter. As the qubit will depend on these two parameters of compact domain, we can represent them as the zenith and azimuthal coordinates of a sphere respectively. As a result we get the geometrical scheme of Fig. ( C.1). Note that a qubit is a mathematical representation, it can describe many different

systems as a two different photon polarization , an electron with spin aligned to an external magnetic field or an atom with nuclear spin aligned to an external magnetic field.
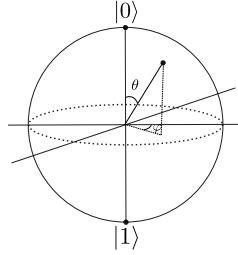


**Figure C.1 |** Bloch Sphere.

Another central concept in quantum information theory is Von Neumann entropy.

**Definition C.0.2.** *Let* $\mathcal{H}$ *a Hilbert space and* $\rho \in \mathcal{D}(\mathcal{H})$. *The* **Von Neumann entropy** *of* $\rho$, *denoted by* $S(\rho)$ *is defined as*

$$S(\rho) = -Tr(\rho \ \log_2(\rho))$$

To understand the physical meaning of Von Neumann entropy, it's convenient to write $\rho$ in its spectral decomposition $\rho = \sum_i p_i |e_i\rangle\langle e_i|$, where $\{|e_i\rangle\}$ is an orthonormal basis. If we apply spectral decomposition to definition C.0.2 we get:

$$S(\rho) = -\sum_i p_i \log_2 p_i, \tag{C.2}$$

which is just the definition of Shannon entropy A.0.1. Von Neumann entropy is the quantum analogous to Shannon entropy $H$. It measures the uncertainty about the state $\rho$. The unit of Von Neumann entropy is not the bit, but the *qubit*.

We are interested in composite systems which we are going to realize measurements in just one subsystem. If we have a composite system $AB$ with subsystems $A$ and $B$ and we are interested just in quantities of subsystem $A$, the observable of interest needs to be in the form $O_A \otimes \mathbb{1}_B$, where $O_A$ is the observable in subspace $A$ and $\mathbb{1}_B$ is the identity of subspace B. We look for a procedure that aloud us to ignore subsystem $B$ instead of carrying on in calculations quantities of it. The procedure that gives the correct statistics for observable quantities is the *partial trace*.

**Definition C.0.3.** *Let* $\rho_{AB} = \sum_{ij} c_{ij} |a_{1i}b_{1j}\rangle\langle a_{2i}b_{2j}| = \sum_{ij} c_{ij} |a_{1i}\rangle\langle a_{2i}| \otimes |b_{1j}\rangle\langle b_{2j}|$. *The* **partial trace** *over the subsystem* $B$ *is defined as*

$$Tr_B(\rho_{AB}) = \sum_{ij} c_{ij} |a_{1i}\rangle\langle a_{2i}| Tr(|b_{1j}\rangle\langle b_{2j}|)$$

*and the* **reduced density operator** *for system $A$ is defined as*

$$\rho_A = Tr_B(\rho_{AB}).$$

Now, we should note that, if $\{e_i\}$ is an orthonormal basis of Hilbert space of subsystem $B$, we have:

$$Tr(|b_{1j}\rangle\langle b_{2j}|) = \sum_i \langle e_i|b_{1j}\rangle\langle b_{2j}|e_i\rangle = \sum_i \langle b_{2j}|e_i\rangle\langle e_i|b_{1j}\rangle = \langle b_{2j}| \left(\sum_i |e_i\rangle\langle e_i|\right) |b_{1j}\rangle = \langle b_{2j}|b_{1j}\rangle.$$

(C.3)

And the most important property of the reduced density matrix is that we get the correct averages of the measurements made in the subsystem:

$$Tr((O_A \otimes \mathbb{1}_B)\rho_{AB}) = Tr(O_A\rho_A).$$

(C.4)

In order to see that, let $\{f_i\}$ an orthonormal basis of Hilbert space of subsystem $A$ and $\{e_j\}$ an orthonormal basis of Hilbert space of subsystem $B$. We get:

$$
\begin{aligned}
Tr((O_A \otimes \mathbb{1}_B)\rho_{AB}) &= \sum_{ij} \langle f_i e_j|(O_A \otimes \mathbb{1}_B) \left(\sum_{nm} c_{nm}|a_{1n}b_{1m}\rangle\langle a_{2n}b_{2m}|\right) |f_i e_j\rangle \\
&= \sum_{ij} \langle f_i e_j| \left(\sum_{nm} c_{nm}O_A|a_{1n}\rangle\langle a_{2n}| \otimes \mathbb{1}_B|b_{1m}\rangle\langle b_{2m}|\right) |f_i e_j\rangle \\
&= \sum_{ij} \sum_{nm} c_{nm}\langle f_i|O_A|a_{1n}\rangle\langle a_{2n}|f_i\rangle\langle e_j|b_{1m}\rangle\langle b_{2m}|e_j\rangle \\
&= \sum_i \sum_{nm} c_{nm}\langle f_i|O_A|a_{1n}\rangle\langle a_{2n}|f_i\rangle \left(\sum_j \langle e_j|b_{1m}\rangle\langle b_{2m}|e_j\rangle\right) \\
&= \sum_i \sum_{nm} c_{nm}\langle f_i|O_A|a_{1n}\rangle\langle a_{2n}|f_i\rangle Tr(|b_{1m}\rangle\langle b_{2m}|) \\
&= \sum_i \langle f_i|O_A \left(\sum_{nm} c_{nm}|a_{1n}\rangle\langle a_{2n}|Tr(|b_{1m}\rangle\langle b_{2m}|)\right) |f_i\rangle \\
&= \sum_i \langle f_i|O_A\rho_A|f_i\rangle = Tr(O_A\rho_A).
\end{aligned}
$$

(C.5)

The partial trace turns out to be the unique operation over the density matrix that satisfy the above condition [63].

Now, lets define the relative entropy in the quantum context:

**Definition C.0.4.** *Let $\rho_1$ and $\rho_2$ two density matrix over the same Hilbert space. The* **quantum relative entropy** *is defined as:*

$$S(\rho_1 \parallel \rho_2) = Tr\left(\rho_1(\log_2 \rho_1 - \log_2 \rho_2)\right).$$

As in classical case the relative entropy is a measure of how different can two density operators be. It expresses the statistical distinguishability between $\rho_1$ and $\rho_2$. Analogous to its classical version, quantum relative entropy satisfies [63]:

$$S(\rho_1 \parallel \rho_2) \geq 0. \tag{C.6}$$

With Eq. ( C.6) we obtain important results analogous to classical information theory. Let $\rho$ a density matrix and $\rho_n = (1/n)\mathbb{1}$ where $\mathbb{1}$ is the identity in Hilbert space of interest. This is the density matrix with spectral form of constant probability. We obtain:

$$S(\rho \parallel \rho_n) = -S(\rho) + Tr(\rho \log_2 n) = -S(\rho) + \log_2 n \geq 0. \tag{C.7}$$

Then, as in the classical context, we have

$$S(\rho) \leq \log_2 n. \tag{C.8}$$

Equality holds if $\rho = (1/n)\mathbb{1}$. This implies that also in quantum information theory the uniform probability state is the state with greater uncertainty.

Another important result is that the entropy of a bipartite state is less than or equal to the entropy of its individual parts. To put this precisely, lets fix some notation until the end of this section. Let $AB$ be a composite system with subsystems $A$ and $B$. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ the Hilbert space of the subsystems and $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Let $\rho_A = Tr_B(\rho_{AB})$ and $\rho_B = Tr_A(\rho_{AB})$. Now let us calculate:

$$
\begin{aligned}
S(\rho_{AB} \parallel \rho_A \otimes \rho_B) &= -S(\rho_{AB}) - Tr(\rho_{AB} \log_2(\rho_A \otimes \rho_B))) \\
&= -S(\rho_{AB}) - Tr(\rho_{AB}((\log_2 \rho_A) \otimes \mathbb{1}_B)) - Tr(\rho_{AB}(\mathbb{1}_A \otimes (\log_2 \rho_B))) \\
&= -S(\rho_{AB}) - Tr(\rho_A \log_2 \rho_A) - Tr(\rho_B \log_2 \rho_B) = -S(\rho_{AB}) + S(\rho_A) + S(\rho_B) \geq 0. \quad \text{(C.9)}
\end{aligned}
$$

So we find

$$S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B). \tag{C.10}$$

The equality holds if $\rho_{AB} = \rho_A \otimes \rho_B$ *i.e.* if $\rho_{AB}$ is separable. Now, the fact that the uncertainty of the whole system is smaller then the uncertainty of the individual if the state is not separable suggests that we can find correlations in non-separable states. Still looking for the relation between non-separable states and correlations, we will define the *quantum mutual information*. Before that we will show the analogue of the conditional information in quantum context:

**Definition C.0.5.** *Let $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. The* **conditional information** *between A and B is defined as:*

$$S(A|B) = S(\rho_{AB}) - S(\rho_B).$$

And we also define the quantum mutual information:

**Definition C.0.6.** *Let $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. The* **quantum mutual information** *between $A$ and $B$ is defined as:*

$$\mathcal{I}(A : B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}).$$

From Eq. ( C.10) we conclude that:

$$\mathcal{I}(A : B) \geq 0. \tag{C.11}$$

The equality holds if $\rho_{AB} = \rho_A \otimes \rho_B$. Equation ( C.10) tells us that the uncertainty of the whole system is less than or equal to the sum of the uncertainty of the individual parts. The equality holds when the systems are separable, *i.e.* completely independent. As in classical context, this suggests that the mutual information is a measure of total correlations of the subsystems $A$ and $B$. This will be important to the study of entanglement.

# Appendix D

# Entanglement

One of the most highlighted quantum phenomenons is the entanglement. This property was used by Einstein, Podolski and Rosen to, assuming locality and realism, try to prove the incompleteness of quantum theory [64]. However, Bell proved later that those same entangled quantum systems couldn't satisfy locality and realism simultaneously [65]. Entanglement, therefore, is one of the phenomenons which distinguishes quantum theory from classical theory. The origin of entanglement lies in postulate 4 presented in section B. Since we postulate that we shall describe composite systems through tensor products of the states of the individual parts, the properties of tensor product shall describe physical phenomena.

In order to understand the phenomenon of entanglement we shall point an remarkable property of tensor algebra. Let $V$ and $W$ be vector spaces and $\{v_i\}$, $\{w_j\}$ two orthonormal basis. We can think of tensor space $V \otimes W$ as all the vectors of the form $l = \sum_{ij} l_{ij} v_i \otimes w_j$ [66]. Not all vectors $l \in V \otimes W$ can be written in the form $l = v \otimes w$, $v \in V$ $w \in W$. For example, take the vector $l = v_i \otimes w_j + v_j \otimes w_i$, $i \neq j$. Suppose there is a $v \in V$ and a $w \in W$ such that $l = v \otimes w$. We have $v = \sum_n c_n v_n$ and $w = \sum_m d_m w_m$, where $c_n$, $d_m$ are coefficients that belong to the field of the vector space. Then $v \otimes w = \sum_{nm} c_n d_m v_n \otimes w_m$ such that $l = v \otimes w$ implies $c_i d_j = d_j c_i = 1$ and $c_i d_i = c_j d_j = 0$ what is just impossible. Therefore $l$ is not separable.

As stated by postulate 4, a composite quantum system is described by a tensor product of its constituent parts. Let $L$ be a system which consists of component systems $V_1, \cdots, V_n$. If the state of the whole system $|l\rangle$ can be written as a tensor product of prepared states of the subsystems $|l\rangle = |v_1\rangle \otimes \cdots \otimes |v_n\rangle$, $|v_i\rangle \in V_i$, then $|l\rangle$ is called a *separable* state. The state that is not separable is an *entangled state*.

The definitions above refer to pure states, but the generalization of the concept to mixture states is just a consequence of above definitions:

**Definition D.0.1.** *Let a composite system with $n$ subsystems, each one with associated Hilbert space $\mathcal{H}_i$. A given state $\rho \in \mathcal{D}(\bigotimes_{i=1}^n \mathcal{H}_i)$ is a* **separable state** *if it can be written as*

$$\rho = \sum_{j=1}^m p_j \rho_1^j \otimes \cdots \otimes \rho_n^j,$$

*where $\rho_i^j \in \mathcal{D}(\mathcal{H}_i) \forall i \in \{1, \cdots, n\}, j \in \{1, \cdots, m\}$, $p_j \geq 0$, $\sum_{j=1}^{m} p_j = 1$. If a state is not separable it's* **entangled**.

A natural question that arises is if we can quantify entanglement. As a matter of fact we can. We shall present just entanglement measures of composite systems with two parts (or bipartite system) which will be enough for our purposes. For pure states there is a natural and unique way to quantify entanglement. For mixture states there is not such a natural way and there are some different ways to quantify it.

We begin with pure states. Consider a bipartite system $AB$ with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Let $|\psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}| \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. Note that $\rho_{AB}$ is the density matrix associated to a pure state and there is no uncertainty about it, *i.e.* $S(\rho_{AB}) = 0$. However, denoting $\rho_A = Tr_B(\rho_{AB})$ and $\rho_B = Tr_A(\rho_{AB})$, we have that $S(\rho_A)$ and $S(\rho_B)$ vanish just in the case where $\rho_{AB} = \rho_A \otimes \rho_B$, otherwise the entropy of the subsystems differs from zero.

We have already discussed that mutual information can be regarded as a measure of correlation. From definition C.0.6, $\mathcal{I}(A : B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$, we notice that the greater the uncertainty about the subsystems, the greater the correlation between the two subsystems. Then the uncertainty of the subsystems presents itself as a good candidate to quantify entanglement.

Before defining definitively the entanglement of a pure state, we need to show a remarkable property of the uncertainty of the subsystems of a pure state. In order to accomplish that we have to enunciate the Schmidt decomposition theorem:

**Theorem D.0.1.** *Let $|\psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. There exist orthonormal states $\{|e_A^i\rangle\}$ of $\mathcal{H}_A$ and $\{|e_B^i\rangle\}$ of $\mathcal{H}_B$, $i \in \{1, \cdots, n\}$, $n \leq \min\{dim(\mathcal{H}_A), dim(\mathcal{H}_B)\}$ such that*

$$|\psi_{AB}\rangle = \sum_{i=1}^{n} \lambda_i |e_A^i e_B^i\rangle,$$

*where $|e_A^i e_B^i\rangle = |e_A^i\rangle \otimes |e_B^i\rangle$, $\lambda_i \geq 0$ and $\sum_{i=1}^{n} \lambda_i^2 = 1$. The basis $\{|e_A^i\rangle\}$ and $\{|e_B^i\rangle\}$ are called Schmidt basis and $\lambda_i$ are called Schmidt numbers.*

Proof of this theorem is found in [63].

Now, suppose we write $\rho_{AB}$ using the form of theorem D.0.1:

$$\rho_{AB} = \sum_{ij} \lambda_i \lambda_j |e_A^i e_B^i\rangle\langle e_A^j e_B^j|. \tag{D.1}$$

Then we have:

$$S(\rho_A) = S(Tr_B(\rho_{AB})) = S(\sum_{ij} \lambda_i \lambda_j |e_A^i\rangle\langle e_A^j| \langle e_B^j | e_B^i\rangle)$$

$$= S(\sum_{ij} \lambda_i \lambda_j |e_A^i\rangle\langle e_A^j| \delta_{ij}) = S(\sum_i \lambda_i^2 |e_A^i\rangle\langle e_A^i|) = \sum_i \lambda_i^2 \log_2(\lambda_i^2). \tag{D.2}$$

On the other hand

$$S(\rho_B) = S(Tr_A(\rho_{AB})) = S(\sum_{ij} \lambda_i\lambda_j|e_B^i\rangle\langle e_B^j|\langle e_A^j|e_A^i\rangle)$$

$$= S(\sum_{ij} \lambda_i\lambda_j|e_B^i\rangle\langle e_B^j|\delta_{ij}) = S(\sum_i \lambda_i^2|e_B^i\rangle\langle e_B^i|) = \sum_i \lambda_i^2 \log_2(\lambda_i^2). \tag{D.3}$$

So, we obtain the result:

$$S(\rho_A) = S(\rho_B). \tag{D.4}$$

As for a pure state of a bipartite system the uncertainty of the individual subsystems are equal, then, instead of using the sum of the entropies we can just define:

**Definition D.0.2.** *Let $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ the density operator associated to a* **pure** *state. The* **entanglement of pure state** $E(\rho_{AB})$ *is defined as:*

$$E(\rho_{AB}) = S(\rho_A) = S(\rho_B).$$

Entanglement's unit is the qubit and for separable states $E(\rho_{AB}) = 0$ while the maximum entanglement is, from Eq. ( C.8), $\log_2 n$ where $n$ is the dimension of the system. From the definition above we note that the maximum entanglement of a two qubit system is obtained in the state of Eq. ( **??**).

Now, we already discussed that the for mixture states there is no natural definition of entanglement and there are some definitions that are useful in specific contexts. However there are some properties that any measure of entanglement should satisfy. We list three properties for which there is a consensus that any measure of entanglement should satisfy:

1. The entanglement of a separable state $\rho$ should vanish

$$E(\rho) = 0$$

2. Changes in local basis representation, *i.e.* unitary transformations of the form $U_A \otimes U_B$ does not change the entanglement of the state. Let $\rho \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$

$$E(\rho) = E(U_A \otimes U_B \rho U_A^\dagger \otimes U_B^\dagger)$$

3. Local operations and classical communication (LOCC) and subselection should not increase the state entanglement. Let $O_A$ and $O_B$ denote LOCC in the subsystems $A$ and $B$ respectively and $\rho \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. This means that

$$E(\rho) \geq E(O_A \otimes O_B \rho O_A^\dagger \otimes O_B^\dagger),$$

where the form $O_A \otimes O_B$ denotes the fact that operations $O_A$ and $O_B$ are performed locally.

Condition 1 above guarantees that a composite state with two independent subsystems $A$ and $B$ should not have entanglement since there is no correlations between them. Condition 2 states that two density operators that are physically equivalent, since they differ only by a change of basis representation, should have the same entanglement. The last condition states that LOCC can not increase quantum entanglement, otherwise entanglement would not be considered a quantum resource.

One of the entanglement measures which satisfies all the conditions presented above is the *entanglement of formation*. That's the measure we'll use in this study. For other possible measures see [67].

**Definition D.0.3.** *Let* $\{p_i,|\psi_i\rangle\}$ *a mixture ensemble, where* $|\psi_i\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$*, and* $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ *the state associated to the ensemble. The* **entanglement of formation** *is defined as*

$$E_f(\rho_{AB}) = \inf \left\{ \sum_i p_i E(|\psi_i\rangle\langle\psi_i|) \right\},$$

*where the infimum is taken over all the mixtures* $\{p_i,|\psi_i\rangle\}$ *which state is* $\rho_{AB}$*.*

The calculation of the infimum is, in general a very hard task. However, there is an analytical expression for the case $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) = 2$ [69]. Let $\sigma_1$, $\sigma_2$ and $\sigma_3$ the usual Pauli matrices. Let's denote by $\{|0\rangle,|1\rangle\}$ an eigenbasis of the single qubit related to $\sigma_3$ and let $\rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be written in the basis $\{|ab\rangle\}$ where $|ab\rangle = |a\rangle \otimes |b\rangle$, $|a\rangle \in \mathcal{H}_A$, $|b\rangle \in \mathcal{H}_B$, $a,b \in \{0,1\}$. We define the operator $\zeta_{AB} = \rho_{AB}(\sigma_2 \otimes \sigma_2)(\rho_{AB})^*(\sigma_2 \otimes \sigma_2)$ where the first Pauli matrix of $\sigma_2 \otimes \sigma_2$ acts on qubit $A$ and the second acts in qubit $B$. $(\rho_{AB})^*$ is the matrix obtained by taking the complex conjugate of all elements of $\rho_{AB}$. Now we define the *concurrence* of $\rho_{AB}$:

**Definition D.0.4.** *Let* $\lambda_i$*,* $i \in \{1,2,3,4\}$*, the eigenvalues of* $\zeta_{AB}$ *such that* $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$*. The* **concurrence** *of* $\rho_{AB}$ *is defined as:*

$$C(\rho_{AB}) = max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}.$$

Concurrence is such that $0 \leq C(\rho_{AB}) \leq 1$. Now, we can obtain the expression of entanglement of formation. Let us define:

$$g(\rho_{AB}) = \frac{1 + \sqrt{1 - C^2(\rho_{AB})}}{2}, \tag{D.5}$$

the entanglement of formation in the case $\dim(\mathcal{H}_A) = \dim(\mathcal{H}_B) = 2$ is given by:

$$E_f(\rho_{AB}) = -g(\rho_{AB}) \log_2 g(\rho_{AB}) - (1 - g(\rho_{AB})) \log_2(1 - g(\rho_{AB})). \tag{D.6}$$

As the entanglement of formation is a monotonic increasing function of concurrence, the last quantity is usually used as the entanglement measure. This will be our case in this study.

# Appendix E

# Probability Space

In this appendix we define probability spaces and related concepts. Let $\mathcal{X}$ be a nonempty set and $\mathcal{A} \subset 2^{\mathcal{X}}$ where $2^{\mathcal{X}}$ denotes the set of all subsets of $\mathcal{X}$. Then

**Definition E.0.1.** *$\mathcal{A}$ is a $\sigma$-**algebra** if satisfies:*

1. *$\mathcal{X} \in \mathcal{A}$.*

2. *$\mathcal{A}$ is closed under complements.*

3. *$\mathcal{A}$ is closed under countable unions.*

We define a measurable space as:

**Definition E.0.2.** *Let $\mathcal{X}$ be a nonempty set and $\mathcal{A}$ its $\sigma$-algebra. the pair $(\mathcal{X}, \mathcal{A})$ is called a* **measurable space***.*

Now, let $\mathcal{A} \subset 2^{\mathcal{X}}$. If $\mu : \mathcal{A} \to [0, \infty)$ we define:

**Definition E.0.3.** *$\mu$ is $\sigma$-**additive** if $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ for any set $\{A_i\}$ of countably many disjoin sets $A_i \in \mathcal{A}$ such that $\bigcup_{i=1}^{\infty} A_i = \mathcal{A}$.*

Now we can define a measure and a probability measure.

**Definition E.0.4.** *Let $\mathcal{A}$ be a $\sigma$-algebra. $\mu : \mathcal{A} \to [0, \infty)$ is a* **measure** *if $\mu$ is $\sigma$-additive and $\mu(\emptyset) = 0$.*

**Definition E.0.5.** *Let $\mu : \mathcal{A} \to [0, 1]$ a measure. $\mu$ is a* **probability measure** *if $\mu(\mathcal{X}) = 1$.*

We can finally define a probability space:

**Definition E.0.6.** *Let $\mathcal{X}$ be a nonempty set, $\mathcal{A}$ its $\sigma$-algebra and $\mathbf{Pr} : \mathcal{A} \to [0, 1]$ a probability measure. The triple $(\mathcal{X}, \mathcal{A}, \mathbf{Pr})$ is called a* **probability space***.*

Now we have established a probability space we can move on to define a random variable. First we have to talk about measurable maps:

**Definition E.0.7.** *Let $(\mathcal{X},\mathcal{A})$ and $(\mathcal{X}',\mathcal{A}')$ be measurable spaces. We call $X : \mathcal{X} \to \mathcal{X}'$ a* **measurable map** *if $X^{-1}(A') \in \mathcal{A} \; \forall A' \in \mathcal{A}'$.*

**Definition E.0.8.** *Let $(\mathcal{X},\mathcal{A})$ and $(\mathcal{X}',\mathcal{A}')$ be measurable spaces. We call $X : \mathcal{X} \to \mathcal{X}'$ a* **random variable** *with values in $(\mathcal{X}',\mathcal{A}')$ if $X$ is measurable. We also denote $X : (\mathcal{X},\mathcal{A}) \to (\mathcal{X}',\mathcal{A}')$.*

For our purposes, random variables will always be defined over measurable spaces that will be also probability spaces $(\mathcal{X},\mathcal{A},\mathbf{Pr})$. We also have $(\mathcal{X}',\mathcal{A}') = (\mathbb{R},\mathcal{B}(\mathbb{R}))$ where $\mathcal{B}(\mathbb{R})$ is the Borel algebra [70] of $\mathbb{R}$ *i.e.* the smallest $\sigma$-algebra that contains the open sets of $\mathbb{R}$. It will refer to observation space the triple $(\mathcal{Y},\bar{A},\bar{\mathbf{Pr}})$ where $\mathcal{Y} = X(\mathcal{X}) \subset \mathbb{R}$, $\bar{A}$ is the $\sigma$-algebra of $\mathcal{Y}$ and $\bar{\mathbf{Pr}} = \mathbf{Pr} \circ X^{-1}$. The random variable $X$ will be discrete if $\mathcal{Y}$ is discrete. We will denote the random variable with values in $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ simply by a random variable.

**Definition E.0.9.** *Let $X$ be a random variable. The probability measure $P_X \equiv \mathbf{Pr} \circ X^{-1}$ is called the* **distribution** *of $X$. If we denote $P_X(X \leq x) \equiv \mathbf{Pr} \circ X^{-1}((-\infty,x])$, we call the* **cumulative distribution function** *of the random variable $X$ the function $F_X(x) : x \mapsto P(X \leq x)$.*

Now we are able to define the probability density function of a random variable $X$:

**Definition E.0.10.** *Let $X$ be a random variable and $F_X(x)$ it's cumulative distribution function. If $F_X(x)$ is continuous $X$ is said to be continuous. If the derivative of $F_X(x)$ exists, $p_X(x) = F_X'(x)$ is called the* **probability density function** *of the continuous random variable $X$.*

For simplicity of notation we'll denote the probability density as $p(x)$.

**Definition E.0.11.** *Let $(\mathcal{X}_i,\mathcal{A}_i)$ be a measurable space for all $i \in I$. Let $X_i : (\mathcal{X},\mathcal{A}) \to (\mathcal{X}_i,\mathcal{A}_i)$ be a family of random variables with $\sigma$-algebra generated by $X_i^{-1}(\mathcal{A}_i)$ i.e. the smallest $\sigma$-algebra of $\mathcal{X}$ which contains $X_i^{-1}(\mathcal{A}_i)$ [70]. If for any finite set $J \subset I$*

$$P_{X_j}\left(\bigcap_{j \in J} A_j\right) = \prod_{j \in J} P_{X_j}(A_j)$$

$\forall A_j \in \mathcal{A}_j, j \in J$ *we call $(X_i)_{i \in I}$ a family of* **independent random variables**. *Additionally, if $(\mathcal{X}_i,\mathcal{A}_i)$ are all the same measurable space and $P_{X_i} = P_{X_\ell} \forall i,\ell \in I$ then we call $(X_i)_{i \in I}$ a family of* **independent and identically distributed** *random variables.*

for shorthand we will write "i.i.d." for independent and identically distributed.

**Definition E.0.12.** *An* **indexed family of probability densities** *is a family $\{p(x,\theta)\}$, $\theta \in \Theta$. $\Theta$ is called the set of parameters.*

As an example of indexed family of probability densities we can point out the family of normal distributions with unit standard deviation $\{N(\theta,1)\}$.

# Appendix F

# Cramér-Rao Theorem Demonstration

Here we present the proof of Cramér-Rao theorem presented in section 3.1. We enunciate it again:

**Theorem F.0.1** (Cramér-Rao Theorem). *The mean squared error of an unbiased estimator of the parameter $\theta$, $\mathcal{E}(X)$, is lower bounded by the inverse of the Fisher information*

$$Var(\mathcal{E}) \geq \frac{1}{F(\theta)}$$

**Proof**

Let $V : \mathcal{X} \to \mathbb{R}$ a score function. By Cauchy-Schwarz inequality we obtain:

$$(E_\theta[(V - E_\theta(V))(\mathcal{E} - E_\theta(\mathcal{E}))])^2 \leq E_\theta[(V - E_\theta(V))^2]E_\theta[(\mathcal{E} - E_\theta(\mathcal{E}))^2]. \quad \text{(F.1)}$$

Remembering Eq. ( 3.1), we have $E_\theta(V) = 0$ and $Var(V) = E_\theta(V^2) = F(\theta)$. Remembering that the $\mathcal{E}$ is unbiased, we obtain:

$$E_\theta[(V - E_\theta(V))(\mathcal{E} - E_\theta(\mathcal{E}))] = E_\theta[V\mathcal{E}]. \quad \text{(F.2)}$$

Then, Eq. ( F.1) turns to:

$$(E_\theta[V\mathcal{E}])^2 \leq F(\theta)Var(\mathcal{E}). \quad \text{(F.3)}$$

Now we calculate:

$$E_\theta[V\mathcal{E}] = \int_\mathcal{X} \frac{\partial_\theta p(x,\theta)}{p(x,\theta)}\mathcal{E}(x)p(x,\theta) = \int_\mathcal{X} [\partial_\theta p(x,\theta)]\,\mathcal{E}(x)$$

$$= \partial_\theta \left[\int_\mathcal{X} p(x,\theta)\mathcal{E}(x)\right] - \int_\mathcal{X} p(x,\theta)\overbrace{[\partial_\theta\mathcal{E}(x)]}^{=0}$$

$$= \partial_\theta E_\theta(\mathcal{E}) = \partial_\theta\theta = 1. \quad \text{(F.4)}$$

Here the interchange between the derivation and integration operators can be justified by the bounded convergence theorem for well behaved probability distribution $p(x,\theta)$. Equation ( F.4) implies the result of the theorem:

$$Var(\mathcal{E}) \geq \frac{1}{F(\theta)}.$$

(F.5)

# Appendix G

# Derivation of Quantum Fisher Information

From definition 3.1.9, the classical Fisher information is given by:

$$F(\xi) = \int_{\mathcal{X}} dx \, \frac{[\partial_\xi p(x,\xi)]^2}{p(x,\xi)}. \tag{G.1}$$

As discussed in section B, measurements are described by a POVM $\{\Pi_x\}$ and the probabilities are computed applying Born's rule:

$$p(x,\xi) = Tr(\Pi_x \rho_\xi). \tag{G.2}$$

combining Eq. (G.2) and Eq. (G.1), we get:

$$F(\xi) = \int_{\mathcal{X}} dx \, \frac{[\partial_\xi Tr(\Pi_x \rho_\xi)]^2}{Tr(\Pi_x \rho_\xi)}. \tag{G.3}$$

As discussed in section 3.2, we define quantum Fisher information (QFI), $J(\xi)$, as the maximum of $F(\xi)$ over all possible POVM's. In order to obtain that maximum we introduce the symmetric logarithm derivative (SLD), $L_\xi$, defined by the Lyapunov equation:

$$\partial_\xi \rho_\xi = \frac{L_\xi \rho_\xi + \rho_\xi L_\xi}{2}. \tag{G.4}$$

From Eq. (G.4) we obtain:

$$\partial_\xi p(x,\xi) = \partial_\xi Tr(\rho_\xi \Pi_x) = Tr(\partial_\xi \rho_\xi \Pi_x) = Tr\left(\frac{L_\xi \rho_\xi + \rho_\xi L_\xi}{2}\Pi_x\right) =$$

$$= Tr\left(\frac{\rho_\xi \Pi_x L_\xi}{2}\right) + Tr\left(\frac{\rho_\xi \Pi_x L_\xi}{2}\right)^* = \text{Re}\, Tr\left(\rho_\xi \Pi_x L_\xi\right), \tag{G.5}$$

such that:

$$F(\xi) = \int_{\mathcal{X}} dx \frac{\text{Re}\left[Tr\left(\rho_\xi \Pi_x L_\xi\right)\right]^2}{Tr\left(\rho_\xi \Pi_x\right)} \leq \int_{\mathcal{X}} dx \left|\frac{Tr\left(\rho_\xi \Pi_x L_\xi\right)}{\sqrt{Tr\left(\rho_\xi \Pi_x\right)}}\right|^2. \tag{G.6}$$

Important to note that the inequality above saturates to equality if

$$\text{Re}\left[Tr\left(\rho_\xi \Pi_x L_\xi\right)\right] = Tr\left(\rho_\xi \Pi_x L_\xi\right) \ \forall \xi \in \Xi, \tag{G.7}$$

where $\Xi$ is the set of all the parameters $\xi$. The last term of Eq. ( G.6) equals:

$$\int_{\mathcal{X}} dx \left|\frac{Tr\left(\rho_\xi \Pi_x L_\xi\right)}{\sqrt{Tr\left(\rho_\xi \Pi_x\right)}}\right|^2 = \int_{\mathcal{X}} dx \left|Tr\left[\left(\frac{\sqrt{\rho_\xi}\sqrt{\Pi_x}}{\sqrt{Tr\left(\rho_\xi \Pi_x\right)}}\right)\left(\sqrt{\Pi_x}L_\xi\sqrt{\rho_\xi}\right)\right]\right|^2. \tag{G.8}$$

Now, since $Tr(M^\dagger N)$ is an inner product in space of matrices, we can define such an inner product in the space of representations of operators over $\mathcal{D}(\mathcal{H})$, $\mathcal{H}$ the Hilbert space of the system of interest. This inner product defines a Frobenius norm and we can apply Schwartz inequality $|Tr(M^\dagger N)|^2 \leq Tr(M^\dagger M)Tr(N^\dagger N)$. Therefore, from the last term of Eq. ( G.8), we get:

$$\int_{\mathcal{X}} dx \left|Tr\left[\left(\frac{\sqrt{\rho_\xi}\sqrt{\Pi_x}}{\sqrt{Tr\left(\rho_\xi \Pi_x\right)}}\right)\left(\sqrt{\Pi_x}L_\xi\sqrt{\rho_\xi}\right)\right]\right|^2 \leq$$

$$\leq \int_{\mathcal{X}} dx \overbrace{Tr\left(\frac{\rho_\xi \Pi_x}{Tr\left(\rho_\xi \Pi_x\right)}\right)}^{=1} Tr\left(\Pi_x L_\xi \rho_\xi L_\xi\right) =$$

$$Tr\left(\overbrace{\left(\int_{\mathcal{X}} dx \Pi_x\right)}^{=1} L_\xi \rho_\xi L_\xi\right) = Tr(\rho_\xi L_\xi^2). \tag{G.9}$$

So we finally obtain:

$$J(\xi) = Tr(\rho_\xi L_\xi^2) = Tr(\partial_\xi \rho_\xi L_\xi). \tag{G.10}$$

Here we should emphasize that the inequality Eq.( G.9) is Schwartz inequality $|Tr(M^\dagger N)|^2 \leq Tr(M^\dagger M)Tr(N^\dagger N)$ which saturates to equality if the vectors are parallel

$$\frac{M}{|M|} = \frac{N}{|N|}. \tag{G.11}$$

This implies that inequality Eq.( G.9) saturates to equality if :

$$\frac{\sqrt{\Pi_x}\sqrt{\rho_\xi}}{\sqrt{Tr(\rho_\xi \Pi_x)}} = \frac{\sqrt{\Pi_x}L_\xi\sqrt{\rho_\xi}}{Tr(\Pi L_\xi \rho_\xi L_\xi)}. \tag{G.12}$$

The above equality is satisfied if and only if the POVM $\{\Pi\}$ consists of projectors over the

eigenstates of $L_\xi$ [73]. So we reach the conditions over the POVM such that the maximization of Fisher information results in quantum Fisher information:

1. $\text{Re}\left[Tr\left(\rho_\xi \Pi_x L_\xi\right)\right] = Tr\left(\rho_\xi \Pi_x L_\xi\right) \ \forall \xi \in \Xi.$

2. The POVM $\{\Pi\}$ should consist of projectors over the eigenstates of $L_\xi$.

But we haven't presented any solution of $L_\xi$ yet. Lyapunov's equation results in

$$L_\xi = 2 \int_0^\infty dt \ e^{-\rho_\xi t} \partial_\xi \rho_\xi e^{-\rho_\xi t}. \tag{G.13}$$

We consider the spectral decomposition of the probe system $\rho_\xi = \sum_j p_j |e_j\rangle\langle e_j|$, $0 < p_j \leq 1$, $\sum_j p_j = 1$. In this basis we have:

$$e^{-\rho_\xi t} = \sum_j e^{-p_j t} |e_j\rangle\langle e_j|, \tag{G.14}$$

such that, combining Eq. ( G.13) and Eq. ( G.14) we get:

$$L_\xi = 2 \int_0^\infty dt \sum_{ij} e^{-(p_i+p_j)t} \langle e_i | \partial_\xi \rho_\xi | e_j \rangle |e_i\rangle\langle e_j| =$$
$$= 2 \sum_{\substack{ij \\ p_i+p_j>0}} \frac{\langle e_i | \partial_\xi \rho_\xi | e_j \rangle}{p_i + p_j} |e_i\rangle\langle e_j|, \tag{G.15}$$

where we emphasize that the summation runs only over the indexes $i,j$ such that $p_i + p_j \neq 0$. Now if we combine Eq. ( G.15) with Eq. ( G.10) we get:

$$J(\xi) = Tr(\partial_\xi \rho_\xi L_\xi) = Tr\left(\partial_\xi \rho_\xi \left[2 \sum_{\substack{ij \\ p_i+p_j>0}} \frac{\langle e_i | \partial_\xi \rho_\xi | e_j \rangle}{p_i + p_j} |e_i\rangle\langle e_j|\right]\right) =$$

$$= \sum_n \langle e_n | \left(2 \sum_{\substack{ij \\ p_i+p_j>0}} \frac{\langle e_i | \partial_\xi \rho_\xi | e_j \rangle}{p_i + p_j} \partial_\xi \rho_\xi |e_i\rangle\langle e_j|\right) |e_n\rangle =$$

$$= \sum_n 2 \sum_{\substack{ij \\ p_i+p_j>0}} \frac{\langle e_i | \partial_\xi \rho_\xi | e_j \rangle}{p_i + p_j} \langle e_n | \partial_\xi \rho_\xi |e_i\rangle \delta_{jn} =$$

$$= 2 \sum_{\substack{ij \\ p_i+p_j>0}} \frac{\langle e_i | \partial_\xi \rho_\xi | e_j \rangle}{p_i + p_j} \langle e_j | \partial_\xi \rho_\xi |e_i\rangle. \tag{G.16}$$

So we obtain the expression:

$$J(\xi) = 2 \sum_{ij} \frac{|\langle e_i | \partial_\xi \rho_\xi | e_j \rangle|^2}{p_i + p_j}, \tag{G.17}$$

where $i,j$ in summation just includes terms such that $p_i + p_j > 0$.

Since $\partial_\xi \rho_\xi = \sum_j \partial_\xi p_j |e_j\rangle\langle e_j| + p_j |\partial_\xi e_j\rangle\langle e_j| + p_j |e_j\rangle\langle \partial_\xi e_j|$, we have:

$$J(\xi) = 2 \sum_{ij} \frac{|\langle e_i | \left(\sum_n \partial_\xi p_n |e_n\rangle\langle e_n| + p_n |\partial_\xi e_n\rangle\langle e_n| + p_n |e_n\rangle\langle \partial_\xi e_n|\right) |e_j\rangle|^2}{p_i + p_j}$$

$$= 2 \sum_{ij} \frac{|\sum_n \partial_\xi p_n \delta_{in}\delta_{nj} + p_n \langle e_i | \partial_\xi e_n\rangle \delta_{nj} + p_n \delta_{ni}\langle \partial_\xi e_n | e_j\rangle|^2}{p_i + p_j}$$

$$= 2 \sum_{ij} \frac{|\partial_\xi p_i \delta_{ij} + p_j \langle e_i | \partial_\xi e_j\rangle + p_i \langle \partial_\xi e_i | e_j\rangle|^2}{p_i + p_j}. \tag{G.18}$$

Now, note that, since $\langle e_i | e_j \rangle = \langle e_j | e_i \rangle = \delta_{ij}$, we have:

$$\partial_\xi \langle e_i | e_j \rangle = \langle \partial_\xi e_i | e_j \rangle + \langle e_i | \partial_\xi e_j \rangle = 0, \tag{G.19}$$

which implies that $\text{Re}[\langle e_i | \partial_\xi e_j \rangle] = 0$ [73]. Taking this equality in consideration, last equality of Eq. ( G.18) becomes:

$$J(\xi) = \sum_i \frac{(\partial_\xi p_i)^2}{p_i} + 2 \sum_{\substack{ij \\ p_i+p_j>0}} \frac{|p_j \langle e_i | \partial_\xi e_j\rangle - p_i \langle e_i | \partial_\xi e_j\rangle|^2}{p_i + p_j}. \tag{G.20}$$

So we finally obtain Eq. ( 3.18):

$$J(\xi) = \sum_i \frac{(\partial_\xi p_i)^2}{p_i} + 2 \sum_{i<j} \frac{2(p_i - p_j)^2}{p_i + p_j} |\langle i | \partial_\xi j\rangle|^2. \tag{G.21}$$

We refer to the first and second terms as the classical and quantum parts of the Fisher information.

# Appendix H

# Derivation of Equation ( 4.9)

Here, we provide a demonstration of Eq. ( 4.9). We keep notation from chapter 2. We reproduce the equation below:

$$\phi(f) = i\left[a([KEf^*]^*) - a^\dagger(KEf))\right].$$

(H.1)

**Proof**

Let's remember that, from Eq. ( 4.8), we have:

$$\phi(f) \equiv \int d^4x \sqrt{-g}\phi(x)f.$$

(H.2)

Let's denote the Klein Gordon inner product as defined in Eq. ( 2.5) as

$$(g_1,g_2)_{KG} = -i\Omega(g_1^*,g_2),$$

(H.3)

such that:

$$\Omega(g_1,g_2) = \int_{\Sigma_t} d^3\mathbf{x}\sqrt{h}\,(g_2\nabla_a g_1 - g_1\nabla_a g_2)n^a.$$

(H.4)

We will first show that:

$$\int d^4x \sqrt{-g}\phi(x)f = \Omega(Ef,\phi).$$

(H.5)

Let's choose $\Sigma_t$ such that it does not contain causal future of supp$(f)$, $J^+(\text{supp}(f))$. By the definition of the Green function $G_A$ we have:

$$\left[\nabla^a\nabla_a - m^2\right]\int d^4x' \sqrt{-g}G_A(x,x')f(x') = f(x).$$

(H.6)

If we denote:

$$Af = \int d^4x' \sqrt{-g}G_A(x,x')f(x'),$$

(H.7)

we have:

$$\int d^4x\sqrt{-g}\phi(x)f(x) = \int_{J^+(\Sigma_t)} d^4x\sqrt{-g}\phi(x)f(x)$$

$$= \int_{J^+(\Sigma_t)} d^4x\sqrt{-g}\phi(x)\left[\nabla^a\nabla_a - m^2\right]Af. \qquad \text{(H.8)}$$

Integrating by parts twice, we have:

$$\int_{J^+(\Sigma_t)} d^4x\sqrt{-g}\phi\left[\nabla^a\nabla_a - m^2\right]Af \qquad \text{(H.9)}$$

$$= \int_{J^+(\Sigma_t)} d^4x\sqrt{-g}\phi\nabla^a\nabla_aAf - m^2\int_{J^+(\Sigma_t)} d^4x\sqrt{-g}\phi Af \qquad \text{(H.10)}$$

$$= \int_{J^+(\Sigma_t)} d^4x\sqrt{-g}[\nabla^a(\phi\nabla_aAf) - \nabla^a\phi\nabla_aAf] - m^2\int_{J^+(\Sigma_t)} d^4x\sqrt{-g}\phi Af \qquad \text{(H.11)}$$

$$= \int_{J^+(\Sigma_t)} d^4x\sqrt{-g}\nabla^a(\phi\nabla_aAf)$$

$$- \left(\int_{J^+(\Sigma_t)} d^4x\sqrt{-g}[\nabla_a(\nabla^a\phi Af) - \nabla_a\nabla^a\phi Af]\right) - m^2\int_{J^+(\Sigma_t)} d^4x\sqrt{-g}\phi Af \qquad \text{(H.12)}$$

$$= \int_{J^+(\Sigma_t)} d^4x\sqrt{-g}\nabla^a(\phi\nabla_aAf - Af\nabla_a\phi) + \int_{J^+(\Sigma_t)} d^4x\sqrt{-g}Af[\nabla^a\nabla_a - m^2]\phi. \qquad \text{(H.13)}$$

Since $\phi$ is solution to Klein-Gordon equation, second integral of Eq. ( H.13) vanishes and we can apply Stokes' theorem to the first integral to obtain:

$$\int_{J^+(\Sigma_t)} d^4x\sqrt{-g}\nabla^a(\phi\nabla_aAf - Af\nabla_a\phi) = \int_{\Sigma_t} d^3\mathbf{x}\sqrt{h}(\phi\nabla_aAf - Af\nabla_a\phi)n^a. \qquad \text{(H.14)}$$

So, from Eq. ( H.8) to Eq. ( H.14) we conclude that:

$$\int d^4x\sqrt{-g}\phi(x)f(x) = \int_{\Sigma_t} d^3\mathbf{x}\sqrt{h}(\phi\nabla_aAf - Af\nabla_a\phi)n^a. \qquad \text{(H.15)}$$

But, as $Ef|_{\Sigma_t} = Af|_{\Sigma_t}$, we obtain the result:

$$\int d^4x\sqrt{-g}\phi(x)f(x) = \Omega(Ef,\phi). \qquad \text{(H.16)}$$

Now, since we can decompose a quantum field as Eq. ( 2.13):

$$\widehat{\phi}(x) = \sum_{i\in\mathcal{I}} a(f_i^*)f_i + a^\dagger(f_i)f_i^*, \qquad \text{(H.17)}$$

if we multiply Eq. ( H.17) by a test function $f$ and integrate over the spacetime, using Eqs. ( H.16), ( H.3) and ( 2.14), we would obtain Eq. ( H.1). However Eq. ( H.17) is not well defined [1]. The most rigorous derivation of Eq. ( H.1) consists in an identification. Since for each

test function $f$, the operator over Klein-Gordon solution space, $\Omega(Ef,\cdot)$, acts averaging each solution by $f$ in all spacetime. So, in the quantum case, we identify the quantum field operator averaged by the test function $f$, $\widehat{\phi}(f)$, with the operator over $\mathcal{F}(\mathcal{H}_{KG})$, $\widehat{\Omega}(Ef,\cdot)$, defined by:

$$\widehat{\Omega}(Ef,\cdot) \equiv i\left[a([KEf^*]^*) - a^\dagger(KEf))\right],\tag{H.18}$$

such that

$$\widehat{\phi}(f) = \widehat{\Omega}(Ef,\cdot) \equiv i\left[a([KEf^*]^*) - a^\dagger(KEf))\right].\tag{H.19}$$

This way we obtain Eq. ( H.1). To a complete discussion about the definition of $\widehat{\Omega}(Ef,\cdot)$ see chapter 2 and 3 of  [1].

# Appendix Ⅰ

# Quantum Fisher Information Expression for the Single Detector

The quantum Fisher information expression for the single detector case is given by:

$$J_T = \frac{A}{B_+} + \frac{A}{B_-} + \frac{C_1}{C_2}. \tag{I.1}$$

We will need to describe term by term. First $A$.

$$A = e^{2\omega/T}\mu^2\omega^2(a_0 + a_1 + a_2 + a_3)^2, \tag{I.2}$$

where

$$a_0 = 24 - 24e^{\omega/T} + 8\mu + 8e^{\omega/T}\mu + 2\mu^2 - 2e^{\omega/T}\mu^2. \tag{I.3}$$

$$a_1 = -\mu(18 + e^{\omega/T}(-18 + \mu) + \mu)\cos(2\eta). \tag{I.4}$$

$$a_2 = 2(4 + 4\mu - \mu^2 + e^{\omega/T}(-4 + 4\mu + \mu^2))\cos(4\eta). \tag{I.5}$$

$$a_3 = +(2\mu - 2e^{\omega/T}\mu + \mu^2 + e^{\omega/T}\mu^2)\cos(6\eta). \tag{I.6}$$

Now, $B_-$:

$$B_- = 2T^4 b_0^3 \Gamma(b_1 - \sqrt{2\Gamma}), \tag{I.7}$$

where

$$b_0 = -2 + \mu + e^{\omega/T}(2 + \mu) + (-1 + e^{\omega/T})\mu\cos(2\eta), \tag{I.8}$$

$$b_1 = -4 + 4e^{\omega/T} + 2\mu + 2e^{\omega/T}\mu - 2\mu\cos(2\eta) + 2e^{\omega/T}\mu\cos(2\eta), \tag{I.9}$$

and $\Gamma$ is given by

$$\Gamma = e^{2\omega/T}(\gamma_1 + \gamma_2 + \gamma_3), \tag{I.10}$$

where $\gamma_1$, $\gamma_2$ and $\gamma_3$ are defined by Eq. ( 4.32), Eq. ( 4.33) and Eq. ( 4.34) respectively. We

reproduce it here:

$$\gamma_1 = e^{-2\omega/T}(8 + \mu(4 + 3\mu)) - 2e^{-\omega/T}(8 + \mu^2) + (8 + \mu(-4 + 3\mu)), \qquad (\text{I.11})$$

$$\gamma_2 = 4\mu f_\omega(-2 + \mu + e^{-\omega/T}(2 + \mu))\cos(2\eta), \qquad (\text{I.12})$$

$$\gamma_3 = \mu(1 + e^{-\omega/T})(-4 + \mu + e^{-\omega/T}(4 + \mu))\cos(4\eta). \qquad (\text{I.13})$$

Now, $B_+$ is given by:

$$B_+ = 2T^4 b_0^3 \Gamma(b_1 + \sqrt{2\Gamma}). \qquad (\text{I.14})$$

The term $C_1$ is given by:

$$C_1 = \frac{65536 e^{2\omega/T}(-1 + e^{\omega/T})^2 \mu^2 \omega^2 \cos^2(2\eta)}{\cos^4(\eta)\sin^4(\eta)} \Gamma(c_0 + \sqrt{\Gamma})^2, \qquad (\text{I.15})$$

where

$$c_0 = \sqrt{2}\left(\mu(-1 + e^{\omega/T}) + 2\cos(2\eta)(1 - e^{\omega/T}) + \mu\cos(2\eta)(1 + e^{\omega/T})\right). \qquad (\text{I.16})$$

Finally $C_2$ is given by:

$$C_2 = T^4 b_0^2 D_+ D_-^3, \qquad (\text{I.17})$$

where

$$D_+ = d_0 + d_1 + d_2 + d_3 + d_4 + d_5\sqrt{\Gamma}, \qquad (\text{I.18})$$

and

$$d_0 = 32 - 64e^{\omega/T} + 32e^{2\omega/T}, \qquad (\text{I.19})$$

$$\begin{aligned} d_1 = \big(&32 - 64e^{\omega/T} + 32e^{2\omega/T} + 32\mu - 32e^{2\omega/T}\mu \\ &+8\mu^2 + 16e^{\omega/T}\mu^2 + 8e^{2\omega/T}\mu^2\big)\cot^2(2\eta), \end{aligned} \qquad (\text{I.20})$$

$$d_2 = \left(-64\mu + 128e^{\omega/T}\mu - 64e^{2\omega/T}\mu - 32\mu^2 + 32e^{2\omega/T}\mu^2\right)\cot(2\eta)\csc(2\eta), \qquad (\text{I.21})$$

$$\begin{aligned} d_3 = \big(&32 - 64e^{\omega/T} + 32e^{2\omega/T} + 16\mu - 16e^{2\omega/T}\mu \\ &+20\mu^2 - 24e^{\omega/T}\mu^2 + 20e^{2\omega/T}\mu^2\big)\csc^2(2\eta), \end{aligned} \qquad (\text{I.22})$$

$$d_4 = \left(4\mu - 4e^{2\omega/T}\mu + \mu^2 + 2e^{\omega/T}\mu^2 + e^{2\omega/T}\mu^2\right)\cos(4\eta)\csc^2(\eta)\sec^2(\eta), \qquad (\text{I.23})$$

$$\begin{aligned} d_5 = \sqrt{2}\big(&2\mu - 2\mu e^{\omega/T} - 4\cos(2\eta) + 4\cos(2\eta)e^{\omega/T} \\ &-2\mu\cos(2\eta) - 2\mu\cos(2\eta)e^{\omega/T}\big)\csc^2(\eta)\sec^2(\eta). \end{aligned} \qquad (\text{I.24})$$

Finally

$$D_- = d_0 + d_1 + d_2 + d_3 + d_4 - d_5\sqrt{\Gamma} \qquad (\text{I.25})$$

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