Relativistic fluctuation relation

Lecture 3 — The mathematical basis of relativity

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Differential Geometry

The postulates of GR

General relativity is based on two postulates:

- Spacetime is a four-dimensional differentiable manifold with a Lorentzian metric defined on it.
- Einstein's field equations hold

$$\mathrm{R}_{ab} - \frac{1}{2}R\mathrm{g}_{ab} + \lambda\mathrm{g}_{ab} = 8\pi G\mathrm{T}_{ab}$$

We need differential geometry in order to properly understand these postulates.

Topological spaces

Let $\mathbb{X}$ be a set and $\mathbb{T}$ a collection of subsets of $\mathbb{X}$ satisfying the following properties.

1. The union of an arbitrary collection of subsets, each of which is in $\mathbb{T}$, is in $\mathbb{T}$. If $\mathbb{O}_\alpha \in \mathbb{T}$ for all α, then $\cup_\alpha \mathbb{O}_\alpha \in \mathbb{T}$.
2. The intersection of a finite number of subsets of $\mathbb{T}$ is in $\mathbb{T}$. If $\{\mathbb{O}_\alpha\}_{\alpha=1}^n \in \mathbb{T}$, then $\cap_{\alpha=1}^n \mathbb{O}_\alpha \in \mathbb{T}$.
3. The entire set $\mathbb{X}$ and the empty set $\emptyset$ are in $\mathbb{T}$.

$\mathbb{X}$ is said to be a topological space, and $\mathbb{T}$ provides a topology to $\mathbb{X}$.

Examples

- The real line $\mathbb{R}$ where the open sets are unions of open intervals $(a, b) \subset \mathbb{R}$.

- $\mathbb{X} = \{a, b, c\}$ and $\mathbb{T} = \{\mathbb{X}, \emptyset, \{a, b\}, \{b\}, \{b, c\}\}$. The pair $(\mathbb{X}, \mathbb{T})$ is a topological space. The collection of all subsets of $\mathbb{X}$ is called the discrete topology, while the set $\{\mathbb{X}, \emptyset\}$ is called the trivial topology.

- If a metric[1] $d(x, y)$, with $x, y \in \mathbb{X}$, is defined on $\mathbb{X}$, its open sets are given by the open discs $\mathbb{O}_\epsilon = \{y \in \mathbb{X} \,|\, d(x, y) < \epsilon\}$ and all their possible unions. Such topology is called a metric topology determined by d and the topological space $(\mathbb{X}, \mathbb{T})$ is called a **metric space**.

[1] $d : \mathbb{X} \times \mathbb{X} \mapsto \mathbb{R}$ under the conditions $i)$ $d(x, y) = d(y, x)$, $ii)$ $d(x, y) \geq 0$, with the equality holding only for $x = y$ and $iii)$ $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in \mathbb{X}$.

What is a manifold?

Manifold

is just a continuous set of points that can exhibit complex global properties like curvature or torsion, but that *locally* looks like the Euclidean space. In a small enough neighborhood of any point on the manifold, Euclidean geometry applies. The surface of a sphere is a manifold!.

The meaning of locally flat space

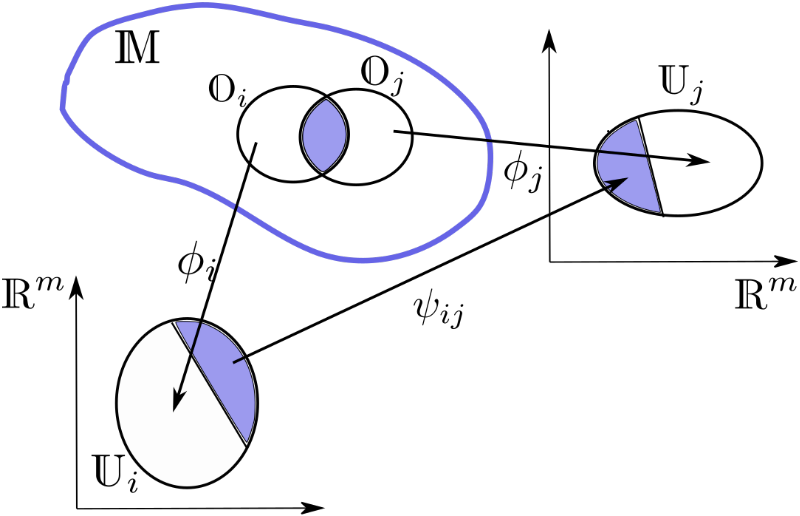
A manifold is a generalization of the concepts of curves and surfaces to objects with arbitrary dimensions. In the same way a curve is locally homeomorphic to $\mathbb{R}$ a manifold is a topological space which is locally homeomorphic to $\mathbb{R}^m$. This is what enables us to define local coordinate systems!

Formal definition

$\mathbb{M}$ is an m-dimensional differentiable manifold if we have a set of pairs $\{(\mathbb{O}_i, \phi_i)\}_i$ of open subsets (local neighborhoods) $\mathbb{O}_i \subset \mathbb{M}$ and local homeomorphisms ϕ_i from $\mathbb{O}_i$ to open subsets $\mathbb{U}_i \subset \mathbb{R}^m$, such that the following conditions hold:

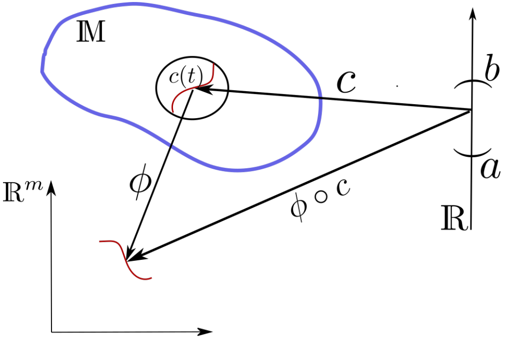
- $\mathbb{M}$ is a topological space.
- The set of $\mathbb{O}_i$ covers $\mathbb{M}$: $\cup_i \mathbb{O}_i = \mathbb{M}$.
- Given two neighbourhoods $\mathbb{O}_i$ and $\mathbb{O}_j$ such that $\mathbb{O}_i \cap \mathbb{O}_j \neq \emptyset$ for $i \neq j$, the map $\psi_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j (\mathbb{O}_i \cap \mathbb{O}_j) \mapsto \phi_i (\mathbb{O}_i \cap \mathbb{O}_j)$ is infinitely differentiable.

Differentiable structure



Curves

An open curve on $\mathbb{M}$ is the map $c : (a,b) \mapsto \mathbb{M}$, where $(a,b) \subset \mathbb{R}$. $c(t)$ has the coordinate representation $x = \phi \circ c : \mathbb{R} \mapsto \mathbb{R}^m$.



Vectors

Tangent space

On a manifold, a vector is defined in terms of a **tangent vector** to a curve, which is the generalization of the tangent line to a curve in the usual two-dimensional plane. However, in an n-dimensional manifold, there are many curves that are indeed tangent to a given vector and, therefore, a tangent vector is an equivalence class of curves.

Vectors

Let us consider a function $f : \mathbb{M} \mapsto \mathbb{R}$. In a local coordinate system

$$\left.\frac{\mathrm{d}f(c(t))}{\mathrm{d}t}\right|_{t=0} = \frac{\partial f}{\partial x^\mu} \left.\frac{\mathrm{d}x^\mu(c(t))}{\mathrm{d}t}\right|_{t=0} \qquad \frac{\partial f}{\partial x^\mu} \equiv \frac{\partial}{\partial x^\mu}\left[f \circ \phi^{-1}(x)\right]$$

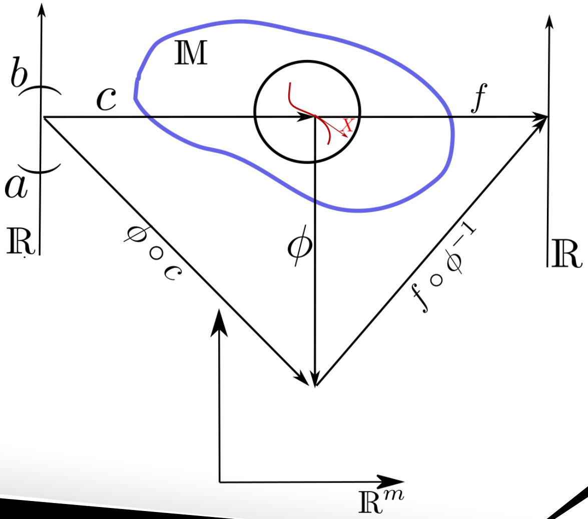
The rate $(\mathrm{d}f/\mathrm{d}t)|_{t=0}$ is obtained by the application of the differential operator

$$\mathrm{X} = \mathrm{X}^\mu \left(\frac{\partial}{\partial \mathrm{x}^\mu}\right) \quad \text{with} \quad \mathrm{X}^\mu = \left.\frac{\mathrm{d}\mathrm{x}^\mu(\mathrm{c}(\mathrm{t}))}{\mathrm{d}\mathrm{t}}\right|_{\mathrm{t}=0}$$

to the map f, denoted as $X[f]$.

Vectors

We define X as the tangent vector to $\mathbb{M}$ at the point $p = c(0)$ along the direction given by the curve $c(t)$.



The tangent space

Given two curves c_1 and c_2 such that $c_1(0) = c_2(0) = p$ and

$$\left.\frac{\mathrm{d}x^\mu(c_1(t))}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\mathrm{d}x^\mu(c_2(t))}{\mathrm{d}t}\right|_{t=0}$$

then both curves define the same vector X at p ($c_1 \sim c_2$). X is the equivalence class of curves

$$[c(t)] = \left\{ \tilde{c}(t) \,|\, \tilde{c}(0) = c(0) \quad \text{and} \quad \left.\frac{\mathrm{d}x^\mu(c_1(t))}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\mathrm{d}x^\mu(c_2(t))}{\mathrm{d}t}\right|_{t=0} \right\},$$

All the equivalence classes of curves at $p \in \mathbb{M}$ (all the tangent vectors at p) form a vector space called the **tangent space** of $\mathbb{M}$ at p, denoted by $\mathrm{T}_p\mathbb{M}$.

The coordinate basis

We have

$$\frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu \Rightarrow X[x^\mu] = \left.\frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t}\right|_{t=0}$$

The coordinate basis is then defined by the vectors

$$e_\mu \equiv \frac{\partial}{\partial x^\mu}$$

We have a tangent space associated with each point on the manifold. The collection of all these tangent spaces for all points on the manifold is called the **tangent bundle**, which is a manifold in its own right.

Transformation laws

The vector is coordinate independent, but its components are not. Let $p \in \mathbb{O}_i \cap \mathbb{O}_j$ and $\{x^\mu\} = \phi_i(p)$ and $\{y^\nu\} = \phi_j(p)$ be the two coordinate functions. We then have two expressions for the vector $\mathrm{X} \in \mathrm{T_p}\mathbb{M}$

$$\mathrm{X} = \mathrm{X}^\mu \frac{\partial}{\partial \mathrm{x}^\mu} = \tilde{\mathrm{X}}^\mu \frac{\partial}{\partial \mathrm{y}^\mu}.$$

This implies that the components of the vector in the two coordinate bases must be related by

$$\tilde{X}^\mu = X^\nu \frac{\partial y^\mu}{\partial x^\nu}.$$

It is important to observe here that this transformation law is such that the vector itself is left invariant.

The dual space

We now associate the **dual vector space** (cotangent space) $\mathsf{T}_p^*\mathbb{M}$ to every $\mathsf{T}_p\mathbb{M}$.

$$\omega \in \mathsf{T}_p^*\mathbb{M} : \mathsf{T}_p\mathbb{M} \mapsto \mathbb{R}$$

ω is called the cotangent vector, the dual vector, or the one-form, with the simplest example being the differential $\mathrm{d}f$ of a function $f \in \mathbb{F}(\mathbb{M})$. The action of $\mathrm{d}f \in \mathsf{T}_p^*\mathbb{M}$ on $V \in \mathsf{T}_p\mathbb{M}$ is defined as

$$\langle \mathrm{d}f, V\rangle = V[f] = V^\mu \frac{\partial f}{\partial x^\mu} \in \mathbb{R}$$

Since $\mathrm{d}f = (\partial f/\partial x^\mu)\mathrm{d}x^\mu$, we take $\mathrm{d}x^\mu$ as the elements of the basis in $\mathsf{T}_p^*\mathbb{M}$, which is the dual basis since

$$\left\langle \mathrm{d}x^\nu, \frac{\partial}{\partial x^\mu} \right\rangle = \frac{\partial x^\nu}{\partial x^\mu} = \delta^\nu_\mu \qquad \Rightarrow \qquad \omega = \omega_\mu \mathrm{d}x^\mu$$

The inner product

The inner product $\langle\,,\rangle : \mathrm{T}_p^*\mathcal{M} \times \mathrm{T}_p\mathcal{M} \mapsto \mathbb{R}$ is defined as

$$\langle \omega, \mathrm{V} \rangle = \omega_\nu V^\mu \left\langle \mathrm{d}x^\nu, \frac{\partial}{\partial x^\mu} \right\rangle = \omega(V) = \omega_\mu V^\mu.$$

Let us consider two coordinate systems $\{x^\mu\} = \phi_i(p)$ and $\{y^\nu\} = \phi_j(p)$. We thus have

$$\omega = \omega_\mu \mathrm{d}x^\mu = \tilde{\omega}_\nu \mathrm{d}y^\nu$$

From which follows the transformation law for the components of the one-form

$$\tilde{\omega}_\nu = \omega_\mu \frac{\partial x^\mu}{\partial y^\nu}$$

Tensors

A tensor of type (q, r) is a multilinear map

$$\mathrm{T} : \left[\times^{q} \mathsf{T}_p^{*}\mathbb{M}\right] \left[\times^{r} \mathsf{T}_p\mathbb{M}\right] \mapsto \mathbb{R}$$

In terms of the basis vectors defined before, we can write the tensor $\mathrm{T} \in \mathbb{T}^{q}_{r,p}(\mathbb{M})$ as

$$\mathrm{T} = T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} \frac{\partial}{\partial x^{\mu_1}} \dots \frac{\partial}{\partial x^{\mu_1}} \mathrm{d}x^{\nu_1} \dots \mathrm{d}x^{\nu_r}$$

The action of a tensor on ω_i $(1 \leq i \leq r)$ and V_j $(1 \leq j \leq q)$ results in the number

$$\mathrm{T}(\omega_1, \dots, \omega_r; V_1, \dots, V_q) = T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} \omega_{1,\mu_1} \dots \omega_{q,\mu_q} V_1^{\nu_1} \dots V_r^{\nu_r}$$

Tensors

Given two coordinate systems x and x', the components of the tensor T change as

$$T^{\mu_1' \dots \mu_q'}{}_{\nu_1' \dots \nu_r'} = T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} \frac{\mathrm{d}x^{\mu_1'}}{\mathrm{d}x^{\mu_1}} \cdots \frac{\mathrm{d}x^{\nu_1}}{\mathrm{d}x^{\nu_1'}},$$

which is the general transformation law for tensors.

We can let the tensor not act on all of its slots, resulting in different tensors. For instance, a map between two vector spaces can be seen as a tensor

$$\mathrm{T^a{}_b} : \mathrm{V^b} \mapsto \mathrm{T^a{}_b V^b}$$

The metric tensor

We impose that the determinant of the metric does not vanish, $\mathrm{Det}[\mathrm{g}_{ab}] \equiv g \neq 0$. Then

$$g^{\mu\nu}g_{\alpha\nu} = \delta^{\mu}_{\alpha}.$$

One of the many applications of the metric tensor is to provide the notion of the length of a path in spacetime, the line element

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} = g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}.$$

The metric is employed to raise and lower indices

$$\mathrm{x}_a = \mathrm{g}_{ab}x^b \qquad \mathrm{x}^a = \mathrm{g}^{ab}x_b$$

The covariant derivative

Vectors and tensors are coordinate-independent objects. The laws of physics must be written in terms of them!

But we need derivatives. The partial derivative is not a tensor since it does not transform like a tensor does. If we want the laws of physics to be coordinate-independent, we need to define the notion of the covariant derivative on manifolds.

From this, the important concept of a **connection** emerges. The curvature of a manifold depends on this guy!

The covariant derivative

In flat space in inertial coordinates, the partial derivative operator is a map between tensor fields

$$\partial_\mu : (k, l) \to (k, l+1)$$

that obeys linearity and the Leibniz rule.

We would like to define an operator ∇ that plays the role of the partial derivative but that is coordinate independent! We require that $\nabla : (k, l) \to (k, l+1)$ that obeys:

- Linearity: $\nabla(\mathrm{T} + \mathrm{S}) = \nabla\mathrm{T} + \nabla\mathrm{S}$
- Leibniz: $\nabla(\mathrm{T} \otimes \mathrm{S}) = \nabla(\mathrm{T}) \otimes \mathrm{S} + \mathrm{T} \otimes \nabla(\mathrm{S})$

The connection

From Leibniz's rule, ∇ is written as a partial derivative and a linear correction to make it covariant. There must be a set of n matrices $(\Gamma_\mu)^\alpha_\beta$ for each direction μ such that

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\alpha} V^\alpha$$

$\Gamma^\nu_{\mu\alpha}$ are the **connection coefficients**. By demanding that it commutes with contraction and that it reduces to the partial derivative in flat spacetimes, we obtain

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\alpha_{\mu\nu} \omega_\alpha$$

From this we can construct the derivative of higher tensors.

The compatible connection

There are many distinct connections! Which one should we pick for GR?

Although $\Gamma^{\alpha}_{\mu\nu}$ is not a tensor, the difference between two of them is the torsion tensor

$$S^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} - \Gamma^{\alpha}_{\gamma\beta}$$

A unique connection arises in GR by imposing:

- Torsion free: $S^{\alpha}{}_{\beta\gamma} = 0$
- Metric compatibility: $\nabla_{\mu} g_{\alpha\beta} = 0$

From which the Christoffel connection follows

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\sigma} \left(\partial_{\beta} g_{\gamma\sigma} + \partial_{\gamma} g_{\beta\sigma} - \partial_{\sigma} g_{\beta\gamma} \right)$$

What do we have done so far?

- We started with the basic notion of a set.
- We introduced the concept of open subsets. The set became a topological space.
- By demanding that each open set look like a region of $\mathbb{R}^n$ and that the coordinate charts be smoothly sewn together, the topological space became a manifold.
- A manifold comes equipped naturally with a tangent bundle, tensor bundles, the ability to take exterior derivatives, and so forth.
- We then proceeded to put a metric on the manifold.
- Independently of the metric we could introduce a connection, allowing us to take covariant derivatives.

How to compare vectors from different spaces?

We need to understand how to move vectors along paths while *keeping them constant*. This is the job of **parallel transport**, which is defined by the connection!

> **Path dependence**
> The key difference between transporting a vector in flat and curved spacetimes is that it is path-dependent in the curved case!

> There is no way to define relative velocities for events happening at distinct points of spacetime!

The geodesic equation

The **geodesic** is defined as the curve along which the tangent vector is parallel transported (in the same way the tangent vector is transported along a straight line!)

$$\frac{D}{\mathrm{d}\lambda}\frac{\mathrm{d}x^{\mu}(\lambda)}{\mathrm{d}\lambda} = 0$$

This implies the **geodesic equation**

$$\frac{\mathrm{d}^2 x^{\alpha}(\lambda)}{\mathrm{d}\lambda^2} + \Gamma^{\alpha}_{\mu\nu}\frac{\mathrm{d}x^{\alpha}(\lambda)}{\mathrm{d}\lambda}\frac{\mathrm{d}x^{\beta}(\lambda)}{\mathrm{d}\lambda} = 0$$

> Extremals of the length functional are curves that parallel transport their tangent vector with respect to the Christoffel connection associated with that metric.

The curvature

The curvature of a manifold is quantified by the Riemann curvature tensor, which is derived from the connection!

Manifestations of flatness

- Parallel transport around a closed loop leaves a vector unchanged.
- Covariant derivatives of tensors commute.
- Initially parallel geodesics remain parallel.

The Riemann tensor arises when we study how any of these properties are altered in more general contexts!

The curvature

Let us consider an infinitesimal closed loop specified by the vectors A^μ and B^ν.

A vector V^μ is first parallel transported along A^μ, then along B^ν, then backward along A^μ and B^ν.

Parallel transport is independent of coordinates, so there must be a tensor telling us how the vector changes when it comes back to its starting point. It will be a linear transformation on a vector, and therefore involve one upper and one lower index. It also depends on the two vectors A^μ and B^ν, implying two lower indices. The change δV^σ must be written as

$$\delta V^\sigma = R^\sigma{}_{\alpha\beta\gamma} V^\alpha A^\beta B^\gamma$$

The curvature

This is the definition of the Riemann tensor $R^{\sigma}{}_{\alpha\beta\gamma}V^{\alpha}A^{\beta}B^{\gamma}$. However, since we know how covariant derivatives work, we can compute the commutator

$$[\nabla_{\mu},\nabla_{\nu}]V^{\alpha}=\nabla_{\mu}\nabla_{\nu}V^{\alpha}-\nabla_{\nu}\nabla_{\mu}V^{\alpha}$$

From which we obtain the formula for the curvature tensor in terms of the connection

$$R^{\rho}{}_{\sigma\mu\nu}=\partial_{\mu}\Gamma^{\rho}_{\nu\sigma}-\partial_{\nu}\Gamma^{\rho}_{\mu\sigma}+\Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma}-\Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}$$

From which it follows three important quantities

$$R_{\mu\nu}=R^{\rho}{}_{\mu\rho\nu}\qquad R=R^{\mu}_{\mu}=g^{\mu\nu}R_{\mu\nu}\qquad G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}$$

That is all, Folks!

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