Relativistic fluctuation relation

Lecture 2 — What is general relativity?

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The Newtonian World

Old ideas on space and time

The structure of space and time

The laws of mechanics

- **The first law**: There exists **inertial reference frames**. Isolated particle remains at rest or in uniform motion in a straight line.
- **The second law**: In any inertial frame it holds

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \vec{F}$$

- **The third law**: To every action there corresponds an equal and opposite reaction

$$\vec{F}_{i,j} = -\vec{F}_{j,i},$$

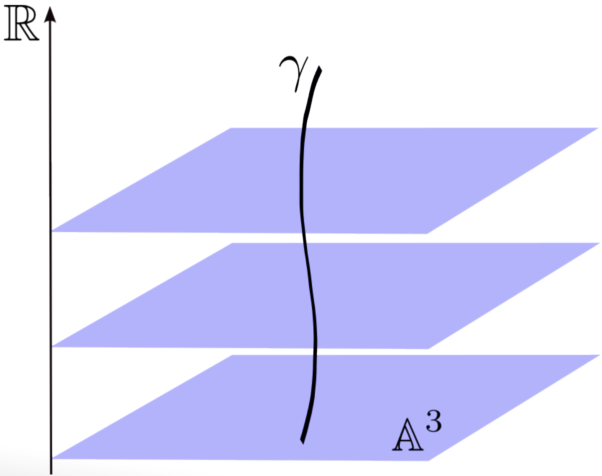
Where did this come from?

Experimental facts!

- Space is three-dimensional and **Euclidean**, while time is one-dimensional.
- We know Galileo's principle of relativity. All the inertial reference frame are equivalent. Homogeneity of spacetime and Isotropy of space!
- We also know the principle of determinacy: Initial position and velocity determines the motion. Newton's law is second order!

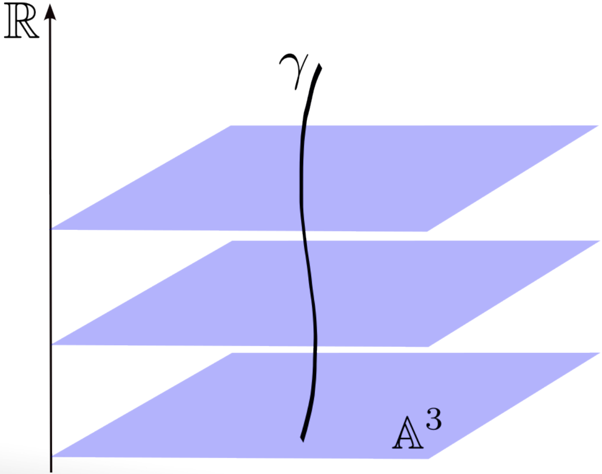
Newtonian spacetime

The Newtonian universe is a four-dimensional **affine space** $\mathbb{A}^4$. $\mathbb{R}^4$ acts as the group of parallel displacements $a \to a + \vec{v}$ where a, $(a + \vec{v}) \in \mathbb{A}^4$ and $\vec{v} \in \mathbb{R}^4$. From this, we see that the difference of two points of $\mathbb{A}^4$ is a vector in $\mathbb{R}^4$, while the sum is not defined. Each element of $\mathbb{A}$ is called **event**.



Newtonian spacetime

Time is defined as a linear map $t : \mathbb{A}^4 \mapsto \mathbb{A}^1$. The kernel of which defines the simultaneous hypersurfaces $\mathbb{A}^4$. On each one, we can define a proper distance function $d : \mathbb{A}^3 \times \mathbb{A}^3 \mapsto \mathbb{R}$. Spacial distances are deeply linked to simultaneity.



The Galilean spacetime

A positive bilinear symmetric form $\langle x, y\rangle$, called a scalar product on $\mathbb{R}^4$, defines the *Euclidean* structure and allows us to define the distance function as

$$d(a,b) \equiv ||a-b|| = \sqrt{\langle a-b, a-b\rangle}$$

between points a and b of the corresponding simultaneity hyperspace. Since the difference of two events in $\mathbb{A}^4$ is a vector in $\mathbb{R}^4$, it is clear that distances on the spatial hypersurfaces are defined by the kernel of t.

We call a **Galilean spacetime** (or Euclidean) the set $\left(\mathbb{A}^4, t, d\right)$. The set of affine transformations that preserve time intervals and distances between simultaneous events forms the *Galilean group*.

Reference frames

The laws of physics are expressed in terms of differential equations, which means that, in order to do physics, we should be able to employ calculus. We do this by introducing reference frames, which are ways to unambiguously label the points of $\mathbb{A}$.

Reference frames are maps from the set $\mathbb{A}^4$ to the set $\mathbb{R}^4$, where we understand how calculus works. In the case of Newtonian space and time, a *single map* is able to cover the entire set $\mathbb{A}^4$. However, this is not possible when gravity comes into play! Moreover, we demand that such maps are of class C^∞.

Reference frames

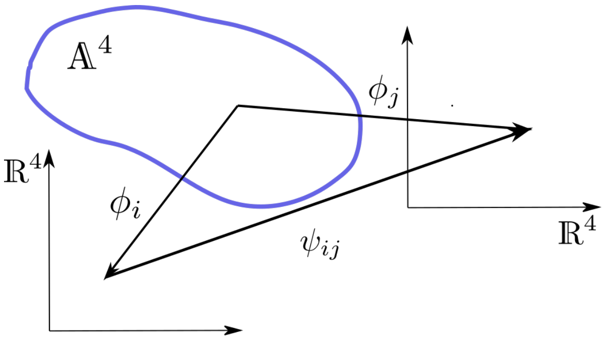
A *Galilean coordinate system* is the map

$$\phi : \mathbb{A}^4 \mapsto \mathbb{R}^3 \times \mathbb{R},$$

Each event is labeled by a set of four numbers. A Galilean transformation

$$\psi_{i,j} \equiv \phi_i \circ \phi_j^{-1} : \mathbb{R}^3 \times \mathbb{R} \mapsto \mathbb{R}^3 \times \mathbb{R}$$

takes us from one reference frame to the other.



Reference frames

Coordinates and Distances

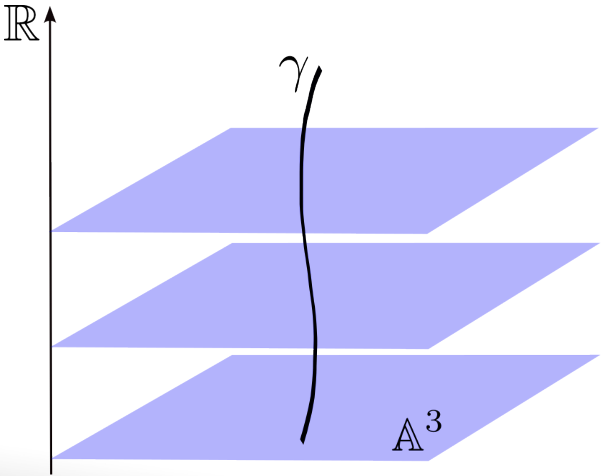
It is important here to make a clear distinction between distances and time intervals —as measured by rulers and clocks— and the set of coordinates. The main goal of the map ϕ_i is to attribute a set of four numbers to each point of the set $\mathbb{A}^4$. In order to define a distance, we need to define the metric on $\mathbb{R}^4$. **Coordinates have no physical meaning**. Moreover, note that the coordinates are not in the set $\mathbb{A}^4$, but in $\mathbb{R}^4$. Since $\mathbb{A}^4$ is isomorphic to $\mathbb{R}^4$, these observations make no important difference here. However, this will be fundamental in GR.

How about dynamics?

The **motion** in $\mathbb{R}^3$ is the image of the differentiable map

$$\vec{x} : \mathbb{I} \mapsto \mathbb{R}^3,$$

which is a **curve** (trajectory) in $\mathbb{R}^3$. The motion $\vec{x}$ defines a curve in $\mathbb{R}^3 \times \mathbb{R}$ called **world-line** γ. $\mathbb{R}^3$ is the **configuration space**. Velocity and acceleration are defined as the first and second time derivatives of the motion.



Back to the postulates

Principle of determinism: $\vec{x}_0 \in \mathbb{R}^3$ and $\dot{\vec{x}}_0 \in \mathbb{R}^3$, at time t_0, uniquely determine the motion at all times. In particular, they determine the **acceleration**

$$\ddot{\vec{x}} = \vec{F}\left(\vec{x}, \dot{\vec{x}}, t\right) \qquad \vec{F}\left(\vec{x}, \dot{\vec{x}}, t\right) : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \mapsto \mathbb{R}^3$$

Galilean invariance implies that $\vec{F}$ must be independent of time and also must depend only on the relative coordinates and velocities. This is a consequence of the homogeneity of space and time.

$$\ddot{\vec{x}}_i = \vec{F}_i\left(\left\{\vec{x}_j - \vec{x}_k, \dot{\vec{x}}_j - \dot{\vec{x}}_k\right\}\right) \quad \text{for} \quad i, j, k = 1, 2, 3.$$

We interpret $\vec{F}$, the force, as the definition of the system under consideration.

Acceleration with respect to what?

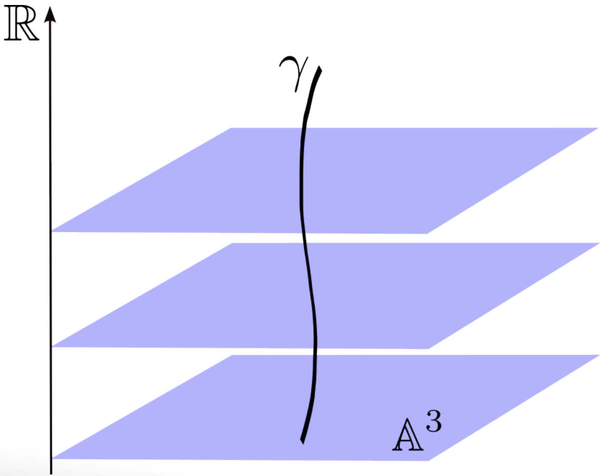
The role of geometry

Such Galilean invariance implies that it is impossible to label the events with a preferred spatial position. That is why we need an affine space! Positions and velocities can only be defined relative to something else.

In other words, given two events happening at different times, it has no meaning to say that they happened at the same position unless we specify a reference object. The only thing that is absolute is the **acceleration that is defined with respect to the static geometry of space and time**. This is the physical meaning of the Galilean transformation.

The causal structure of Newtonian spacetime

Let $p \in \mathbb{A}^4$ be the crossing point of the world-line and some hypersurface $\mathbb{A}^3$. This set defines the present of p. The past (future) of p are all the points in $\mathbb{A}^4$ below (above) p, with respect to the direction defined by time. Only the past can influence p. This is universal! Time and space are absolute!



Special Relativity

Simultaneity

- Newtonian spacetime $\rightarrow$ Absolute meaning of simultaneity.
- Special relativity $\rightarrow$ Simultaneity hypersurface depends on the state of motion of the observer. Simultaneity is not absolute!
- Physical quantities must be independent of coordinates:
    - Newton: space intervals and time intervals are invariant!
    - Special Relativity: Only the spacetime interval is invariant!

The postulates of Special Relativity

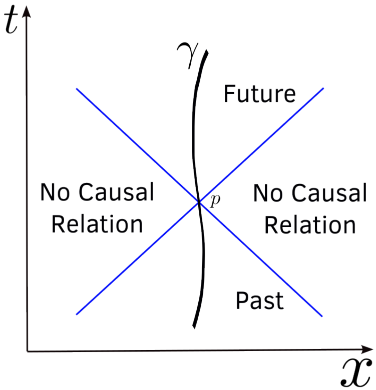
- **Principle of relativity**: All the laws of physics are the same in all inertial frames.
- **Universality of the speed of light**: The speed of light in vacuum is the same for all observers, regardless of their state of motion.

> The first principle is quite natural and it was already presented in physics since Galileo. Einstein realized the second principle due to the electromagnetic field equations.

> In Newtonian physics, if a particle is in a certain point in space at a given instance of time, it can influence any other point in space in the subsequent instant. In SR, an observer at a given event cannot be everywhere in a subsequent one. The notions of past, present, and future must change.

The causal structure of SR

At each event we define a **light-cone**, which determines the locus of paths that point particles can follow. Space-time is split into three regions: The future light-cone; the past light-cone; and the causally disconnected set. The *present* of p is not defined.



The causal structure of SR

Such structure is specified by the spacetime metric, which can be written as

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2,$$

Lorentzian metric

Three kinds of intervals between two points p and q:

- **space-like**: $\mathrm{d}s^2 > 0$. p and q hold no causal relation.
- **null-like**: $\mathrm{d}s^2 = 0$. Not allowed for massive particles.
- **time-like**: $\mathrm{d}s^2 < 0$. Allowed paths for all massive particles. **proper time** is defined as $\mathrm{d}\tau^2 = -\mathrm{d}s^2$.

While past and future are well defined, the simultaneity hypersurface depends on the observer!

How to compute physical intervals?

Let us consider a curve parameterized by λ, with coordinates $x^\mu(\lambda)$. The space-like and time-like lengths of the path are

$$l = \int \sqrt{\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\mathrm{d}\lambda \quad \text{and} \quad \tau = \int \sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\mathrm{d}\lambda,$$

The meaning of Lorentz transformations

The symmetry group of SR is the **Poincaré group**, which is the set of translations (in space and time) and the Lorentz transformations, which say that identical clocks moving with respect to one another measure distinct times. It is meaningless to say that two events happening in different locations occurred at the same time unless we specify with respect to what time is determined.

The general theory

What is the theory about?

Newtonian and SR spacetimes share the same basic structure. Newton is the low-velocity approximation of SR. Both exist independently of any other physical entity, but neither possesses any inherent dynamics.

General Relativity

Recognized that spacetime is a physical entity with a dynamic nature. It is a physical field that describes both the geometry of spacetime and gravity. The static metric η_{ab} is replaced by g_{ab}, that obeys Einstein's field equations and interacts with matter.

Why a field theory?

The Coulomb force between two stationary charges q_1 and q_2 is

$$\vec{F} = \frac{q_1 q_2}{r^3}\vec{r}$$

Action at a distance
Electric interaction acts at a distance, as a force. If we move q_2 suddenly, the charge q_1 would feel this change instantaneously. This is incompatible with special relativity!

What is actually happening here?

Fields provide local interactions

Coulomb's law is only valid in the **static limit**. Electromagnetic interaction is local, propagating via the field.

Maxwell's theory

It is a field theory with three components:

- The Maxwell potential $\mathrm{A}_a(\mathrm{x})$ (the field)
- Lorentz force: governs the motion of particles interacting with fields

$$\ddot{\mathrm{x}}^a = \frac{e}{m}\mathrm{F}^a{}_b\dot{\mathrm{x}}^b \qquad \mathrm{F}_{ab} = \partial_a \mathrm{A}_b - \partial_b \mathrm{A}_a$$

- Maxwell's field equations

$$\nabla_a \mathrm{F}^{ab} = 4\pi \mathrm{J}^b \qquad F_{[ab,c]} = 0$$

How about gravity?

Newton describes the gravitational force

$$\vec{F} = \frac{m_1 m_2}{r^3}\vec{r}$$

This also reveals the same issue, indicating that there must be a field theory that accounts for the degrees of freedom of the physical system carrying the gravitational interaction. This field should reduce to Newton's law in the static, non-relativistic limit. This field is the gravitational field.

The field theory of gravity

General relativity

Is the field theory of the gravitational field.

- The gravitational field g_{ab}
- The geodesic equation

$$\ddot{\mathrm{x}}^a = -\Gamma^a_{bc}\dot{\mathrm{x}}^b\dot{\mathrm{x}}^c$$

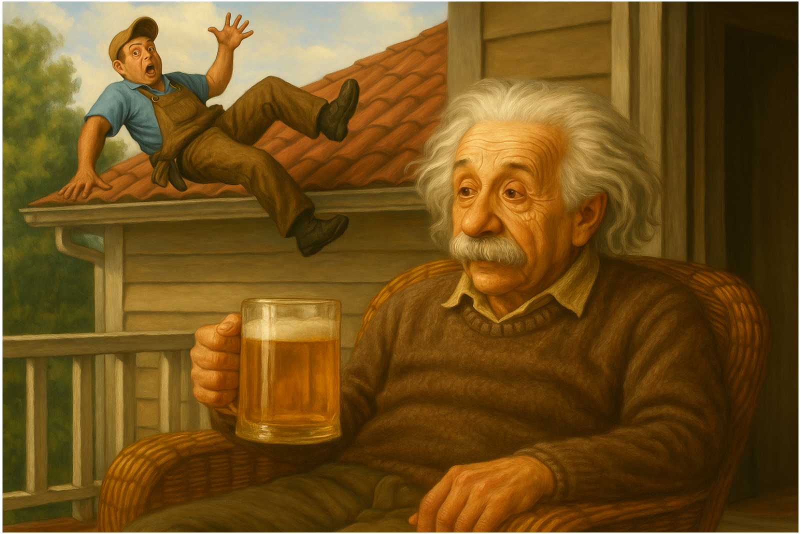
- Einstein's field equation

$$\mathrm{R}_{ab} - \frac{1}{2}R\mathrm{g}_{ab} + \lambda\mathrm{g}_{ab} = 8\pi G\mathrm{T}_{ab}$$

OK, but WTH is the spacetime?

In SR, inertial forces result from acceleration with respect to the fixed geometry of spacetime. Essentially, spacetime defines what is accelerating and what is not.

The equivalence principle



The role of gravity

Redefinition of acceleration
Inside a free-falling laboratory, the laws of physics appear exactly as they would in an inertial reference frame. This is a consequence of the fact that everything in the laboratory experiences gravity in exactly the same way.

Spacetime equals gravity
The role of Newtonian spacetime is to define inertial reference systems, and this is true in Special Relativity as well. However, gravity also plays a role in determining inertial systems. Therefore, spacetime and gravity must be the same thing.

Spacetime is curved

Spacetime must not be Euclidean

Einstein concluded that both Newtonian and special relativistic spacetimes are simply specific configurations of the gravitational field. In more general cases, the geometry of spacetime is curved.

Inertial reference frames

The spatial relations as determined by rigid rods that remain at rest in the system are Euclidean, and there is a universal time in terms of which massive particles remain at rest or in uniform motion on a straight line. Gravity breaks down the Euclidean character of space!

That is all, Folks!

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