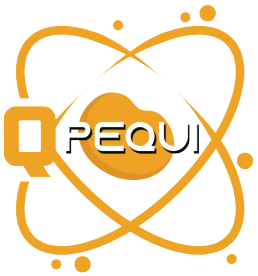

Institute of Physics — Federal University of Goiás

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Relativistic Fluctuation Relations

Lecture Notes

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Notation and Convention

Before we begin, let us clarify some notations and conventions. First, we use the Einstein summation convention, where repeated indices are implicitly summed over:

$$\sum_a \mathrm{x}^a \mathrm{y}_a \equiv \mathrm{x}^a \mathrm{y}_a.$$

Moreover, we employ the abstract index notation, in which Latin indices $\{a, b, c, ...\}$ in tensors do not represent their basis components, but only their slots. In this notation, a tensor T of rank (k, l) will be denoted by

$$\mathrm{T} \equiv \mathrm{T}^{a_1,...,a_k}{}_{b_1,...,b_l}.$$

Therefore, the indices denote how many contravariant (k) and how many covariant (l) slots define the tensor T. This is very important since we need to distinguish between equations that hold in a specific coordinate system, which are written in terms of the components of the tensors in that system, and equations that are independent of the choice of basis, which are the ones written in terms of the abstract index notation.

To write down the equations for the components of the tensor for which a specific basis is required, we employ a Greek set of indices. In this way, $T^{\alpha\beta\gamma}{}_{\delta\epsilon}$ are the components of the tensor $\mathrm{T}^{abc}{}_{de}$ in some specific coordinate system. In this sense, Latin indices denote the number and type of variables the tensor acts on, and not its coordinate components.

It is also convenient to introduce a notation for the totally symmetric

$$\mathrm{T}_{(ab)} = \frac{1}{2}\left(\mathrm{T}_{ab} + \mathrm{T}_{ba}\right)$$

and the totally anti-symmetric

$$\mathrm{T}_{[ab]} = \frac{1}{2}\left(\mathrm{T}_{ab} - \mathrm{T}_{ba}\right)$$

parts of a tensor.

Note that we also employed the notation T to represent a tensor, while T is used

for its components. This notation will also be employed for vectors. In this way, x or x^a represents the coordinate vector, while x^μ stands for its coordinates in a specific reference frame. Three-vectors in space will be denoted by a letter with an arrow, like $\vec{x}$.

We also employ metric signature $(+,-,-,-,)$ and natural units in which G (Newton's gravitational constant), c (the speed of light), k (Coulomb's constant), k_B (Boltzmann constant) and $\hbar$ (Planck's constant) are set to one.

Sets will be denoted by a letter containing a double line as in $\mathbb{V}$, like the sets of vectors in a vector space.

Contents

Chapter 1

Introduction

The profound interplay between thermodynamics, statistical mechanics, and gravity has captivated physicists for more than a century. As early as the 19th century, thermodynamics established itself as a powerful and universal theory, independent of microscopic details [5]. The advent of statistical mechanics brought clarity to thermodynamic laws by grounding them in the behavior of microscopic constituents. In the 20th century, the marriage between thermodynamics and information theory ——most notably in the work of Shannon [45], Jaynes [25], and Landauer [28]—— further enriched our understanding of entropy, irreversibility, and the arrow of time.

Parallel to these developments, the general theory of relativity (GR), formulated by Einstein in 1915, revolutionized our understanding of space and time. GR describes gravity as a manifestation of spacetime curvature, replacing the Newtonian concept of a force by a field [48]. However, integrating thermodynamic concepts into a relativistic setting remains a subtle and intricate endeavor. Unlike classical thermodynamics, which relies on notions of equilibrium and simultaneity, GR challenges these foundations by rendering time and space observer-dependent.

The challenge of formulating thermodynamics in curved spacetimes emerged prominently in the 1970s, with the discovery of the laws of black hole mechanics [1] and Hawking's prediction of black hole radiation [20]. These insights unveiled a deep and unexpected connection between gravity, quantum theory, and thermodynamics, including the thermodynamic interpretation of the Einstein field equations themselves [23].

More recently, the field of nonequilibrium statistical mechanics has produced remarkable results in the form of fluctuation relations. The Jarzynski equality [24] and Crooks fluctuation theorem [11] reveal that equilibrium free energy differences can be extracted from irreversible processes, highlighting a fundamental symmetry between forward and time-reversed trajectories. These results have since been extended

to quantum systems [19], where they underpin a growing understanding of thermodynamic irreversibility in the quantum domain.

These short lectures aim to discuss these topics. In the first part, we provide a comprehensive introduction to classical and quantum fluctuation relations, including their physical motivation, mathematical derivations, and experimental implications. The second part introduces the necessary geometric tools from differential geometry and general relativity. Finally, the third part explores a relativistic formulation of quantum fluctuation relations, as developed by Basso, Maziero, and Céleri [2, 3].

This synthesis of fields not only deepens our understanding of time's arrow in relativistic contexts, but also has implications for quantum gravity, cosmology, and quantum information in curved spacetimes. We hope that these notes serve as an accessible entry point for researchers interested in the thermodynamics of the universe.

1.1 Foundations of Thermodynamics

Thermodynamics is the study of the universal principles governing systems composed of a large number of degrees of freedom. It provides a macroscopic description of such systems using a small number of variables and is grounded in experimental laws that have been abstracted into an axiomatic framework. While thermodynamics originated from practical considerations in engineering and heat engines, its formal development has uncovered a profound connection with statistical mechanics, geometry, and relativity. This section presents a self-contained, formal, and deeply motivated exposition of the foundations of thermodynamics.

A thermodynamic system in equilibrium is fully characterized by a finite set of macroscopic variables. These variables fall into two fundamental categories: extensive variables, such as internal energy U, entropy S, volume V, and particle number N, which scale with the size of the system; and intensive variables, such as temperature T, pressure P, and chemical potential μ, which are independent of scale. The equilibrium state of a system is represented as a point in a state space whose coordinates are these macroscopic quantities. The relationships among these variables are described by equations of state and thermodynamic potentials, derived from a small number of empirically justified postulates.

The first postulate asserts the existence of equilibrium states that are completely specified by a finite number of extensive parameters. This foundational idea allows us to associate each equilibrium state with a unique set of thermodynamic variables. The second postulate establishes the existence of entropy as a continuous, differentiable function that is maximized for equilibrium states. Moreover, entropy is required to

be a first-order homogeneous function of its extensive variables and concave, thereby ensuring consistency under rescaling and stability of equilibrium. Another property imposed on entropy is that it is additive over non-interacting subsystems, reflecting the empirical observation that entropy measures a form of compositional extensive information. Finally, the last postulate expresses the second law of thermodynamics, which states that for any spontaneous process in an isolated system, the total entropy does not decrease. In this sense, entropy is defined up to a multiplicative and an additive constant. This last one can be put equal to zero by demanding that S goes to zero when $T \to 0$.

From these postulates, one derives the fundamental thermodynamic relation

$$\mathrm{d}U = T\mathrm{d}S - P\mathrm{d}V + \mu\mathrm{d}N,$$

where T, P, and μ are defined as the partial derivatives of the internal energy with respect to its natural extensive variables. The internal energy $U(S, V, N)$ is a fundamental thermodynamic potential. However, in many contexts, it is more convenient to work with potentials that have different natural variables. These are introduced via Legendre transformations.

The Legendre transformation provides a systematic method to switch from one thermodynamic potential to another by replacing a natural variable with its conjugate. For example, the Helmholtz free energy $F(T, V, N) = U - TS$ replaces entropy S with temperature T as a natural variable. Its differential form is

$$\mathrm{d}F = -S\mathrm{d}T - P\mathrm{d}V + \mu\mathrm{d}N,$$

and it is especially useful in systems at fixed temperature and volume. Similarly, the enthalpy $H(S, P, N) = U + PV$ and the Gibbs free energy $G(T, P, N) = U + PV - TS$ serve in situations where pressure and/or temperature are controlled. These potentials generate Maxwell relations through the equality of mixed partial derivatives, providing thermodynamic identities and constraints that hold near equilibrium.

1.1.1 Statistical description

Statistical mechanics offers a microscopic foundation for the phenomenological laws of thermodynamics just described. In the microcanonical ensemble, the entropy is defined by Boltzmann's relation

$$S = \ln \Omega,$$

where Ω is the number of accessible microstates compatible with the macroscopic constraints. In the canonical ensemble, one considers a system in contact with a heat reservoir at temperature β^{-1}, and the probability of occupying a microstate of energy E_i is given by

$$p_i = \frac{e^{-\beta E_i}}{Z} \quad Z = \sum_i e^{-\beta E_i}.$$

The average energy and entropy in this ensemble are

$$\langle E \rangle = -\frac{\partial \ln Z}{\partial \beta}, \quad S = -\sum_i p_i \ln p_i,$$

and the Helmholtz free energy is identified as

$$F = -\beta^{-1} \ln Z,$$

which connects thermodynamics to the partition function Z. Other ensembles, such as the grand canonical or isothermal-isobaric ensembles, similarly relate other thermodynamic potentials to generating functions in statistical mechanics.

1.1.2 A geometric view

Beyond its analytical structure, thermodynamics exhibits a rich geometric framework. The state space of a thermodynamic system can be understood as a differentiable manifold, with thermodynamic potentials viewed as generating functions. The differential form of the first law defines a Pfaffian form that introduces a contact structure on the phase space. A contact manifold is a $(2n+1)$-dimensional manifold equipped with a 1-form θ such that $\theta \wedge (d\theta)^n \neq 0$. In thermodynamics, this 1-form reads

$$\theta = \mathrm{d}U - T\mathrm{d}S + P\mathrm{d}V - \mu \mathrm{d}N.$$

The submanifolds on which this form vanishes correspond to equilibrium states satisfying the first law. Legendre submanifolds of the contact manifold correspond to surfaces of constant potential and reflect constraints imposed on the system.

Riemannian geometry also plays a crucial role in the geometric interpretation of thermodynamics, particularly in the theory of fluctuations. Ruppeiner geometry constructs a thermodynamic metric from the negative Hessian of the entropy with respect to extensive variables [42]

$$g^{(R)}_{ij} = -\frac{\partial^2 S}{\partial X^i \partial X^j},$$

where X^i are extensive quantities. The scalar curvature associated with this metric has been interpreted as a measure of underlying microscopic interactions: a flat geometry corresponds to non-interacting particles (as in the ideal gas), while non-zero curvature signals correlations and possibly critical behavior. Weinhold geometry, which uses the Hessian of internal energy,

$$g_{ij}^{(W)} = \frac{\partial^2 U}{\partial X^i \partial X^j},$$

is conformally related to Ruppeiner's metric by the temperature. These geometric insights provide a unified language to describe both equilibrium and near-equilibrium thermodynamics, and they are especially useful in analyzing phase transitions.

1.1.3 Relativistic considerations

Thermodynamics also extends to relativistic settings. In special relativity, physical laws must be consistent with Lorentz invariance. The central object in relativistic thermodynamics is the energy-momentum tensor $T^{\mu\nu}$, which encodes energy density, momentum flux, and stress. Its conservation is expressed by the covariant continuity equation

$$\nabla_\mu T^{\mu\nu} = 0.$$

The entropy current s^μ satisfies

$$\nabla_\mu s^\mu \geq 0,$$

to reflect the second law. For a perfect relativistic fluid, the energy-momentum tensor is

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu + p g^{\mu\nu},$$

where ε is the energy density, p is pressure, and u^μ is the four-velocity. The entropy current is typically modeled as $s^\mu = s u^\mu$, ensuring local entropy production is non-negative.

In general relativity, spacetime curvature affects the behavior of thermodynamic systems. A classic result by Tolman states that, in a static gravitational field, thermal equilibrium requires that the local temperature varies with the gravitational potential according to

$$T\sqrt{-g_{00}} = \text{const.} \tag{1.1}$$

This expression captures the gravitational redshift of temperature and follows from extremizing the total entropy of a system under energy conservation in curved spacetime.

Perhaps the most striking connection between thermodynamics and gravity arises

in black hole physics. Bekenstein and Hawking established that the area of a black hole's event horizon behaves as entropy, leading to the Bekenstein-Hawking relation

$$S_{\text{BH}} = \frac{A}{4\ell_P^2}, \quad T_{\text{H}} = \frac{\kappa}{2\pi}, \tag{1.2}$$

where A is the area of the event horizon, ℓ_P is the Planck length, and κ is the surface gravity. These results are encapsulated in the laws of black hole mechanics, which parallel the classical laws of thermodynamics.

A profound insight by Jacobson in 1995 demonstrated that Einstein's field equations can be derived from the thermodynamic relation $\delta Q = TdS$ applied to local Rindler horizons. This suggests that spacetime dynamics may emerge from thermodynamic principles, thereby unifying gravity and thermodynamics at a fundamental level.

Thermodynamics, therefore, is not merely a phenomenological theory but a deep structure underpinning many branches of physics. Its formulation through differential geometry, its compatibility with special and general relativity, and its statistical origin provide a compelling and unifying view of physical law. The study of nonequilibrium thermodynamics, quantum extensions, and thermodynamic fluctuation theorems continues to build upon these foundational ideas.

1.2 Foundations of relativity

The first question we need to address is why we require a field theory for gravity. The answer lies in special relativity. Let's explore why.

Consider the Coulomb force between two stationary charges q_1 and q_2:

$$\vec{F} = \frac{q_1 q_2}{r^3}\vec{r},$$

where $\vec{r}$ is the vector pointing along the line connecting the two charges. This implies that the electric interaction acts at a distance, as a force. If we move q_2 suddenly, the charge q_1 would feel this change instantaneously. This is incompatible with special relativity, which asserts that interactions must propagate at a finite speed, no greater than the speed of light.

The resolution to this issue is recognizing that Coulomb's law is only valid in the static limit, where the charges are not in motion relative to each other. In more general scenarios, we must use Maxwell's theory, which is a field theory, providing field equations governing the electromagnetic interaction. In this framework, there is no action

at a distance (there is no force), and the electromagnetic interaction becomes a local interaction that propagates at a finite speed (the speed of light) via the electromagnetic field.

To properly describe the electromagnetic interaction, we require three components: i) The Maxwell potential $\mathrm{A}_a(\mathrm{x})$ (the field); ii) The Lorentz force, which governs the motion of particles with mass m and charge e interacting with the field:

$$\ddot{\mathrm{x}}^a = \frac{e}{m}\mathrm{F}^a{}_b\dot{\mathrm{x}}^b,$$

with $\mathrm{F}_{ab} = \partial_a\mathrm{A}_b - \partial_b\mathrm{A}_a$ being Maxwell's tensor; iii) Maxwell's equation

$$\nabla_a\mathrm{F}^{ab} = 4\pi\mathrm{J}^b,$$

with ∇_a representing the covariant derivative, and J_a being the four-current.

A similar situation arises with gravity. Newton describes the gravitational force between two masses m_1 and m_2 as:

$$\vec{F} = \frac{m_1 m_2}{r^3}\vec{r},$$

where $\vec{r}$ is the vector pointing along the line connecting the two masses. This also reveals the same issue, indicating that there must be a field theory that accounts for the degrees of freedom of the physical system carrying the gravitational interaction. This field should reduce to Newton's law in the static, non-relativistic limit. This field is the gravitational field.

Drawing a parallel with electrodynamics, General Relativity is also characterized by three components: i) The gravitational field g_{ab}; ii) The geodesic equation:

$$\ddot{\mathrm{x}}^a = -\Gamma^a{}_{bc}\dot{\mathrm{x}}^b\dot{\mathrm{x}}^c,$$

which governs the motion of particles under the influence of the gravitational field; iii) Einstein's field equation:

$$\mathrm{R}_{ab} - \frac{1}{2}R\mathrm{g}_{ab} + \lambda\mathrm{g}_{ab} = 8\pi G\mathrm{T}_{ab}.$$

The specific definitions of these quantities will be provided in later chapters. In this sense, General Relativity is the field theory of the gravitational field.

Now we need to understand one of the most profound facts of GR, that this field

theory also describes the geometry of the spacetime.

Before Isaac Newton introduced the concepts of absolute space and absolute time, philosophers typically thought of them as being intrinsically linked to the things of the world. Space was understood as a relational concept between various objects, such as rocks, people, animals, trees, and so on. In this sense, space did not exist independently but was defined by the relationships between things. Similarly, time was understood as fundamentally related to the changes we observe in the world, such as the cycle of day and night or the phases of the moon. Just as it would be meaningless to talk about space without referring to objects, it was also impossible to talk about time without changes taking place. Space and time, therefore, were not considered as independent entities but were intrinsically tied to the material world.

Newton transformed this view by positing that there is a three-dimensional Euclidean structure underlying the relational connections between objects. In this space, we can assign coordinates to label the points of the space. For time, he considered the real line $\mathbb{R}$ as its structure. In this way, both space and time gained a metric meaning that did not depend on any objects. They were now understood to have physical existence independent of the things in the world. This represented a profound shift in conceptual thinking. The metric structure could now be used to define the measurements of roads and clocks. This also changed the interpretation of Euclidean geometry, which was previously seen as concerned with the properties of objects but now was understood as the geometry of space itself.

When Einstein discovered Special Relativity, he recognized that it would be more appropriate to describe space and time as a four-dimensional entity, which he called spacetime. However, the structure of space and time did not change significantly compared to the Newtonian concepts. While there are differences, they do not concern the basic structure. Space and time are still considered physical, but they lack dynamics in Special Relativity. Space remains Euclidean, and time retains the metric structure of the real line.

General relativity revolutionized this view by proposing that space and time are not just physical entities but rather a dynamic physical field. This field is known as the gravitational field. In this framework, the readings of rulers and clocks are understood as manifestations of this dynamic field, rather than as the effect of gravity on these devices. Consequently, General Relativity replaces the static Minkowski metric of Special Relativity with a field that depends on the spacetime point. This means that the geometry of spacetime is no longer Minkowskian, and the geometry of space is not Euclidean.

In the context of Newtonian space and time and special relativistic spacetime, acceleration is considered absolute —acceleration with respect to the fixed geometry of

spacetime! In this context, acceleration is a consequence of inertial forces. Let us think about Newton's bucket experiment[1], which demonstrates absolute space by showing that rotational motion is not merely relative. When the bucket first starts moving, the water remains still and its surface is flat, even though the bucket is rotating relative to the water. As the water speeds up, it takes on a concave shape, despite being at rest relative to the bucket at that point. The shape of the water's surface depends not on its motion relative to the bucket but on its rotation relative to absolute space. Crucially, the concavity persists even when the water and bucket rotate together, proving that rotation is not relative to the bucket but to an external frame: the absolute space. In other words, there must be a privileged frame of reference, the absolute space, against which rotation is measured. If only relative motion mattered, then the shape of the water's surface should depend only on its motion relative to the bucket, which it does not.

From the above discussion we realize the fundamental role of the geometric structure of the spacetime in order to define acceleration and, equivalently, to define what is an inertial frame. Recognizing that acceleration does not depend on the mass of the bodies involved, Einstein realized that during free fall, all objects should behave as though they are in an inertial reference frame. This insight led Einstein to conclude that the role of gravity is essentially to redefine inertial reference frames, the same role played by the spacetime in Special Relativity. Therefore, these two entities must be the same. *Causes assigned to effects of the same type must be, as much as possible, the same!* From this, Einstein inferred that Newtonian space and time, as well as the spacetime of special relativity, are specific configurations of the gravitational field. In general, spacetime must be curved. This is the essence of General Relativity, and it is going to be the main focus of these lectures.

For completeness, we review the main ideas behind the Newtonian concepts of space and time and the special relativity spacetime.

1.2.1 The Newtonian space and time

Let us start by postulating the basic laws of Newtonian mechanics, known as Newton's laws.

- **The first law**: There exists **inertial reference frames** with respect to which every

[1]The experiment goes as follows: A bucket filled with water is suspended by a rope. The bucket is twisted and then released, causing it to spin. Initially, the water remains still while the bucket rotates. Over time, the water starts spinning along with the bucket due to friction, eventually matching its rotational speed. As this happens, the surface of the water forms a concave shape (a parabolic curve) due to centrifugal force. If the bucket is suddenly stopped, the water continues spinning, maintaining the concave shape temporarily.

isolated particle remains at rest or in uniform motion in a straight line.

- **The second law**: In any inertial frame, the motion of a particle is governed by the following set of differential equations

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \vec{F}, \tag{1.3}$$

with $\vec{p}$ being the linear momentum of the particle while $\vec{F}$ stands for the total force acting on the particle.

- **The third law**: To every action there corresponds an equal and opposite reaction

$$\vec{F}_{i,j} = -\vec{F}_{j,i}, \tag{1.4}$$

where $\vec{F}_{i,j}$ is the force on particle j due to particle i. The forces are directed along the line joining both particles.

It is important to note that Newton's third law is not valid in general. It is violated, for instance, for moving charged particles due to the fact that the speed of propagation of the electromagnetic interaction is finite.

Such postulates are based on our experimental observations of the natural world. For instance, we know that space is three-dimensional and Euclidean, while time is one-dimensional. Moreover, we understand that there are special reference frames, called inertial frames, on which Newton's second law takes the same form, and that all other frames that are at rest or in rectilinear uniform motion with respect to one of these frames are also inertial frames. This is usually known as Galileo's principle of relativity. Finally, if we specify the position and velocity of a particle at a given instant of time, we should be able to tell the motion of the particle at any other time, thus justifying the second-order derivative appearing in Newton's equation of motion. This last fact is known as the principle of determinacy.

The Newtonian universe is a four-dimensional **affine space**[2] $\mathbb{A}^4$. $\mathbb{R}^4$ acts as the group of parallel displacements $a \to a + \vec{v}$ where a, $(a + \vec{v}) \in \mathbb{A}^4$ and $\vec{v} \in \mathbb{R}^4$. From this, we see that the difference of two points of $\mathbb{A}^4$ is a vector in $\mathbb{R}^4$, while the sum is not defined. Each element of $\mathbb{A}$ is called **event**.

Time is defined as a linear map $t : \mathbb{A}^4 \mapsto \mathbb{A}^1$. The kernel of this map —the set of vectors for which $t(a - b) = 0$— defines the simultaneous hypersurfaces, which are 3-dimensional subspaces of $\mathbb{A}^4$. On each of these spatial hypersurfaces, we can define a distance function $d : \mathbb{A}^3 \times \mathbb{A}^3 \mapsto \mathbb{R}$ such that, for all a, b, $c \in \mathbb{A}^3$, it is positive

[2]See Appendix A for further details

semidefinite $d(a,b) \geq 0$, with the equality holding if and only if $a = b$, symmetric, $d(a,b) = d(b,a)$, and fulfills the triangle inequality, $d(a,c) \geq d(a,b) + d(b,c)$. Observe here the strong connection between the notion of simultaneity —the kernel of the map t— and spatial distances.

A positive bilinear symmetric form $\langle x, y\rangle$, called a scalar product on $\mathbb{R}^4$, defines the Euclidean structure and allows us to define the distance function as

$$d(a,b) \equiv ||a-b|| = \sqrt{\langle a-b, a-b\rangle} \tag{1.5}$$

between points a and b of the corresponding simultaneity hyperspace. Since the difference of two events in $\mathbb{A}^4$ is a vector in $\mathbb{R}^4$, it is clear that distances on the spatial hypersurfaces are defined by the kernel of t. We call a **Galilean spacetime** (or Euclidean) the set $(\mathbb{A}^4, t, d)$.

The set of affine transformations that preserve time intervals and distances between simultaneous events forms the Galilean group[3], which is the symmetry group of Newtonian mechanics.

Since we defined the mathematical structure of space and time, we are now in a position to properly assign meaning to the laws of Newton that were stated earlier. We know that the laws of physics are expressed in terms of differential equations, which means that, in order to do physics, we should be able to employ calculus. We do this by introducing reference frames, which are ways to unambiguously label the points of $\mathbb{A}$. Reference frames are maps from the set $\mathbb{A}^4$ to the set $\mathbb{R}^4$, where we understand how calculus works. It is important to observe here that, in the case of Newtonian space and time, a single map is able to cover the entire set $\mathbb{A}^4$. However, this is not possible when gravity comes into play, but the definition of a reference frame is exactly the same, and we just need more than one of them to cover the entire spacetime. Moreover, we physicists usually deal with well-behaved functions, and we then demand that such maps are of class C^∞. Strictly speaking, it does not need to be C^∞, but it must be sufficiently smooth.

In order to introduce a reference frame, note that all Galilean spaces are isomorphic[4] to each other and are also isomorphic to $\mathbb{R}^3 \times \mathbb{R}$. Therefore, we can use such isomorphism to define a coordinate system as the map

$$\phi : \mathbb{A}^4 \mapsto \mathbb{R}^3 \times \mathbb{R}, \tag{1.6}$$

which is called a Galilean coordinate system. In this way, we just labeled each one of

[3]See Appendix **??** for further details.
[4]See Appendix A for further details.

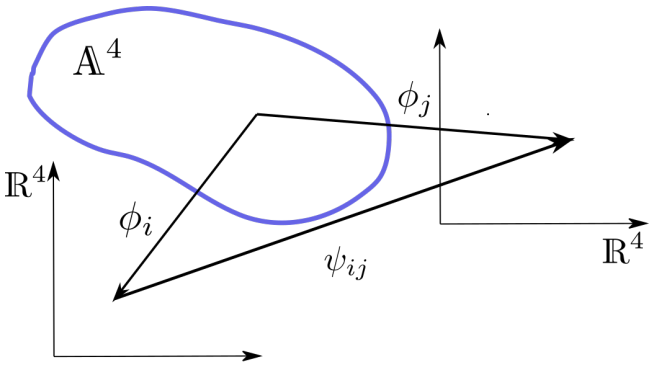

Figure 1.1: **Inertial reference frames**. ϕ_i and ϕ_j are two Galilean coordinate systems, defined by the isomorphisms $\phi_i : \mathbb{A} \mapsto \mathbb{R}^3 \times \mathbb{R}$ and $\phi_j : \mathbb{A} \mapsto \mathbb{R}^3 \times \mathbb{R}$, respectively. A Galilean transformation, or coordinate transformation, is defined by the map $\psi_{ij} \equiv \phi_j \circ \phi_i^{-1} : \mathbb{R}^3 \times \mathbb{R} \mapsto \mathbb{R}^3 \times \mathbb{R}$. Therefore, ψ_{ij} takes us from the set of coordinates defined by ϕ_i to the set of coordinates defined by ϕ_j.

the events in $\mathbb{A}^4$ with four real numbers, which are called the coordinates of the events. Now, a Galilean transformation takes us from one inertial coordinate system to some other. Mathematically, if a coordinate system ϕ_j moves with zero acceleration with respect to the coordinate system ϕ_i, the map $\psi_{i,j} \equiv \phi_i \circ \phi_j^{-1} : \mathbb{R}^3 \times \mathbb{R} \mapsto \mathbb{R}^3 \times \mathbb{R}$ is a Galilean transformation. We also demand that $\phi_i \circ \phi_j^{-1}$ to be C^∞. Both ϕ_i and ϕ_j give $\mathbb{A}^4$ the same Galilean structure. This idea is illustrated in Fig. 1.2.

It is important here to make a clear distinction between distances and time intervals —as measured by rulers and clocks— and the set of coordinates. The main goal of the map ϕ_i is to attribute a set of four numbers to each point of the set $\mathbb{A}^4$. In order to define a distance, we need to define the metric on $\mathbb{R}^n$. In principle, coordinates have no physical meaning. Moreover, note that the coordinates are not in the set $\mathbb{A}^4$, but in $\mathbb{R}^4$. Since $\mathbb{A}^4$ is isomorphic to $\mathbb{R}^4$, these observations make no important difference here. However, this will be fundamental in GR.

Dynamics

We start by defining the **motion** in $\mathbb{R}^3$, which is the image of the differentiable map $\vec{x} : \mathbb{I} \mapsto \mathbb{R}^3$, with $\mathbb{I} \subset \mathbb{R}$ being an open interval of the real line. Therefore, the motion is a **curve** in $\mathbb{R}^3$. It is also called a trajectory. The motion $\vec{x}$ defines a curve in $\mathbb{R}^3 \times \mathbb{R}$ called **world-line** γ. Figure 1.2 illustrates the concept of a motion along with the spacetime on which classical mechanics is built. $\mathbb{R}^3$ is called the **configuration space**. Each one of these subspaces is a Cauchy hypersurface[5]. The velocity and the acceleration vectors are defined as the first and second time derivatives of the motion, respectively.

Now, according to Newton's principle of determinism, the initial position $\vec{x}_0 \in \mathbb{R}^3$ and velocity $\dot{\vec{x}}_0 \in \mathbb{R}^3$, at time t_0, uniquely determine the motion of the system at all times. In particular, they determine the acceleration, which implies that there is a func-

[5]Intuitively, we can interpret a Cauchy hypersurface as the one defining an instant of time. This concept will be made clear after we introduce the Lorentzian manifold.

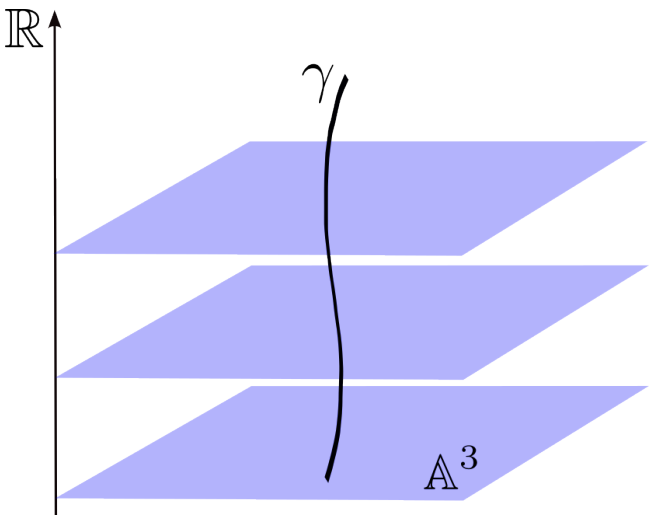

Figure 1.2: **World-line in Newtonian spacetime**. The trajectory $\vec{x} : \mathbb{I} \mapsto \mathbb{R}^3$ defines the motion of the system in the three dimensional space $\mathbb{R}^3$ (which is isomorphic to $\mathbb{A}^3$). The graph of this motion is a curve in $\mathbb{R}^3 \times \mathbb{R}$, called world-line γ, that intercepts each one of the simultaneity hypersurfaces $\mathbb{A}^3$ precisely once. The space is them an equivalent class of simultaneous hypersurfaces defined by the kernel of the time map t. The properties of $\vec{x}$ tell us that each one of these hypersurfaces is a Cauchy hypersurface.

tion $\vec{F}\left(\vec{x}, \dot{\vec{x}}, t\right) : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \mapsto \mathbb{R}^3$ such that

$$\ddot{\vec{x}} = \vec{F}\left(\vec{x}, \dot{\vec{x}}, t\right), \tag{1.7}$$

According to the Galilean invariance principle, this last equation must be invariant under a Galilean transformation, that takes us from one reference frame to another. In particular, since this set of transformations includes time and space translations, we see that $\vec{F}$ must be independent of time and also must depend only on the relative coordinates and velocities of the particles composing the system. Such symmetry is a consequence of the homogeneity of space and time. Mathematically

$$\ddot{\vec{x}}_i = \vec{F}_i\left(\left\{\vec{x}_j - \vec{x}_k, \dot{\vec{x}}_j - \dot{\vec{x}}_k\right\}\right) \quad \text{for} \quad i, j, k = 1, 2, 3. \tag{1.8}$$

We interpret $\vec{F}$, the force, as the definition of the system under consideration.

The fact that Newtonian mechanics is invariant under a Galilean transformation thus implies that it is impossible to label the events with a preferred spatial position. That is why we need an affine space! Positions and velocities can only be defined relative to something else. In other words, given two events happening at different times, it has no meaning to say that they happened at the same position, unless we specify a reference object. The only thing that is absolute is the acceleration that, as mentioned earlier, is defined with respect to the static geometry of the space and time. This is the physical meaning of the Galilean transformation.

The isotropy of space implies that the force must obey $\vec{F}(G\vec{x}, G\dot{\vec{x}}) = G\vec{F}(\vec{x}, \dot{\vec{x}})$ for every orthogonal transformation G (a rotation in space). Since every Galilean transformation can be written as a combination of a translation (in time and in space), a

uniform motion, and a rotation, we have considered all the possible symmetries.

Figure 1.2 also defines the causal structure of the Newtonian world. Let $p \in \mathbb{A}^4$ be the crossing point of the world-line and some hypersurface $\mathbb{A}^3$. This set defines the present of p, i.e. all the points that are simultaneous to p. This means that no observer can be at any other event in this set if it is at p. The past (future) of p are all the points in $\mathbb{A}^4$ below (above) p, with respect to the direction defined by time. Events that lie below this set can influence p, while events above can be influenced by p. Such causal structure is universal in the sense that it is the same for all observers. Time and space are absolute!

Let us see how this structure changes in Special Relativity.

1.2.2 Special Relativity

We saw that the Newtonian concept of spacetime (space and time taken together) is described by a continuous set of events and we can think about each one of these events as a point in space at a given instant of time. Moreover, we can unambiguously label every event of this set with four numbers, called the coordinates of the point. Additionally, given a specific point in space at a specific moment in time, there is an absolute meaning of simultaneity, defined by the set of points associated with the same instant of time. This last property was changed by Einstein in his special theory of relativity. In Newtonian physics, all the simultaneous events to a given one form a three-dimensional one, while in SR, it is much more than this. An observer can still define a three-dimensional hypersurface containing the events that occur at the same time as a given one. However, such hypersurface depends on the state of motion of the observer. Simultaneity is not absolute!

Such a structure leads to the fact that two distinct inertial observers will assign different coordinates to the events of the spacetime. This makes clear that such coordinates have no physical meaning. We must then look for quantities that are observer independent. In other words, we need to find functions of the coordinates that are observer independent. In Newtonian physics, we do have two of these quantities (and functions of them), the time interval between two events and the space interval between two simultaneous events. Given the above discussion regarding the nature of simultaneity, we can anticipate that none of these quantities will be invariant in SR.

Before introducing the corresponding quantities in SR, let us start with the physical principles behind this theory, which can be stated as follows.

- **Principle of relativity**: All the laws of physics are the same in all inertial frames.

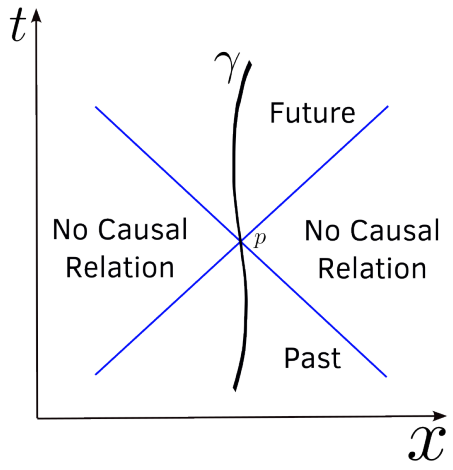

Figure 1.3: **Spacetime in special relativity**. At each event p we define the light-cone which determined the past, future and the set of events that holds not causal relation with that event. This is represented by the blue lines in the figure. The paths fallowed by material particles are constrained to the interior of the cone while light walks only on the surface of the cone, which are called null surfaces. Nothing can have a path whose inclination is bigger than the surface of the cone.

- **Universality of the speed of light**: The speed of light in vacuum is the same for all observers, regardless of their state of motion.

The first principle is quite natural and it was already presented in physics since Galileo. Einstein realized the second principle due to the electromagnetic field equations. The solution to Maxwell's equations in vacuum is a propagating wave whose velocity is constant no matter the observer. Although simple, it presents profound consequences. For instance, in Newtonian physics, if a particle is in a certain point in space at a given instance of time, it can be at any other point in space in the subsequent instant, since there is no limit for its speed. In this way, simultaneity defines a three-dimensional hypersurface which is orthogonal to the time line. In SR, due to the finite speed of light, an observer at a given event cannot be everywhere in a subsequent one. Therefore, the notions of past, present and future must change.

At each event of spacetime, we can define a **light-cone**, which determines the locus of paths that point particles can follow, as illustrated in Fig. 1.3. At each event p, spacetime is split into three regions. The first one is the locus of the events that can be influenced by p. This is the future light-cone. The second one contains the events that can influence p and is called the past light-cone. Both of these regions form a three-dimensional set. All the events lying outside the light cone do not have any causal relation with p. The notion of the present of p is not defined.

Such structure is specified by the spacetime metric, which can be written as

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2, \tag{1.9}$$

in the dual coordinate basis. The minus sign in this metric indicates that the line element $\mathrm{d}s^2$ can be negative. This Lorentzian metric determines three kinds of intervals between two points p and q: i) $\mathrm{d}s^2 > 0$ indicates that p and q hold no causal relation and

the interval is called **space-like**; ii) Paths for which $\mathrm{d}s^2 = 0$ are not allowed for massive particles and they are called **null-like** (or light-like) interval; iii) $\mathrm{d}s^2 < 0$, determining a **time-like** interval, are the allowed paths for all massive particles. The path γ shown in Fig. 1.3 is an example of a time-like curve. Remember that, when considering time-like intervals, the **proper time**[6] is defined as $\mathrm{d}\tau^2 = -\mathrm{d}s^2$.

Note the difference between this causal structure and the one in the Newtonian world. Considering p as the event defining the light-cone shown in Fig. 1.3, events that can be influenced by p lie in the future light-cone, while the past light-cone defines those events that can influence p. It is important to observe that no material particle can travel at the boundaries of this light-cone, since these are null-surfaces, which are regions only allowed for non-massive particles. The set of events that holds no causal relation to p defines a four-dimensional set, instead of a three-dimensional one defined in Newton's spacetime. It is still possible to define a three-dimensional set that constitutes those events forming the *present* of p. However, such a set will depend on the state of motion of the observer defining it. This is a fundamental difference between Newton and Einstein. Simultaneity is relative in SP, while it is absolute in the Newtonian world (see Appendix **??** for further details).

In Newtonian spacetime the space interval ($\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$) is conserved under rotations. This is a consequence of the fact that the notion of a three-dimensional space at a single instant of time is independent of coordinates. The notion of rotation in time is not defined. In Special Relativity, where the notion of simultaneity is observer dependent, this is not true anymore and we need to consider rotations of the entire four-dimensional set, called the **Minkowski spacetime**. This implies that we rotate time and space into each other. Under this sort of transformation, the spacetime interval (1.9) is invariant.

We can consider a curve parameterized by λ, with coordinates $x^\mu(\lambda)$ and, from the line element 1.9 it is possible to compute the length of the path as

$$s = \int \sqrt{\eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} \mathrm{d}\lambda \quad \text{and} \quad \tau = \int \sqrt{-\eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} \mathrm{d}\lambda, \tag{1.10}$$

for space-like and time-like intervals, respectively. The precise notion of derivatives and coordinates will be discussed in the next chapter.

While the Newtonian spacetime interval is invariant under a Galilean transformation, the symmetry group of special relativity is the **Poincaré group**, which is the set of translations (in space and time) and the Lorentz transformations. Appendix **??** presents

[6] The proper time is the time measured by a clock carried by an observer moving along the path between the considered events.

some details on the structure of this group.

Although Lorentz transformations were known before Einstein, the meaning of the transformation of time became clear only within SR. The physical content of this transformation is that identical clocks moving with respect to one another measure distinct times. Therefore, it is meaningless to say that two events happening in different locations occurred at the same time unless we specify with respect to what time is determined.

1.2.3 General Relativity

Both Newtonian and special relativistic spacetimes share the same basic structure. Newtonian spacetime can be considered as the low-velocity approximation of special relativistic spacetime. Both exist independently of any other physical entity, but neither possesses any inherent dynamics.

Einstein recognized that spacetime is a physical entity and, importantly, that it must have a dynamic nature. He argued that it is a physical field, which we now identify as the gravitational field. He replaced the static metric η_{ab} with a general one, g_{ab}, that varies depending on the point in spacetime. The metric g_{ab} represents a physical field whose behavior is governed by field equations (Einstein's equations) and interacts with matter. It defines the geometry of spacetime and is, at the same time, the gravitational field.

To understand why this is the case, remember that inertial forces arise due to the acceleration of a reference frame. As Newton pointed out, this acceleration must be defined with respect to the space itself. In other words, inertial forces result from acceleration with respect to the fixed geometry of spacetime. This holds true in Special Relativity as well. Essentially, Newtonian spacetime defines what is accelerating and what is not.

Einstein made a striking observation about gravity. Inside a free-falling laboratory, the laws of physics appear exactly as they would in an inertial reference frame. This follows from the fact that everything in the laboratory experiences gravity in exactly the same way. This insight is the core of the equivalence principle, a remarkable realization. Gravity's role is to redefine what constitutes an inertial reference frame —specifically, those that are in free fall[7].

As noted, the role of Newtonian spacetime is to define inertial reference systems, and this is true in Special Relativity as well. However, gravity also plays a role in determining inertial systems. Therefore, spacetime and gravity must be the same thing. Einstein concluded that both Newtonian and special relativistic spacetimes are simply

[7]Note that this holds true only locally, a point that will be clarified in later chapters.

specific configurations of the gravitational field. In more general cases, the geometry of spacetime is curved.

Based on the above discussions, we can rewrite the definition of an inertial frame as follows. The spatial relations as determined by rigid rods that remain at rest in the system are Euclidean and there is a universal time in terms of which massive particles remain at rest or in uniform motion on a straight line. The role of gravity is to break down the Euclidean character of space!

Chapter 2

Fluctuation Relations

Equilibrium thermodynamics usually ignores fluctuations. To understand why this is the case, let us consider a box filled with an ideal monoatomic gas of N particles at thermal equilibrium at inverse temperature β. The equipartition theorem tells that the system energy and variance are

$$U = \frac{3N}{2\beta} \qquad \text{and} \qquad \text{Var}(U) = \frac{3N}{2\beta^2}.$$

This implies that the relative fluctuations are

$$\frac{\sqrt{\text{Var}(U)}}{U} \approx \frac{1}{\sqrt{N}}.$$

Therefore, fluctuations vanish at the thermodynamic limit. However, there are cases where fluctuations matter. For instance, when we are dealing with small systems or quantum systems. It is important to observe here that, although the system is small, it still possesses a large number of degrees of freedom, which forbids us from completely describing it. In this situation, we need to employ the rules of statistical mechanics in order to properly describe the system. The so-called stochastic thermodynamics treats quantities like internal energy, work, and heat as stochastic variables, allowing the treatment of finite-time processes and non-equilibrium states.

We have no intention here to provide a complete historical development of the field, but just mention some key developments that lead to the establishment of the quantum fluctuation theorems. The field started in the early years of the last century, with the work of Sutherland [46, 47], Einstein [13, 12, 14] and, later, of Johnson [26] and Nyquist [32], when it was found that the linear response of a system in thermal equi-

librium, as it is driven out of equilibrium by an external force, is determined by the fluctuation properties of the system in the initial equilibrium state.

Sutherland [46, 47] and Einstein [13, 12, 14] found a relation between the mobility of a Brownian particle, which encodes information about its response to an externally applied force, and its diffusion constant, which encodes information about its equilibrium fluctuations. Johnson [26] and Nyquist [32] discovered the corresponding relation between the resistance of a circuit and the spontaneous current fluctuations occurring in the absence of an applied electric potential.

Let us consider a Brownian particle of mass m suspended in a fluid at inverse temperature β. The diffusion coefficient D measures how quickly particles spread over time due to random thermal motion

$$\left\langle x^2(t) \right\rangle = 2Dt,$$

with x being the trajectory of the particle and the average is taken over the fluctuations. The mobility μ characterizes the linear response of a particle to an applied force $\vec{F}$, and is defined as

$$\mu = \frac{v}{F},$$

where v is the average drift velocity and $F \equiv \left|\vec{F}\right|$.

Einstein fluctuation relation states that

$$D = \beta^{-1}\mu \tag{2.1}$$

holds at thermal equilibrium.

This equation emerges from the Fokker-Planck equation governing the probability distribution of the position of a particle under diffusion and drift.

Let $\rho(x,t)$ be the probability density of finding the particle at position x at time t. The probability current $J(x,t)$ includes both drift and diffusive contributions

$$J(x,t) = \mu F \rho(x,t) - D\frac{\partial \rho}{\partial x}.$$

The Fokker-Planck equation is the continuity equation for the probability

$$\frac{\partial \rho}{\partial t} = -\frac{\partial J}{\partial x} = -\frac{\partial}{\partial x}\left(\mu F \rho - D\frac{\partial \rho}{\partial x}\right).$$

At equilibrium, $\partial\rho/\partial t = 0$, so

$$\mu F\rho(x) - D\frac{\partial\rho}{\partial x} = 0.$$

Now, we assume the system is in a potential $U(x) = -Fx$, so the equilibrium distribution is the Boltzmann distribution

$$\rho(x) = \rho_0 e^{-\beta U(x)} = \rho_0 e^{\beta F x},$$

thus implying

$$\frac{\partial\rho}{\partial x} = \beta F\rho(x).$$

Plug into the current balance equation

$$\mu F\rho - D\beta F\rho = 0,$$

which implies Eq. (2.1).

An alternative approach is via the Langevin equation, which describes the motion of a Brownian particle under viscous drag and random thermal noise

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -\gamma v + \eta(t),$$

where γ is the friction coefficient and $\eta(t)$ is a random force satisfying

$$\langle\eta(t)\rangle = 0, \qquad \langle\eta(t)\eta(t')\rangle = 2\gamma\beta^{-1}\delta(t-t').$$

In the overdamped limit ($m \to 0$), the inertial term is negligible, and

$$v(t) = \frac{1}{\gamma}\eta(t).$$

Now, the position evolves as

$$x(t) = x(0) + \int_0^t v(s)\mathrm{d}s,$$

while the mean squared displacement takes the form

$$\begin{aligned}\langle [x(t) - x(0)]^2 \rangle &= \frac{1}{\gamma^2} \int_0^t \int_0^t \langle \eta(s)\eta(s') \rangle \mathrm{d}s\mathrm{d}s' \\ &= \frac{1}{\gamma^2} \int_0^t \int_0^t 2\gamma\beta^{-1}\delta(s-s')\mathrm{d}s\mathrm{d}s' \\ &= \frac{2}{\beta\gamma} \int_0^t \mathrm{d}s = \frac{2}{\beta\gamma} t.\end{aligned}$$

Thus

$$D = \frac{1}{\beta\gamma}, \quad \text{and since} \quad \mu = \frac{1}{\gamma}, \quad \Rightarrow \quad D = \frac{\mu}{\beta}.$$

For a spherical particle of radius R in a fluid with viscosity η, the drag force from Stokes' law is

$$F_{\text{drag}} = -6\pi\eta R v.$$

Therefore, the friction coefficient is determined by

$$\gamma = 6\pi\eta R, \qquad \Rightarrow \qquad \mu = \frac{1}{\gamma} = \frac{1}{6\pi\eta R}.$$

Einstein relation takes the form

$$D = \frac{\mu}{\beta} = \frac{1}{6\beta\pi\eta R}.$$

which is known as the Stokes-Einstein relation.

The Johnson–Nyquist theorem quantifies the mean-square voltage fluctuations across a resistor R in thermal equilibrium at temperature T. These fluctuations, known as Johnson noise, are due to the thermal agitation of electrons in the conductor. This phenomenon illustrates a deep connection between thermal energy and dissipative electrical elements, as predicted by the fluctuation-dissipation theorem.

Consider an ideal resistor of resistance R connected to an ideal voltmeter. Even in the absence of a net current, microscopic charge carriers in the resistor are constantly undergoing thermal motion. These stochastic movements cause random fluctuations in voltage measurable across the terminals.

The thermal voltage fluctuations $V(t)$ are characterized by their power spectral density $S_V(\omega)$, which quantifies the contribution to the mean-square voltage from different frequency components

$$\langle V^2 \rangle = \int_0^\infty S_V(\omega) \frac{\mathrm{d}\omega}{2\pi}.$$

Nyquist's approach to derive $S_V(\omega)$ is based on comparing the thermal energy of circuit modes with the observed fluctuations. Consider a transmission line or resonant circuit composed of a resistor R and a capacitor C (or in some variations, also an inductor L). These circuits support harmonic voltage oscillations which are in thermal equilibrium with the resistor.

Each mode of frequency ω in the circuit has an average energy per degree of freedom given by the equipartition theorem

$$\langle E \rangle = \frac{1}{2}\beta^{-1}.$$

A single mode of frequency ω in an LC resonator can be modelled as a harmonic oscillator with energy split equally between the electric field in the capacitor and the magnetic field in the inductor. For purely resistive elements, the energy is dissipated, and the power delivered by thermal fluctuations must be accounted for via the fluctuation-dissipation relation.

The power dissipated in a resistor due to voltage fluctuations is given by

$$P = \frac{\langle V^2 \rangle}{2R}.$$

But the fluctuations arise from thermal agitation, which means their spectrum is flat (white noise), i.e.,

$$S_V(\omega) = \text{constant}.$$

From the general linear response theory (see bellow), the power spectral density of voltage across an impedance $Z(\omega)$ at equilibrium is given by

$$S_V(\omega) = 4\beta^{-1} \operatorname{Re}[Z(\omega)].$$

For a pure resistor, $Z(\omega) = R$, thus

$$S_V(\omega) = 4\beta^{-1} R,$$

which is the Johnson–Nyquist noise formula.

This shows that the spectral density is frequency-independent (white noise) and proportional to temperature and resistance.

Let us now discuss the general fluctuation dissipation theorem, derived by Kubo [27] and Callen and Welton [6].

2.1 The fluctuation-dissipation theorem

The fluctuation-dissipation theorem (FDT) provides a fundamental connection between the response of a physical system to external perturbations and the internal fluctuations that occur in thermal equilibrium. Originally formulated by Kubo and generalized by Callen and Welton, it plays a foundational role in both classical and quantum statistical mechanics.

Consider a system with equilibrium Hamiltonian H_0 and a time-dependent perturbation coupled to an observable A:

$$H(t) = H_0 - h(t)A,$$

where $h(t)$ is a small external field. The system is initially in thermal equilibrium at temperature β^{-1} (with density matrix $\rho_0 = e^{-\beta H_0}/Z$). Our goal is to compute the linear response of another observable B due to the perturbation.

In the interaction picture, the time evolution of the density operator $\rho(t)$ satisfies

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H_I(t), \rho(t)],$$

where $H_I(t) = -h(t)A(t)$ and $A(t)$ is the Heisenberg-picture operator.

We solve this to first order in $h(t)$

$$\rho(t) = \rho_0 + \delta\rho(t), \quad \text{with } \delta\rho(t) = \frac{i}{\hbar}\int_{-\infty}^{t} \mathrm{d}t' h(t')[\rho_0, A(t')].$$

Then the expectation value of B becomes

$$\begin{aligned}\langle B(t)\rangle &= \mathrm{Tr}[\rho(t)B(t)] = \mathrm{Tr}[\rho_0 B(t)] + \mathrm{Tr}[\delta\rho(t)B(t)] \\ &= \langle B(t)\rangle_0 + \frac{i}{\hbar}\int_{-\infty}^{t} \mathrm{d}t' h(t')\langle[B(t), A(t')]\rangle_0.\end{aligned}$$

Now, let us define the **response function** as

$$\chi_{BA}(t-t') = \frac{i}{\hbar}\theta(t-t')\langle[B(t), A(t')]\rangle_0,$$

such that

$$\delta\langle B(t)\rangle = \int_{-\infty}^{\infty} \mathrm{d}t'\, \chi_{BA}(t-t')h(t').$$

This describes the linear response of B to a perturbation through A.
The equilibrium (symmetrized) correlation function is defined as

$$C_{BA}(t) = \frac{1}{2}\langle\{B(t), A(0)\}\rangle_0 = \frac{1}{2}\left(\langle B(t)A(0)\rangle_0 + \langle A(0)B(t)\rangle_0\right).$$

The corresponding power spectral density is then

$$S_{BA}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, e^{i\omega t} C_{BA}(t).$$

Let $\chi_{BA}(\omega)$ denote the Fourier transform of the response function

$$\chi_{BA}(\omega) = \int_{-\infty}^{\infty} \mathrm{d}t\, e^{i\omega t} \chi_{BA}(t).$$

The Kubo fluctuation-dissipation theorem then relates the imaginary part of the response function to the spectral density as

$$\mathrm{Im}[\chi_{BA}(\omega)] = \frac{1}{2\hbar}\left(1 - e^{-\beta\hbar\omega}\right) S_{BA}(\omega) \tag{2.2}$$

Let us now move to the theorems beyond linear response.

2.2 Crooks fluctuation theorem

The Crooks fluctuation theorem [11] is a central result in nonequilibrium statistical mechanics that quantifies the probability of observing entropy production (or work performed) in forward and time-reversed thermodynamic processes. It provides a rigorous connection between nonequilibrium dynamics and equilibrium free energy differences.

Consider a classical system with phase space coordinates x, evolving under a time-dependent external parameter $\lambda(t)$ over a time interval $[0, \tau]$. The system is initially prepared in thermal equilibrium at inverse temperature β under the parameter value λ_0. Its dynamics are assumed to be Markovian and microscopically reversible.

We define two protocols: The **forward process**, where the system evolves under $\lambda_F(t)$ from λ_0 to λ_τ and the **reverse process**, defined by the time-reversed protocol $\lambda_R(t) = \lambda_F(\tau - t)$, starting from equilibrium at λ_τ.

A trajectory is denoted by $\Gamma = \{x(t)\}_{0\le t\le\tau}$. The probability of observing Γ in the forward process is denoted as $P_F[\Gamma]$, and $\tilde{\Gamma} = \{\tilde{x}(t) = x(\tau - t)\}$ has probability $P_R[\tilde{\Gamma}]$ in

the reverse process.

Now, let $H(x, \lambda)$ be the Hamiltonian. The system starts in the canonical distribution

$$p_0(x) = \frac{1}{Z_0} e^{-\beta H(x,\lambda_0)}, \quad Z_0 = \int \mathrm{d}x\, e^{-\beta H(x,\lambda_0)}.$$

The work $W[\Gamma]$ performed along a single trajectory in the forward protocol is defined as

$$W[\Gamma] = \int_0^\tau dt\, \dot{\lambda}_F(t) \frac{\partial H(x(t),\lambda)}{\partial \lambda}\Big|_{\lambda=\lambda_F(t)}.$$

This definition coincides with the change in energy of the system minus the heat exchanged with the environment.

Let $\Delta E = H(x_\tau, \lambda_\tau) - H(x_0, \lambda_0)$ denote the total energy change. Then, the heat transferred to the environment is simply

$$Q[\Gamma] = W[\Gamma] - \Delta E.$$

We now assume that the dynamics satisfy a local detailed balance condition

$$\frac{P_F[\Gamma]}{P_R[\tilde{\Gamma}]} = \frac{p_0(x_0)}{p_\tau(x_\tau)} e^{-\beta Q[\Gamma]}.$$

Using $Q = W - (H(x_\tau, \lambda_\tau) - H(x_0, \lambda_0))$ and the Boltzmann weights

$$\frac{p_0(x_0)}{p_\tau(x_\tau)} = \frac{Z_\tau}{Z_0} e^{-\beta[H(x_0,\lambda_0) - H(x_\tau,\lambda_\tau)]},$$

we obtain

$$\begin{aligned}\frac{P_F[\Gamma]}{P_R[\tilde{\Gamma}]} &= \frac{Z_\tau}{Z_0} e^{-\beta[H(x_0,\lambda_0)-H(x_\tau,\lambda_\tau)]} e^{-\beta[W[\Gamma]-(H(x_\tau,\lambda_\tau)-H(x_0,\lambda_0))]} \\ &= \frac{Z_\tau}{Z_0} e^{-\beta W[\Gamma]} = e^{-\beta(W[\Gamma]-\Delta F)},\end{aligned}$$

where $\Delta F = F(\lambda_\tau) - F(\lambda_0) = -\beta^{-1} \ln(Z_\tau/Z_0)$ is the equilibrium free energy difference.

Now we ask ourselves about the probability distribution of the work itself. The rules of statistical mechanics tell us that the work probability density in the forward process is given by

$$P_F(W) = \int \mathcal{D}\Gamma\, P_F[\Gamma]\, \delta(W - W[\Gamma]),$$

and similarly in the reverse process

$$P_R(-W) = \int \mathcal{D}\Gamma \, P_R[\tilde{\Gamma}] \, \delta(W + W[\Gamma]).$$

This can be rewritten by using the trajectory probability ratio as

$$\begin{aligned} P_R(-W) &= \int \mathcal{D}\Gamma \, P_R[\tilde{\Gamma}] \, \delta(W + W[\Gamma]) \\ &= \int \mathcal{D}\Gamma \, P_F[\Gamma] \, e^{-\beta(W[\Gamma] - \Delta F)} \delta(W + W[\Gamma]) \\ &= e^{\beta(W + \Delta F)} P_F(-W). \end{aligned}$$

Therefore, we obtain

$$\frac{P_F(W)}{P_R(-W)} = e^{\beta(W - \Delta F)}. \tag{2.3}$$

This is the Crooks fluctuation theorem, which rigorously quantifies irreversibility in terms of measurable fluctuations of work. It shows that, while trajectories violating the second law are possible, they are exponentially suppressed. The crossing point of $P_F(W)$ and $P_R(-W)$ identifies the equilibrium free energy difference.

This theorem plays a central role in non-equilibrium thermodynamics, particularly in the analysis of small systems, single-molecule pulling experiments, and numerical free energy estimation.

It is interesting to note that the stochastic thermodynamic entropy production is simply

$$\sigma = \beta(W - \Delta F)$$

Jarzynski equality [24] can be derived from this equation by means of the normalization of the probability density. Multiplying both sides by $P_R(-W)e^{-\beta W}$ and integrating, we obtain

$$\int \mathrm{d}W \, P_F(W) e^{-\beta W} = e^{-\beta \Delta F} \int \mathrm{d}W \, P_R(-W) = e^{-\beta \Delta F},$$

which gives us Jarzynski equality

$$\left\langle e^{-\beta W} \right\rangle_F = e^{-\beta \Delta F}. \tag{2.4}$$

2.2.1 Quantum Crooks fluctuation theorem

The Crooks fluctuation theorem also holds for quantum systems, with appropriate reformulations to accommodate quantum measurements and non-commutative observables. In this section, we provide a derivation of the quantum version of the Crooks theorem.

Let the system evolve under a time-dependent Hamiltonian $H(t) = H(\lambda_t)$ over a time interval $[0, \tau]$, where λ_t is an external protocol. The system is initially in equilibrium at inverse temperature β, but evolves unitarily during the protocol[1].

The quantum definition of work requires care. The most common notion of work is the so-called **two-point projective measurement (TPM) scheme** that can be summarized as follows:

- At $t = 0$ we perform a projective energy measurement in the basis of H_0, collapsing the system into an eigenstate $|\epsilon_n^0\rangle$ with probability

$$p_n^0 = \frac{e^{-\beta\epsilon_n^0}}{Z_0} \qquad Z_0 = \mathrm{Tr} e^{-\beta H_0}$$

- Evolve the system unitarily under $U = \mathcal{T} \exp\left(-\frac{i}{\hbar}\int_0^\tau H(t)\mathrm{d}t\right)$, with $\mathcal{T}$ being the time-ordering operator.

- At $t = \tau$, the energy of the system is again measured and the energy ϵ_m^τ (associated with the eigenstate $|\epsilon_m^\tau\rangle$ of H_τ is obtained with conditional probability

$$p_{m|n}^\tau = \mathrm{Tr}\left[\Pi_m^\tau U \Pi_n^0 U^\dagger\right],$$

  with Π_m^τ (Π_n^0) being the projector into the final (initial) Hamiltonian eigenbasis.

- The joint probability of obtaining ϵ_n^0 and ϵ_m^{tau} is then given by

$$p_{m,n} = p_n^0 p_{m|n}^\tau.$$

In this scheme, the stochastic work performed on the system along the trajectory defined by the two measurements is defined as

$$W_{m,n} = \epsilon_m^\tau - \epsilon_n^0.$$

[1] More general evolutions can also be considered with no fundamental changes.

Now, we proceed in the same way we did in the classical case, by defining the forward and the reverse protocols.

Starting from the forward process, the initial density matrix is the thermal one defined by H_0. The protocol follows as specified above. The joint probability can be written as

$$p^F_{m,n} = \text{Tr}\left[\Pi^\tau_m U \Pi^0_n \rho_0 \Pi^0_n U^\dagger\right] = \frac{e^{-\beta\epsilon^0_n}}{Z_0} |\langle \epsilon^\tau_m | U | \epsilon^0_n \rangle|^2.$$

In the **reverse process**, the protocol λ_t is time-reversed: $\tilde{\lambda}_t = \lambda(\tau - t)$. The unitary evolution is $\tilde{U} = \Theta U^\dagger \Theta^{-1}$, where Θ is the anti-unitary time-reversal operator. In this case, the initial state is defined by the Hamiltonian H_τ

$$\rho_\tau = \frac{1}{Z_\tau} e^{-\beta H_\tau}.$$

The reverse transition probability can be computed in the same way, resulting in

$$p^R_{n,m} = \text{Tr}\left[\Pi^0_n \tilde{U} \Pi^\tau_m \rho_\tau \Pi^\tau_m \tilde{U}^\dagger\right] = \frac{e^{-\beta\epsilon^\tau_m}}{Z_\tau} |\langle \epsilon^\tau_m | U | \epsilon^0_n \rangle|^2.$$

From this, we can compute the ratio of the forward and reverse joint probabilities as

$$\frac{p^F_{m,n}}{p^R_{n,m}} = \frac{\frac{e^{-\beta\epsilon^0_n}}{Z_0} |\langle \epsilon^\tau_m | U | \epsilon^0_n \rangle|^2}{\frac{e^{-\beta\epsilon^\tau_m}}{Z_\tau} |\langle \epsilon^\tau_m | U | \epsilon^0_n \rangle|^2} = \frac{Z_\tau}{Z_0} e^{\beta(\epsilon^\tau_m - \epsilon^0_n)} = e^{\beta(W_{m,n} - \Delta F)}.$$

In order to obtain Crooks' theorem, we need to define the work distribution, which, for the forward process, takes the form

$$p^F(W) = \sum_{m,n} p^F_{m,n} \delta(W - W_{m,n}),$$

and in the reverse case, we have

$$p^R(-W) = \sum_{m,n} p^R_{n,m} \delta(W + W_{m,n}). \tag{2.5}$$

Hence, the **quantum Crooks fluctuation theorem** reads

$$\frac{p^F(W)}{p^R(-W)} = e^{\beta(W - \Delta F)}. \tag{2.6}$$

As in the classical case, integration yields the Jarzynski equality

$$\left\langle E^{-\beta W} \right\rangle = \int \mathrm{d}W \, p^f(W) e^{-\beta W} = e^{-\beta \Delta F} \int \mathrm{d}W \, p^R(-W) = e^{-\beta \Delta F},$$

Chapter 3

A Bit of Differential Geometry

As discussed in the previous chapter, space-time is a four-dimensional continuous set of events. Each event then needs four numbers to be characterized. Special Relativity assumes that this fact is true globally, meaning that there is a one-to-one map between the events and points in the $\mathbb{R}^4$ space. This is no longer true in general spacetimes described by General Relativity. If we think about the surface of a sphere, we will readily realize that it is impossible to build such a map. Therefore, we need a more precise definition of a manifold, which is a mathematical space that *locally* looks like the Euclidean space, just like a sphere. The main goal of this chapter is to formally introduce this concept, along with some others that are necessary in order to properly describe the spacetime within General Relativity. Some additional definitions are presented, for completeness, in Appendix A.

3.1 Manifolds

A manifold is just a continuous set of points that can exhibit complex global properties like curvature or torsion, but that *locally* looks like the Euclidean space. This means that in a small enough[1] neighborhood of any point on the manifold, Euclidean geometry applies. For instance, the surface of a sphere is a manifold and if we are on top of a very big sphere, like our planet for instance, our neighborhood will certainly look like the usual flat two-dimensional plane. This is the main mathematical structure behind the theory of relativity since the space-time is postulated to be a differentiable manifold[2].

[1] Small enough means that we cannot, by any means, detect any violation of Euclidean geometry within the precision of the measurements available to us.

[2] To be precise, it is a differentiable manifold that is also Hausdorff, paracompact and connected. See Appendix A for more details.

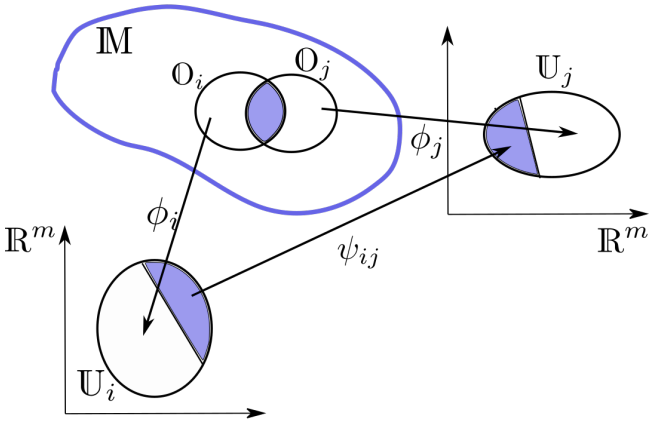

Figure 3.1: **Differentiable Manifold**. Homeomosphisms ϕ_i and ϕ_j mapping open subsets $\mathbb{O}_i$ and $\mathbb{O}_i$ on the manifold $\mathbb{M}$ into open subsets $\mathbb{U}_i$ and $\mathbb{U}_j$ on Euclidean spaces. Such maps specify two different coordinate systems. For points in the overlapping region $\mathbb{O}_i \cap \mathbb{O}_j$, the smooth map ψ_{ij} is called a coordinate transformation. This basic structure allows us to do calculus on manifolds as we normally do in $\mathbb{R}^n$.

Let us start with the formal definition of this object. A manifold can be viewed as a generalization of our concepts of curves and surfaces to objects with arbitrary dimensions. In the same way a curve and a two-dimensional surface are locally homeomorphic to $\mathbb{R}$ and $\mathbb{R}^2$, respectively, a manifold is a topological space which is locally homeomorphic to $\mathbb{R}^m$. This is a very important requirement, especially in the context of relativity since this local homeomorphism enables us to define local coordinate systems. If the manifold is not globally homeomorphic to $\mathbb{R}^m$, it will be impossible to cover it entirely with a single coordinate system. However, we can define several overlapping coordinate systems in order to cover $\mathbb{M}$. By imposing that the transformation between different coordinate systems is continuous, we can develop the usual calculus on manifolds. A precise definition, which is illustrated in Fig. 3.1, can be stated as follows.

Definition 1 (Differentiable Manifold) *$\mathbb{M}$ is an m-dimensional differentiable manifold if we have a set of pairs $\{(\mathbb{O}_i, \phi_i)\}_i$ of open subsets (local neighborhoods) $\mathbb{O}_i \subset \mathbb{M}$ and local homeomorphisms ϕ_i from $\mathbb{O}_i$ to open subsets $\mathbb{U}_i \subset \mathbb{R}^m$, such that the following conditions hold.*

1. *$\mathbb{M}$ is a topological space*[3]

2. *The set of $\mathbb{O}_i$ covers $\mathbb{M}$: $\cup_i \mathbb{O}_i = \mathbb{M}$.*

3. *Given two neighbourhoods $\mathbb{O}_i$ and $\mathbb{O}_j$ such that $\mathbb{O}_i \cap \mathbb{O}_j \neq \emptyset$ for $i \neq j$, the map $\psi_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(\mathbb{O}_i \cap \mathbb{O}_j) \mapsto \phi_i(\mathbb{O}_i \cap \mathbb{O}_j)$ is infinitely differentiable.*

The pair $(\mathbb{O}_i, \phi_i)$ is called a **coordinate chart** while the whole set $\{(\mathbb{O}_i, \phi_i)\}_i$ constitutes an **atlas**. If the atlas is complete, i.e. it is not contained in each other atlas, it is called a **differentiable structure**. Two atlases are said to be compatible if their union is also an atlas. In this sense, a differentiable structure on $\mathbb{M}$ is an equivalence class of

[3]See Appendix A for definitions.

compatible atlases. A topological space equipped with a differential structure is called a differentiable manifold.

The subset $\mathbb{O}_i$ is called the **coordinate neighborhood** and ϕ_i the **coordinate function**. Given a point $p \in \mathbb{M}$, the homeomorphism ϕ_i is represented by the set of coordinates $\{x^\mu(p)\} = \{x^0(p), ..., x^{m-1}(p)\}$. This is what we mean by the statement that the manifold locally looks like Euclidean space. In each coordinate neighbourhood $\mathbb{O}_i$, $\mathbb{M}$ looks like an open set of $\mathbb{R}^m$.

As we can see from Fig. 3.1, two coordinate systems can be assigned to the same point p if two of its neighborhoods $\mathbb{O}_i$ and $\mathbb{O}_j$ overlap. The axioms defining the manifold assure that the transition from one coordinate system to the other is smooth. In other words, the transition map, which is called a **coordinate transformation**, is of class C^∞. The map $\psi_{i,j}$ is explicitly given by the coordinate functions $x^\mu = x^\mu(y^\nu)$, where x^μ and y^ν are the coordinates assigned to p by ϕ_i and ϕ_j, respectively. Therefore, differentiability can be defined in the same way we do in usual calculus, thus justifying the name differential structure[4]. The coordinate transformation is differentiable if each function $x^\mu(y)$ is differentiable with respect to each y^ν.

In order to make these definitions more clear, let us consider some examples. The simplest case is the space $\mathbb{R}^m$ itself, which is a manifold that can be covered by a single coordinate system with the homeomorphism ϕ being the identity.

Moving to less trivial cases, we start with the usual two-dimensional surface of a sphere of unit radius, $\mathbb{S}^2$, which is a submanifold[5] of $\mathbb{R}^3$. We can simply choose the polar coordinates (θ, ϕ) in order to parametrize the surface of $\mathbb{S}^2$. Such coordinates are usually defined by the relations

$$\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z} \quad \text{and} \quad \phi = \tan^{-1} \frac{y}{x}, \tag{3.1}$$

with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$, while (x, y, z) represent the usual Cartesian coordinates. However, we are also free to choose any other coordinate system, like the stereographic one (u, v), which is defined by the projection from the North pole to the equatorial plane by the equations

$$u = \frac{x}{1 - z} \quad \text{and} \quad v = \frac{y}{1 - z}. \tag{3.2}$$

[4]Actually, if the union of two atlases is again an atlas, they are said to be compatible. The differential structure is defined by the compatibility equivalence class

[5]$\mathbb{N} \subset \mathbb{M}$ is a smooth submanifold of $\mathbb{M}$ if every point of $\mathbb{N}$ lies in some chart $(\mathbb{O}_i, \phi_i)$ with $\phi_i(\mathbb{N} \cap \mathbb{O}_i) = \phi_i(\mathbb{O}_i) \cap \mathbb{R}^k$, with $0 < k \leq m$.

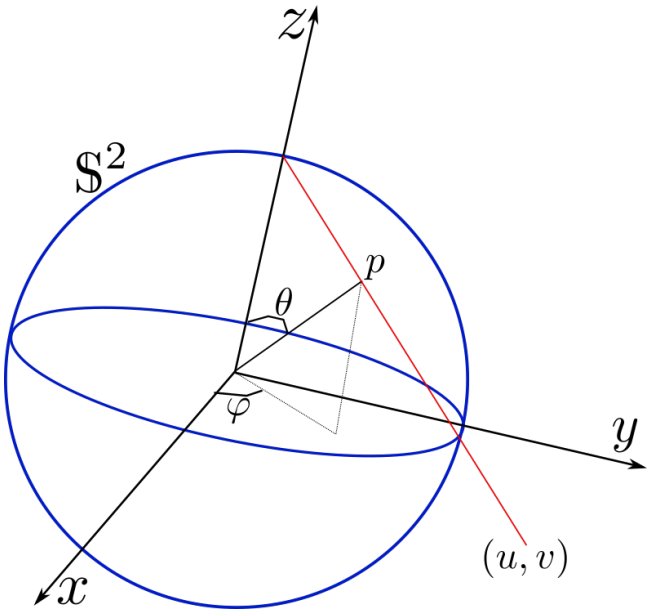

Figure 3.2: **Sphere**. Illustration of the polar (θ, φ) and stereographic (u, v) coordinates for the point $p \in \mathbb{S}^2$ over the $\mathbb{S}^2$ manifold. Such space frequently appears in many physical problems. For instance, the configuration space of a spherical pendulum is $\mathbb{S}^2$.

It is straightforward to see that both coordinate systems are related by the equations

$$u = \cot\frac{\theta}{2}\cos\phi \quad \text{and} \quad v = \cot\frac{\theta}{2}\sin\phi. \tag{3.3}$$

Figure 3.2 illustrates both coordinate systems.

Of course, there are many other coordinate systems that we can choose, and all of them are equally good. However, an important point here is that no coordinate system can be employed everywhere at once. In other words, there is no single coordinate system that is able to uniquely assign a set of coordinates to every point on $\mathbb{S}^2$. This can be illustrated from the stereographic coordinates at the pole or the polar coordinates at the equator ($\theta = \pi/2$). We cannot label the points on the sphere with a single coordinate system such that nearby points always have nearby coordinates. However, we can do this on parts of $\mathbb{S}^2$. We can construct two or more overlapping coordinate systems, such that each one of them covers some part of the manifold by uniquely labeling every one of its points and that nearby points have nearby coordinates (in at least one of them). Specifically, we can consider two stereographic coordinates, one defined by the projections from the North pole and the other one by projections from the South pole. By imposing that the transition from one coordinate system to the other is determined by functions of class C^∞, we have a differentiable manifold.

Another example is the unit circle, which is the submanifold of $\mathbb{R}^2$ defined by $\mathbb{S}^1 = \{(x, y)|x^2 + y^2 = 1; x, y \in \mathbb{R}\}$. Note that there is no way to globally parametrize the circle with a single coordinate function.

Now that we have a space with a differential structure defined on it, it is time to move forward and see how to do calculus on a manifold.

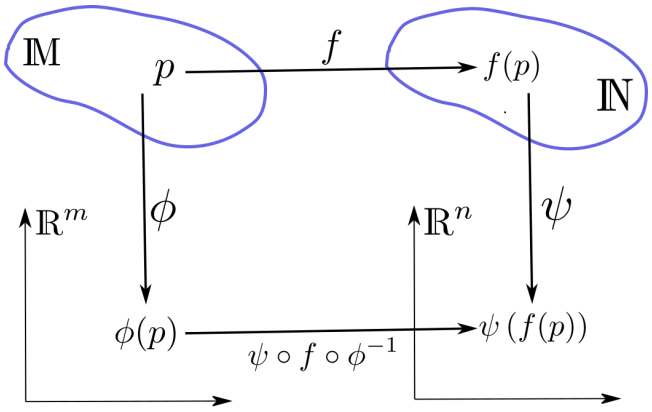

Figure 3.3: **Maps on manifolds**. Illustration of the map $f : \mathbb{M} \mapsto \mathbb{N}$. The maps ϕ and ψ are employed in order to provide a coordinate representations for f in the way discussed in the text. Note that the dimensions of the involved manifolds do not need to be the same.

3.2 Differentiable maps

As it is clear from the last section, the theory of manifolds is based on smoothness, so we can employ the usual calculus developed in $\mathbb{R}^n$. Let us start by defining a very important concept, differentiable maps between manifolds, which are transformations that preserve the structure.

Let $f : \mathbb{M} \mapsto \mathbb{N}$ be a map between an m-dimensional manifold $\mathbb{M}$ and an n-dimensional one $\mathbb{N}$. In this way, a point $p \in \mathbb{M}$ is mapped to the point $f(p) \in \mathbb{N}$, $f : p \mapsto f(p)$, as illustrated in Fig. 3.3. Now, in order to build a coordinate representation of such a map, we define the charts $(\mathbb{O}, \phi)$ and $(\mathbb{P}, \psi)$, in such a way that $p \in \mathbb{O}$ and $f(p) \in \mathbb{P}$. The coordinate representation for f can be written as

$$\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \mapsto \mathbb{R}^n. \tag{3.4}$$

If $\phi(p) = \{x^\mu\}$ and $\psi(f(p)) = \{y^\nu\}$ we have $y = \psi \circ f \circ \phi^{-1}(x)$ and, when we know which coordinate systems are being used, we can write $y = f(x)$ or $y^\mu = f^\mu(x^\alpha)$, considering a certain abuse of notation. If these functions are of class C^∞, then the map f is said to be differentiable. Such a notion is independent of the coordinate system used.

A very important class of maps between manifolds can be defined as follows.

Definition 2 (Diffeomorphism) *Let $f : \mathbb{M} \mapsto \mathbb{N}$ be a homeomorphism[6] and ψ and ϕ coordinate functions. If $\psi \circ f \circ \phi^{-1}(x)$ is invertible, and both $y = \psi \circ f \circ \phi^{-1}(x)$ and $x = \phi \circ f^{-1} \circ \psi^{-1}(y)$ are C^∞, the map f is said to be a diffeomorphism and $\mathbb{M}$ and $\mathbb{N}$ are said to be diffeomorphic, denoted as $\mathbb{M} \cong \mathbb{N}$.*

Clearly, $\dim\mathbb{M} = \dim\mathbb{N}$ if $\mathbb{M} \cong \mathbb{N}$. This notion provides a classification o spaces into equivalence classes according to whether it is possible to smoothly deform one space into another. The set of diffeomorphisms $f : \mathbb{M} \mapsto \mathbb{M}$ is a group denoted by $\mathrm{Diff}(\mathbb{M})$.

[6] See Appendix A for definitions.

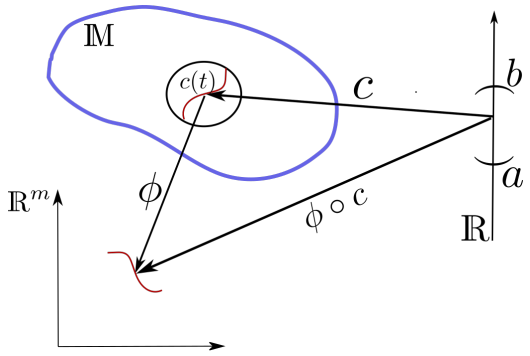

Figure 3.4: **Curves on manifolds**. Illustration of the curve c, parametrized by t, which is a map from an open set $(a, b) \subset \mathbb{R}$ of the real line to the manifold. Given a coordinate function ϕ, we can build the coordinate representation of the curve on $\mathbb{R}^m$.

A very special class of maps that will be employed to introduce the fundamental notion of vector space is called **curve**. Formally, an open curve on an m-dimensional manifold $\mathbb{M}$ is the map $c : (a, b) \mapsto \mathbb{M}$, where (a, b) is an open interval of the real line. We assume that the curve does not intersect with itself, as illustrated in Fig. 3.4. A generalization of such a concept is a closed curve, which is the map $c : \mathbb{S}^1 \mapsto \mathbb{M}$. By defining the chart $(\mathbb{O}, \phi)$, the curve $c(t)$ has the coordinate representation $x = \phi \circ c : \mathbb{R} \mapsto \mathbb{R}^m$.

Another important mapping is the function f on $\mathbb{M}$, which is a smooth map from $\mathbb{M}$ to $\mathbb{R}$. By choosing a chart $(\mathbb{O}, \phi)$, we can build the coordinate representation of f as $f \circ \phi^{-1} : \mathbb{R}^m \mapsto \mathbb{R}$, which is a real-valued function of m variables. We denote the set of smooth functions on $\mathbb{M}$ by $\mathbb{F}(\mathbb{M})$.

3.3 The tangent space

One of the ways we can define a tangent space to a point in a manifold is embedding such a manifold in a higher dimensional Euclidean space and selecting a specific linear subspace as the tangent one. However, it would be much more powerful if we could define all the necessary quantities in an intrinsic way, making reference to the manifold structure only. This is the way modern differential geometry deals with such problems. In this context, a tangent space is understood as an object that is tangent to a curve on the manifold.

The notion of a vector as an arrow connecting some given point to the origin does not work, in general, on a manifold. For instance, how to draw a straight arrow connecting two points on the surface of a sphere? Therefore, we need a more general definition of what a vector is. On a manifold, a vector is defined in terms of a **tangent vector** to a curve, which is the generalization of the tangent line to a curve in the usual two-dimensional plane. However, in an n-dimensional manifold, there are many curves that are indeed tangent to a given vector and, therefore, a tangent vector is an equivalence class of curves. Let us see how this idea works mathematically.

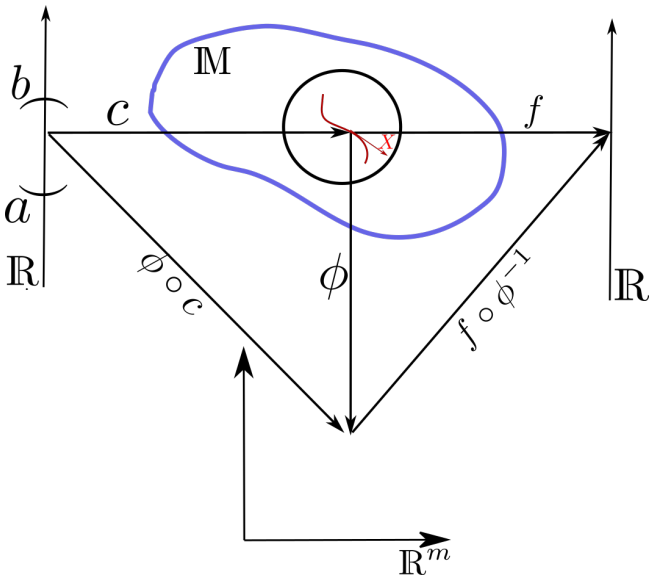

Figure 3.5: **Vector on manifolds**. A curve c along with a function f and a local coordinate system ϕ define the tangent vector X to the manifold $\mathbb{M}$ in the direction determined by the curve $c(t)$.

Let us consider a curve $c : (a, b) \mapsto \mathbb{M}$ and a function $f : \mathbb{M} \mapsto \mathbb{R}$, where (a, b) is an open interval in $\mathbb{R}$ containing the point $t = 0$ (t is an arbitrary parametrization of the curve). By choosing a local coordinate function ϕ, the rate of change of the function f at $t = 0$ along the curve c is given by

$$\left.\frac{\mathrm{d}f(c(t))}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\partial f}{\partial x^\mu}\frac{\mathrm{d}x^\mu(c(t))}{\mathrm{d}t}\right|_{t=0}, \tag{3.5}$$

with

$$\frac{\partial f}{\partial x^\mu} \equiv \frac{\partial}{\partial x^\mu}\left[f \circ \phi^{-1}(x)\right]. \tag{3.6}$$

This means that the rate $(\mathrm{d}f/\mathrm{d}t)|_{t=0}$ is obtained by the application of the differential operator

$$\mathrm{X} = \mathrm{X}^\mu\left(\frac{\partial}{\partial \mathrm{x}^\mu}\right) \quad \text{with} \quad \mathrm{X}^\mu = \left.\frac{\mathrm{dx}^\mu(\mathrm{c(t)})}{\mathrm{dt}}\right|_{\mathrm{t=0}} \tag{3.7}$$

to the map f, denoted as $X[f]$. Therefore, we define X as the tangent vector to $\mathbb{M}$ at the point $p = c(0)$ along the direction given by the curve $c(t)$. Figure 3.5 illustrates this concept.

Since

$$\frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu, \tag{3.8}$$

with δ^μ_ν being the Kronecker delta function, it follows directly

$$X[x^\mu] = \left.\frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t}\right|_{t=0}. \tag{3.9}$$

If t is understood as time, this is simply the μ-th component of the velocity vector[7].

Based on the above definition, let us now define the equivalence class of curves on $\mathbb{M}$. Given two curves c_1 and c_2 such that $c_1(0) = c_2(0) = p$ and

$$\left.\frac{\mathrm{d}x^\mu(c_1(t))}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\mathrm{d}x^\mu(c_2(t))}{\mathrm{d}t}\right|_{t=0}, \tag{3.10}$$

then both curves define the same vector X at p, in which case we have $c_1 \sim c_2$. We therefore identify the tangent vector X with the equivalence class of curves

$$[c(t)] = \left\{\tilde{c}(t)\,|\,\tilde{c}(0) = c(0) \quad \text{and} \quad \left.\frac{\mathrm{d}x^\mu(c_1(t))}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\mathrm{d}x^\mu(c_2(t))}{\mathrm{d}t}\right|_{t=0}\right\}, \tag{3.11}$$

rather than a curve itself. All the equivalence classes of curves at $p \in \mathbb{M}$, i. e. all the tangent vectors at p, form a vector space called the **tangent space** of $\mathbb{M}$ at p, denoted by $\mathrm{T}_p\mathbb{M}$. From Eq. (3.7) it is clear that $e_\mu \equiv \partial/\partial x^\mu$ is a basis vector. Evidently, $\mathrm{dim}\mathrm{T}_p\mathbb{M} = \mathrm{dim}\mathbb{M}$. The basis $\{e_\mu\}$ is called the coordinate basis. Note that a vector is not a line segment going from one point to the other on the manifold. Instead, it is defined at a single point. We have a tangent space associated with each point to the manifold. The collection of all these tangent spaces for all points on the manifold is called the **tangent bundle**, which is a manifold in its own right.

Although we have employed a coordinate system, it is clear from the above discussion that a vector X exists without this definition. We use coordinates only because it is convenient. The fact that the vector is coordinate independent allows us to find the transformation law of the components of the vector. Let $p \in \mathbb{O}_i \cap \mathbb{O}_j$ and $\{x^\mu\} = \phi_i(p)$ and $\{y^\nu\} = \phi_j(p)$ be the two coordinate functions. We then have two expressions for the vector $\mathrm{X} \in \mathrm{T_pM}$

$$\mathrm{X} = \mathrm{X}^\mu \frac{\partial}{\partial \mathrm{x}^\mu} = \tilde{\mathrm{X}}^\mu \frac{\partial}{\partial \mathrm{y}^\mu}. \tag{3.12}$$

The second equality must hold since the vector is the same. This implies that the components of the vector in the two coordinate bases must be related by

$$\tilde{X}^\mu = X^\nu \frac{\partial y^\mu}{\partial x^\nu}. \tag{3.13}$$

This last relation follows from the application of the vector X to the coordinate functions y^ν. It is important to observe here that this transformation law is such that the vector itself is left invariant. So, the components of the vector change, not the vector.

[7]See Appendix B for the usual Euclidean space.

If a vector is smoothly defined at each point of the manifold, we have a **vector field**. Therefore, X is a vector field if $X[f] \in \mathbb{F}(\mathbb{M})$ for any $f \in \mathbb{F}(\mathbb{M})$. The vector field defines, in this way, a map between smooth functions over the manifold.

Now, since we have defined the vector space $\mathrm{T}_p\mathbb{M}$, we can associate with it the dual vector space $\mathrm{T}_p^*\mathbb{M}$ (also called the cotangent space), whose elements are linear functions from $\mathrm{T}_p\mathbb{M}$ to $\mathbb{R}$, i.e. $\omega : \mathrm{T}_p\mathbb{M} \mapsto \mathbb{R}$ for $\omega \in \mathrm{T}_p^*\mathbb{M}$. ω is called the cotangent vector, the dual vector, or the one-form, with the simplest example being the differential $\mathrm{d}f$ of a function $f \in \mathbb{F}(\mathbb{M})$.

The action of $\mathrm{d}f \in \mathrm{T}_p^*\mathbb{M}$ on $V \in \mathrm{T}_p\mathbb{M}$ is defined as

$$\langle \mathrm{d}f, V \rangle = V[f] = V^\mu \frac{\partial f}{\partial x^\mu} \in \mathbb{R}. \tag{3.14}$$

We can clearly see that this expression is bilinear. Now, since $\mathrm{d}f = (\partial f/\partial x^\mu)\mathrm{d}x^\mu$, it is natural to take $\mathrm{d}x^\mu$ as the elements of the basis in $\mathrm{T}_p^*\mathbb{M}$, which is the dual basis since

$$\left\langle \mathrm{d}x^\nu, \frac{\partial}{\partial x^\mu} \right\rangle = \frac{\partial x^\nu}{\partial x^\mu} = \delta^\nu_\mu. \tag{3.15}$$

In this way, an arbitrary one form can be written as $\omega = \omega_\mu \mathrm{d}x^\mu$.

We are now in a position to define the **inner product** $\langle\, , \rangle : \mathrm{T}_p^*\mathcal{M} \times \mathrm{T}_p\mathcal{M} \mapsto \mathbb{R}$ as

$$\langle \omega, \mathrm{V} \rangle = \omega_\nu V^\mu \left\langle \mathrm{d}x^\nu, \frac{\partial}{\partial x^\mu} \right\rangle = \omega(V) = \omega_\mu V^\mu. \tag{3.16}$$

It is important to observe here that the inner product is defined in terms of the action of a dual vector on a vector, and not between two vectors.

Now, let us consider two coordinate systems $\{x^\mu\} = \phi_i(p)$ and $\{y^\nu\} = \phi_j(p)$ for the point $p \in \mathbb{O}_i \cap \mathbb{O}_j$. We thus have

$$\omega = \omega_\mu \mathrm{d}x^\mu = \tilde{\omega}_\nu \mathrm{d}y^\nu, \tag{3.17}$$

From the fact that $\mathrm{d}y^\nu = (\partial y^\nu/\partial x^\mu)\mathrm{d}x^\mu$ we can write down the transformation law for the components of the one-form ω

$$\tilde{\omega}_\nu = \omega_\mu \frac{\partial x^\mu}{\partial y^\nu}. \tag{3.18}$$

Again, the components of the one-form change in such a way that the one-form itself is left invariant.

3.4 Tensors

A tensor of type (q, r) is a multilinear object which maps q elements of $\mathrm{T}_p^*\mathbb{M}$ and r elements of $\mathrm{T}_p\mathbb{M}$ to a real number. The set of all tensors of type (q, r) at $p \in \mathbb{M}$ is denoted by $\mathbb{T}^q_{r,p}(\mathbb{M})$. We can write the tensor $\mathrm{T} \in \mathbb{T}^q_{r,p}(\mathbb{M})$ in terms of the bases defined in the last section as

$$\mathrm{T} = T^{\mu_1 \ldots \mu_q}{}_{\nu_1 \ldots \nu_r} \frac{\partial}{\partial x^{\mu_1}} \ldots \frac{\partial}{\partial x^{\mu_1}} \mathrm{d}x^{\nu_1} \ldots \mathrm{d}x^{\nu_r}. \tag{3.19}$$

Clearly, a tensor of type (q, r) is the multilinear map

$$\mathrm{T} : \left[\times^q \mathrm{T}_p^*\mathbb{M}\right] \left[\times^r \mathrm{T}_p\mathbb{M}\right] \mapsto \mathbb{R}, \tag{3.20}$$

where the symbol $\times^q \mathrm{T}_p^*\mathbb{M}$ means the Cartesian product of the space $\mathrm{T}_p^*\mathbb{M}$ q times, with a similar definition for the vector space $\mathrm{T}_p\mathbb{M}$. The action of a tensor on ω_i $(1 \leq i \leq r)$ and V_j $(1 \leq j \leq q)$ results in the number[8]

$$\mathrm{T}(\omega_1, ..., \omega_r; V_1, ..., V_q) = T^{\mu_1 \ldots \mu_q}{}_{\nu_1 \ldots \nu_r} \omega_{1,\mu_1} \ldots \omega_{q,\mu_q} V_1^{\nu_1} \ldots V_r^{\nu_r}. \tag{3.21}$$

Given two coordinate systems x and x', the components of the tensor T change as

$$T^{\mu_1' \ldots \mu_q'}{}_{\nu_1' \ldots \nu_r'} = T^{\mu_1 \ldots \mu_q}{}_{\nu_1 \ldots \nu_r} \frac{\mathrm{d}x^{\mu_1'}}{\mathrm{d}x^{\mu_1}} \cdots \frac{\mathrm{d}x^{\nu_1}}{\mathrm{d}x^{\nu_1'}}, \tag{3.22}$$

which is the general transformation law for tensors.

Similarly, we did in the case of a vector field, we define a tensor field of type (q, r) by the smooth assignment of an element of $\mathbb{T}^q_{r,p}(\mathbb{M})$ to each point $p \in \mathbb{M}$. The set of the tensor fields of type (q, r) on $\mathbb{M}$ is denoted by $\mathbb{T}^q_r(\mathbb{M})$. For example, $\mathbb{T}^0_1(\mathbb{M})$ is the set of the dual vector fields, while $\mathbb{T}^1_0(\mathbb{M})$ is the tangent bundle.

It is important to observe that we defined a tensor as in Eq. 3.20, we can let the tensor not act on all of its slots, resulting in different tensors. For instance, a map between two vector spaces can be seen as a tensor

$$\mathrm{T}^{\mathrm{a}}{}_{\mathrm{b}} : \mathrm{V}^{\mathrm{b}} \mapsto \mathrm{T}^{\mathrm{a}}{}_{\mathrm{b}} \mathrm{V}^{\mathrm{b}}.$$

[8]Note that the Latin indexes in these expressions are not labelling the components of the vectors, but they are labeling distinct vectors.

3.4.1 The metric tensor

A fundamental tensor for us is the metric tensor g_{ab}, which is a symmetric $(0,2)$ tensor. By imposing that the determinant of the metric does not vanish, $\mathrm{Det}[\mathrm{g}_{ab}] = g \neq 0$, we can properly define the inverse of the metric as

$$g^{\mu\nu} g_{\alpha\nu} = \delta^{\mu}_{\alpha}. \tag{3.23}$$

In the same way we used the Minkowski metric $\eta_{\mu\nu}$ to raise and lower indices in special relativity, in the general theory, we use the metric $g_{\mu\nu}$ to perform such tasks.

One of the many applications of the metric tensor is to provide the notion of the length of a path in spacetime, the line element

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^{\mu} \otimes \mathrm{d}x^{\nu} = g_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu}. \tag{3.24}$$

This equation makes sense because $\mathrm{d}x^{\mu}$ is really a basis dual vector. For instance, the Euclidean line element in the usual three-dimensional space is $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, written in Cartesian coordinates. In this equation, and in most of these lectures, we employ the shorthand notation $\mathrm{d}x^2 \equiv \mathrm{d}x \otimes \mathrm{d}x$, except for the line element itself, $\mathrm{d}s^2$, which is just a new notation for the metric, not representing the square of any quantity.

The Newtonian spacetime is called Euclidean and has a positive metric, with signature[9] $(+,+,+)$. This is also true for the case of Riemannian geometry, where all eigenvalues of the metric are positive. However, the signature of the metric of special and general theory of relativity is $(+,-,-,-)$, which means that it is not positive semi-definite. This kind of metric is called **Lorentzian**.

An important observation here is that the partial derivative is not a true tensor, as can be seen by computing its transformation law. Physics needs derivatives, and if we want the laws of physics to be coordinate independent, we need to define the notion of the covariant derivative on manifolds.

[9] the number of positive and negative engenvalues.

Chapter 4

Relativistic approach to thermodynamic irreversibility

4.1 Introduction

Irreversible processes are ubiquitous in Nature despite the fact that the fundamental laws of Nature respect the time-reverse symmetry [51]. Irreversibility is signaled by thermodynamic entropy production [29] and says that the thermodynamic arrow of time points from low to high entropy [33]. One of the fundamental results in this context is the set of relations known as fluctuation theorems, which state that in every physical process, a positive entropy production is highly likely to be observed, while processes in which entropy decreases are observed with vanishingly small probability [11, 24, 44, 18, 7]. Therefore, an average positive entropy production is typically manifested in Nature.

The first ideas on relativistic thermodynamics were put forward by Einstein and Planck while considering the behavior of thermodynamic properties under changes in reference frames [15, 16, 36]. An important step in the direction to understand the connections between these theories was the development of black hole thermodynamics [49]. Such laws were then employed to show that Einstein's field equations could be interpreted as a thermodynamic equation of state [23]. Many advances in this direction were witnessed, among which we mention investigations about the non-equilibrium properties of spacetime [17], attempts to construct a statistical mechanical theory of the gravitational field, and the conjecture that time can have a thermodynamic origin [38, 8, 40, 39].

In this chapter, we describe a major step in the direction to properly understand

the thermodynamics of localized quantum systems living in a general curved spacetime and, thus, under the action of the gravitational field [2, 3]. Considering linear response, a fluctuation theorem in curved spacetimes was presented in Ref. [31], while the non-equilibrium fluctuations of a black hole horizon were studied in Ref. [22] in the context of the Jarzynski equality [24] along with the generalized second law of thermodynamics [4].

This chapter describes a fully general relativistic detailed quantum fluctuation relation for a localized quantum system [2, 3]. The result is based on the so-called two-point measurement (TPM) scheme [18]. Therefore, we unveil the complete impact of the gravitational field on irreversible processes by explicitly considering the effect of spacetime curvature. One of the most impressive consequences of this result is that thermodynamic entropy is fundamentally observer-dependent, being deeply linked to the time-orientability of the spacetime. We present two paradigmatic examples in which we discuss the role of the equivalence principle and the expansion of the universe on entropy generation.

We use natural units throughout the chapter. The signature of the metric is $(-,+,+,+)$ and $\eta_{ab} = \text{diag}(-1,1,1,1)$ is the Minkowski metric.

4.2 Localized quantum systems in a curved spacetime

Our initial goal is to formulate a description of a localized non-relativistic quantum system in a curved spacetime. This system will be characterized by a set of external degrees of freedom along with one or more internal degrees of freedom. To achieve this, we construct Fermi normal coordinates around a timelike trajectory representing the worldline of our laboratory frame. At last, we analyze the Hamiltonian dynamics of a localized quantum particle in the vicinity of this trajectory [37, 34].

4.2.1 Fermi normal coordinates

To begin with, let $(\mathcal{M}, \mathsf{g})$ be a four-dimensional spacetime, where $\mathcal{M}$ is a differentiable manifold and g is a Lorentzian metric. In this spacetime, the worldline of a laboratory frame is represented by a timelike curve $\gamma : I \subset \mathbb{R} \to \mathcal{M}$, parametrized by its proper time $\tau \in I$, with 4-velocity u^μ which satisfies $u_\mu u^\mu = -1$.

In order to define the Fermi normal coordinates, we begin by setting the proper time τ along the curve γ as the time component of the coordinate system, as illustrated in Fig. 4.1. Next, we define an orthonormal basis $e_a{}^\mu$ at a point $\mathfrak{p} \equiv \gamma(\tau = 0) \in \mathcal{M}$, where $a = 0,1,2,3$ labels the four basis vectors, with $e_0{}^\mu$ identified as the tangent vector u^μ.

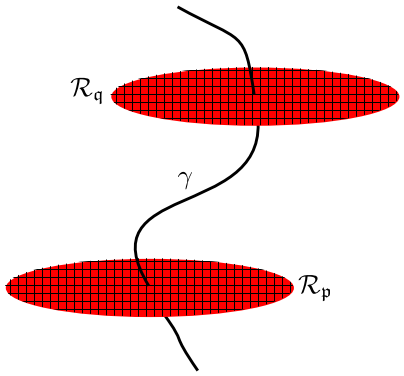

Figure 4.1: The laboratory world-line γ with 4-velocity u^μ and the rest spaces $\mathcal{R}_{\mathfrak{p}}$ and $\mathcal{R}7_{\mathfrak{q}}$ where the Fermi normal coordinates are valid.

At point $\mathfrak{p}$, we have the relation

$$g_{\mu\nu} e_a{}^\mu e_b{}^\nu = \eta_{ab}, \tag{4.1}$$

where $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric. The next step is to extend this orthonormal frame along the curve γ while preserving the orthogonality of the basis vectors. Parallel transport along γ alone does not ensure the vectors in the set $\{e_a{}^\mu\}_{a=0}^3$ will remain mutually orthogonal, and tangent to γ, unless γ is a geodesic. To maintain the orthonormality of the frame along γ, we employ Fermi-Walker transport, governed by the differential equations [21]

$$\frac{\mathrm{D}_F}{\mathrm{d}\tau}(e_a)^\mu \equiv \frac{\mathrm{D}}{\mathrm{d}\tau}(e_a)^\mu + 2a^{[\mu}u^{\nu]}(e_a)_\nu = 0, \tag{4.2}$$

where $\mathrm{D}/\mathrm{d}\tau \equiv u^\mu \nabla_\mu$ is the covariant derivative along the curve γ, $a^\mu = (\mathrm{D}/\mathrm{d}\tau)\, u^\mu = u^\nu \nabla_\nu u^\mu$ is the 4-acceleration of the curve γ while $2a^{[\mu}u^{\nu]} \equiv a^\mu u^\nu - a^\nu u^\mu$. When the above condition is met, the vectors $e_a{}^\mu$ are said to be Fermi transported. We refer to this frame as the laboratory frame.

To define the spacelike Fermi normal coordinates (x^1, x^2, x^3), we first consider the normal neighborhood of the point $\mathfrak{p} = \gamma(\tau = 0)$, which consists of all points that can be reached from $\mathfrak{p}$ by a single geodesic. This neighborhood is denoted by $\mathcal{U}_{\mathfrak{p}}$. Within this region, we define the local rest space $\mathcal{R}_{\mathfrak{p}} \subset \mathcal{U}_{\mathfrak{p}}$ as the set of points connected to $\mathfrak{p}$ by geodesics whose tangent vectors are orthogonal to u^μ, the 4-velocity at $\mathfrak{p}$. Refer to Fig. 4.1 for an illustration.

With this setup, we can assign the coordinates $(\tau = 0, x^1, x^2, x^3)$ to any point $\mathfrak{r} \in \mathcal{R}_{\mathfrak{p}}$ using the exponential map, such that $\mathfrak{r} = \exp_{\mathfrak{p}}(x^a e_a)$. Here, $\exp_{\mathfrak{p}} : T_{\mathfrak{p}}(\mathcal{M}) \to \mathcal{M}$ is the exponential map at $\mathfrak{p}$, $T_{\mathfrak{p}}(\mathcal{M})$ is the tangent space at $\mathfrak{p}$, and $e_a \in T_{\mathfrak{p}}(\mathcal{M})$ represents

the basis vectors. The local rest space of the entire curve γ is then defined as $\mathcal{R} \equiv \cup_{\mathfrak{p}\in\gamma}\mathcal{R}_{\mathfrak{p}}$, which provides a local foliation of the spacetime $\mathcal{M}$ around γ. Any point in $\mathcal{R}$ is described by the Fermi normal coordinates (τ, x^1, x^2, x^3).

As a result of this construction and the decomposition of the metric along γ as $g_{\mu\nu} = -u_\mu u_\nu + \delta_{ij}e^i{}_\mu e^j{}_\nu$, the spatial distance from a point $\mathfrak{r} \in \mathcal{R}$ to the curve γ is given by $r^2 = \delta_{ij}x^i x^j$ [37]. In addition, the Fermi normal coordinates (τ, x^1, x^2, x^3) allow us to express the components of the metric around the curve γ as

$$\begin{aligned} g_{\tau\tau} &= -(1 + a_i(\tau)x^i)^2 - R_{\tau i\tau j}(\tau)x^i x^j + \mathcal{O}(r^3), \\ g_{\tau i} &= -\frac{2}{3}R_{\tau jik}(\tau)x^j x^k + \mathcal{O}(r^3), \\ g_{ij} &= \delta_{ij} - \frac{1}{3}R_{ikjl}(\tau)x^k x^l + \mathcal{O}(r^3), \end{aligned} \tag{4.3}$$

where $a^\mu(\tau)$ and $R_{\mu\nu\alpha\beta}(\tau)$ represent, respectively, the 4-acceleration and components of the Riemann curvature tensor in the Fermi normal coordinates evaluated at the point $\gamma(\tau)$. Finally, the red-shift factor $z(\tau)$, defined by the norm of the 1-form $\mathrm{d}\tau$ [34], is given by

$$z(\tau) = \left|g_{\tau\tau} - g^{ij}g_{\tau i}g_{\tau j}\right|^{1/2}. \tag{4.4}$$

4.2.2 Hamiltonian dynamics of a localized quantum system in curved spacetimes

In this section, we construct the Hamiltonian of a non-relativistic quantum particle with some internal degree of freedom in a curved spacetime around the time-like trajectory γ of the laboratory frame discussed in the previous section. It is worth mentioning that Ref. [34] provides a formal and elegant description of a localized quantum system in a curved spacetime using the Fermi normal coordinates, which we follow closely. However, our construction is also related to the formulation reported in Refs. [52, 35]. We start by giving the classical description of the Hamiltonian of a particle with some internal structure, which we later quantize.

Let us start by considering our quantum particle with some internal structure moving along its worldline, denoted by α, near the trajectory γ of the laboratory frame. Specifically, the timelike curve α lies within the local rest space $\mathcal{R}$ associated with the curve γ, where Fermi normal coordinates hold. See the sketch in Fig. 4.2. The particle's 4-momentum along the worldline α is represented by p^μ in this coordinate system. In contrast, in the particle's rest frame, described by primed coordinates $x^{\mu'}$, it is straight-

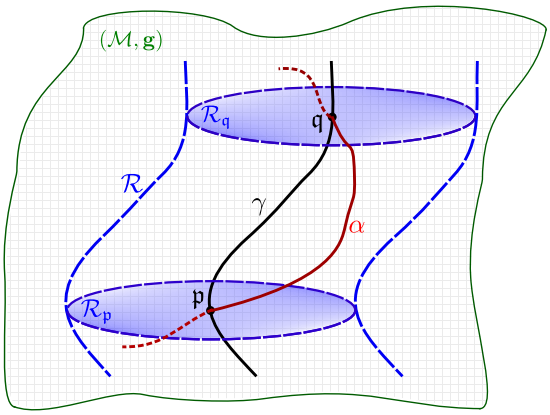

Figure 4.2: The world-line $\alpha \subset \mathcal{R}$ of the system and the laboratory world-line γ.

forward to show that $p^{j'} = (\partial x^{j'}/\partial x^\mu)p^\mu = 0$. This indicates that the total energy, as measured by a commoving observer, is given by $p_{t'}$ ($x^{0'} \equiv t'$). This total energy includes not only the contribution from the system's rest mass but also any kinetic or binding energies associated with the internal degrees of freedom, which are captured by the particle's internal Hamiltonian H_int. Hence

$$p_{t'} = m + H_\mathrm{int} \equiv H_\mathrm{rest}. \tag{4.5}$$

On the other hand, p^τ characterizes the motion of the particle relative to the laboratory frame, as described by the Fermi coordinate system. This includes contributions from both the internal and external degrees of freedom. As a result, it represents the total Hamiltonian of the system with respect to the coordinates (τ, x^1, x^2, x^3), and we denote it as $H = p_\tau$. From $p_\mu p^\mu = p_{\mu'} p^{\mu'}$, the following relation is obtained:

$$H = \sqrt{\frac{g^{t't'} H_\mathrm{rest}^2 - p_j p^j}{g^{\tau\tau}}}. \tag{4.6}$$

By selecting the component $x^{t'}$ associated with the commoving observer as the proper time along the particle's world line, it is possible to express the Hamiltonian as

$$H = \sqrt{-\frac{H_\mathrm{rest}^2 + p_j p^j}{g^{\tau\tau}}} = z(\tau)\sqrt{H_\mathrm{rest}^2 + p_j p^j}, \tag{4.7}$$

once $g^{t't'} = \eta^{tt} = -1$ and $z(\tau) = (-g^{\tau\tau})^{1/2} = |g^{\tau\tau}|^{1/2}$ are the redshift factors given in Eq. (4.4).

Now, for a non-relativistic particle such that the internal and kinetic energies are

much smaller than its rest energy, we can expand the redshift factor as [34]

$$z(\tau) \approx 1 + a_i(\tau)x^i + \frac{1}{2}R_{\tau i\tau j}(\tau)x^i x^j, \tag{4.8}$$

and Hamiltonian as $\sqrt{H_{\text{rest}}^2 + p_j p^j} \approx H_{\text{rest}} + \frac{p^2}{2H_{\text{rest}}}$, where $p^2 \equiv p_j p^j$, which gives us

$$H(\tau) = H_{\text{cm}}(\tau) + \mathcal{Z}(\tau)H_{\text{int}}, \tag{4.9}$$

with $\mathcal{Z}(\tau)$ being defined as

$$\mathcal{Z}(\tau) \equiv 1 - \frac{p^2}{2m^2} + a_i(\tau)x^i + \frac{1}{2}R_{\tau i\tau j}(\tau)x^i x^j. \tag{4.10}$$

Moreover, H_{cm} is the Hamiltonian of the center-of-mass and it is given by

$$H_{\text{cm}}(\tau) = m + \frac{p^2}{2m} + ma_i(\tau)x^i + \frac{m}{2}R_{\tau i\tau j}(\tau)x^i x^j. \tag{4.11}$$

In this limit, we can interpret the last two terms of Eq. (4.11) as perturbations to the (flat space-time) free Hamiltonian $H_0 = m + \frac{p^2}{2m}$. The quantity $\mathcal{Z}(\tau)$ defined by Eq. (4.10) can be interpreted as the time dilation factor between the proper time τ of the laboratory frame and the proper time t' of the system of interest, once

$$\frac{\mathrm{d}t'}{\mathrm{d}\tau} = \sqrt{-g_{\tau\tau} - g_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}} \approx 1 - \frac{p^2}{2m^2} + a_i(\tau)x^i + \frac{1}{2}R_{\tau i\tau j}(\tau)x^i x^j = \mathcal{Z}(\tau). \tag{4.12}$$

If the particle being considered is a quantum system with internal degrees of freedom, we can apply a quantization procedure to the classical framework by associating a Hilbert space $\mathcal{H}_{\text{cm}} \otimes \mathcal{H}_{\text{int}}$, which represents the composite Hilbert space of the center-of-mass and internal degrees of freedom. The total Hamiltonian H, as defined in Eq. (4.9), is a Hermitian operator that acts within this composite Hilbert space. Since the redshift factor $z(\tau)$ is a function of the space-like Fermi coordinates, and the time dilation factor $\mathcal{Z}(\tau)$ is a function of the space-like Fermi coordinates and the momentum of the center of mass, the quantization procedure promotes the space dependence of $z(\tau)$ to a function of the position operator and $\mathcal{Z}(\tau)$ to a function of the position and momentum operators, acting on the Hilbert space of the center-of-mass $\mathcal{H}_{\text{cm}}$ and potentially on the Hilbert space of the internal degrees of freedom $\mathcal{H}_{\text{int}}$. For instance, if the internal degrees of freedom are relative position with respect to the center of mass, then it also depends on the relative position between the world-line of the laboratory

frame and the internal degree of freedom and, therefore, $\mathcal{H}_{\text{int}} \simeq L^2(\mathcal{R}_\tau)$ and $\mathcal{Z}(\tau)$ act on the composite Hilbert space. On the other hand, if the internal degrees of freedom are the spin, then $\mathcal{Z}(\tau)$ will act only in the Hilbert space of the center of mass.

Using this framework, we can express the Schrödinger equation that governs the unitary evolution of our quantum system. Given that the quantum state of our system is represented by $|\psi\rangle \in \mathcal{H}_{\text{cm}}^{(\tau)} \otimes \mathcal{H}_{\text{int}}$, where $\mathcal{H}_{\text{cm}}^{(\tau)} \simeq L^2(\mathcal{R}_\tau)$ with $L^2(\mathcal{R}_\tau)$ being the space of square-integrable functions over the rest space $\mathcal{R}_\tau$. Here, τ denotes the proper time along the worldline γ of the laboratory frame, which is used to label a specific point $\mathfrak{p}$ on this worldline. Therefore, the Schrödinger equation takes the form

$$i\partial_\tau |\psi\rangle = H(\tau) |\psi\rangle = (H_{\text{cm}}(\tau) + \mathcal{Z}(\tau) H_{\text{int}}) |\psi\rangle . \tag{4.13}$$

From the above equation, the global unitary evolution operator U can be defined by $U(\tau) |\psi(\tau = 0)\rangle = |\psi(\tau)\rangle$, such that $U : \mathcal{H}_{\text{cm}}^{(0)} \otimes \mathcal{H}_{\text{int}} \to \mathcal{H}_{\text{cm}}^{(\tau)} \otimes \mathcal{H}_{\text{int}}$, also satisfying the Schrödinger equation.

4.3 Entropy production in a curved spacetime

In this section, we describe the protocol employed to derive the detailed fluctuation theorem under the two-point measurement scheme for a localized quantum system in a curved spacetime, which was first presented in Ref. [3].

Let us begin by recalling that a spacetime $(\mathcal{M}, \mathrm{g})$ is said to be time-orientable if it is possible to consistently distinguish between future-directed and past-directed timelike vectors throughout the entire manifold [48]. In our case, we only require that at least a portion of the spacetime $(\mathcal{M}, \mathrm{g})$ be time-orientable, specifically the local rest space $\mathcal{R} \subset \mathcal{M}$ associated with the curve γ. This assumption holds because we consider γ to be a timelike curve representing the worldline of the laboratory frame, which is oriented toward the future. Furthermore, from the curve γ and its local rest space $\mathcal{R}$, we derived the Hamiltonian (4.9), which governs the system's evolution and establishes a concept of time flow in the sense described by Connes and Rovelli [8], which makes it possible to define local thermal equilibrium states [2, 3, 8, 40, 39].

Moving on, we introduce two protocols: one corresponding to the forward direction of time and the other to the reverse direction. The difference between these two processes will serve as a measure of the irreversibility of the forward process [**?**]. In both cases, the system is initially prepared in an equilibrium state, its energy is measured, it then evolves under a specific quantum map, and finally, its energy is measured again. From the outcomes of these measurements, we can quantify entropy produc-

tion.

As illustrated in Fig. 4.2, the forward process starts at the point $\mathfrak{p} = \gamma(\tau = 0) \in \mathcal{M}$, in the intersection of the curves α and γ, where the observer performs the first projective energy measurement. We assume that the state of our system is given by $\rho_0 = |x_0\rangle\langle x_0| \otimes \sigma_0$, with $|x_0\rangle$ representing the state of the external degrees of freedom while the internal degrees of freedom are described by the thermal state $\sigma_0 = e^{-\beta\mathfrak{h}(0)}/Z_0$, with $\mathfrak{h}(0) = \mathcal{Z}(0)H_{\text{int}}$ and $Z_0 = \text{Tr}\{e^{-\beta\mathfrak{h}(0)}\}$ being the initial Hamiltonian of the internal degrees of freedom and the partition function, respectively. The inverse temperature is represented by β. The measurement performed in the eigenbasis of $\mathfrak{h}(0)$ results in the eigenvalue ϵ_l^0 with probability $p_l = e^{-\beta\epsilon_l^0}/Z_0$. Right after this measurement, the state is updated to a localized non-relativistic quantum system in curved spacetimes: a general characterization of particle detector mode $|\Psi_0\rangle = |x_0\rangle \otimes |\epsilon_l^0\rangle$.

The second step consists in letting the quantum system move along its worldline α, with the evolution dictated by the Hamiltonian (4.9), such that the state of the system evolves to

$$\begin{aligned} |\Psi(\tau)\rangle &= \mathcal{T} e^{-i\int_\alpha (H_{\text{cm}}(\tau)+\mathcal{Z}(\tau)H_{\text{int}})\text{d}\tau} |x_0\rangle \otimes \left|\epsilon_l^0\right\rangle \\ &= \mathcal{T}_{\text{cm}} e^{-i\int_\alpha H_{\text{cm}}(\tau)\text{d}\tau} |x_0\rangle \otimes \mathcal{T}_{\text{int}} e^{-i\int_\alpha \mathfrak{h}(\tau)\text{d}\tau} \left|\epsilon_l^0\right\rangle, \end{aligned}$$

where $\mathcal{T} \equiv \mathcal{T}_{\text{cm}} \otimes \mathcal{T}_{\text{int}}$ is the time-ordering operator and $\mathfrak{h}(\tau) = \mathcal{Z}(\tau)H_{\text{int}}$. The last equality follows from the semiclassical approximation, in which the motion of the quantum particle along its worldline is well-defined. This implies that the internal state evolves accordingly to $U \equiv U(\tau) = \mathcal{T}_{\text{int}} e^{-i\int_\alpha \mathfrak{h}(\tau)\text{d}\tau}$.

The third, and final, step is realized when the system intersects again the laboratory frame at the point $\mathfrak{q} = \gamma(\tau = T) \in \mathcal{M}$, where a final projective measurement with respect to the internal Hamiltonian $\mathfrak{h}(T) = \mathcal{Z}(T)H_{\text{int}}$ is done. Hence, from the definition of work as the stochastic variable $W_{k,l} \equiv \epsilon_k^T - \epsilon_l^0$, we can construct the work probability distribution density of the forward process as $P_{\text{fwd}}(W) = \sum_{k,l} p_{k,l}\delta\left[W - W_{k,l}\right]$, where $p_{k,l} = p_l p_{k|l}$ is the joint probability of obtaining ϵ_l^0 in the first measurement and ϵ_k^T in the second one. It follows that

$$P_{\text{fwd}}(W) = \sum_{j,k} \delta\left(W - (\epsilon_k^T - \epsilon_l^0)\right) \frac{e^{-\beta\epsilon_l^0}}{Z_0} \left|\left\langle\epsilon_k^0\right| U \left|\epsilon_l^0\right\rangle\right|^2. \tag{4.14}$$

For the reverse process, let us remember that, given a time orientation in the region $\mathcal{R} \subset \mathcal{M}$, general relativity does not forbid us to define past-directed curves in $\mathcal{R}$ [**?**]. Since the curves γ and α parameterized by τ in Fig 4.2 are directed towards the future, then we can obtain past-directed curves γ' and α' by making $\tau \to -\tau$. Hence,

the reverse process starts at the point $\mathfrak{q} = \gamma(T) \in \mathcal{M}$, where a first projective energy measurement is realized with the state of the system given by $\rho_T = \Theta \left|x_T\right\rangle\!\left\langle x_T\right| \otimes \sigma_T \Theta^\dagger$. Here, Θ is the anti-unitary time-reversal operator [43]. The internal degrees of freedom are supposed to be in the thermal state $\sigma_T = e^{-\beta \mathfrak{h}(\tau)}/Z_T$, as usual.

Under the assumption that the Hamiltonian (4.9) is time-reversal invariant, the time-reversal evolution is then governed by the micro-reversibility principle, i.e., $\tilde{U} \equiv \tilde{U}(T-\tau) = \Theta(\mathcal{T}_{\text{int}} e^{-i\int_\alpha \mathfrak{h}(\tau)\mathrm{d}\tau})^\dagger \Theta^\dagger$ [7]. The last step is then realized when the system intersects the laboratory frame at point $\mathfrak{p} = \gamma(0) \in \mathcal{M}$, where the final projective energy measurement with respect to the internal Hamiltonian $\mathfrak{h}(0)$ is done. The work probability distribution density of the reverse process is

$$P_{\text{rev}}(-W) = \sum_{j,k} \delta\left((\epsilon_k^T - \epsilon_l^0) - W\right) \frac{e^{-\beta\epsilon_k^T}}{Z_T} \left|\left\langle \epsilon_l^0 \right| \tilde{U} \left|\epsilon_k^0\right\rangle\right|^2, \tag{4.15}$$

which allows us to obtain the Crooks relation in a curved spacetime, i. e.,

$$\frac{P_{\text{fwd}}(W)}{P_{\text{rev}}(-W)} = e^{\beta(W-\Delta F)}. \tag{4.16}$$

Moreover, by integrating Eq. (4.16) over the probability distributions, we obtain the Jarzynski equality

$$\left\langle e^{-\beta W} \right\rangle_{\alpha,\gamma} = e^{-\beta \Delta F}, \tag{4.17}$$

where the subscripts above remind us that the joint probability distribution depends on the path α the system follows through spacetime, on the acceleration of the curve γ, and on the components of the curvature tensor evaluated at the curve γ. Therefore, Eq. (4.17) represents the extent to which the system deviates from this initial equilibrium state during its journey along its trajectory in curved spacetime.

Let us notice that, if H_{int} depends on τ, the results given by Eqs. (4.16) and (4.17) still hold, with the addition of contributions from the driven component of the Hamiltonian, which alters the rate of entropy production. Moreover, by neglecting the internal degrees of freedom, we can derive Eq. (4.17) for the center-of-mass degrees of freedom by considering a localized quantum system within the local rest space $\mathcal{R}$ of the curve γ. Then, the total Hamiltonian is the center-of-mass Hamiltonian given by Eq. (4.11) and the projective energy measurements are realized with respect to $H_{\text{cm}}(\tau)$.

4.4 The newtonian limit and the equivalence principle

To provide more concrete results, we now examine the Newtonian limit, which was first derived in Ref. [2]. This limit applies to the case of a weak and static gravitational field, like the one near Earth's surface, where the gravitational field can be considered uniform. The spacetime metric of this scenario can be expressed as

$$\mathrm{d}s^2 = -(1+2gx)\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \tag{4.18}$$

where $g = GM/R^2$ is Earth's gravitational acceleration at the origin of the laboratory frame ($x = 0$), which is located a distance R from Earth's center.

The equivalence principle states that an observer under uniform acceleration in empty spacetime is equivalent to an observer at rest under the action of a uniform gravitational field. Therefore, in this case, we can consider that the Fermi normal coordinates are just usual Newtonian coordinates $\{\tau = t, x, y, z\}$ such that, from Eq. (4.3), all the components of the Riemann curvature tensor vanish and the only nonzero component of the observer' acceleration is given by $a_x = g$.

Now, let us consider that our quantum system is a free semiclassical system with the center of mass having a well defined trajectory in spacetime while its internal degrees of freedom are described quantum mechanically. Moreover, the internal degrees of freedom are a two level quantum system such that $\mathcal{Z}(\tau)$ will act only in the Hilbert space of the center of mass, as e. g., a spin-$1/2$ particle. In this case, the total Hamiltonian of the system is given by Eq. (4.9), where the Hamiltonian of the center-of-mass is given by

$$H_{\text{cm}} = m + \frac{p^2}{2m} + 2mgx, \tag{4.19}$$

while the time dilation factor is

$$\mathcal{Z} = 1 - \frac{p^2}{2m} + 2gx. \tag{4.20}$$

Going back to the work protocol discussed in the previous section, given that $\mathfrak{h}(0) = H_{\text{int}}$ is the initial internal Hamiltonian at the point $\mathfrak{p} \in \mathcal{M}$, one defines $H_{\text{int}}\left|\epsilon_m^0\right\rangle = \epsilon_m^0\left|\epsilon_m^0\right\rangle$. Since the particle is initially at rest in $x = 0$, we have $\mathcal{Z} = 1$. After the preparation, a projective energy measurement (defined by H_{int}) is performed on the system and, if the eigenvalue ϵ_m^0 is obtained, the state of the system just after the measurement is given by $\left|\epsilon_m^0\right\rangle$. The external degrees of freedom do not enter here since we are considering the semiclassical approximation. We let the system evolve by following the

trajectory α on spacetime. After a certain amount of proper time τ, the second energy measurement is performed, according to the final internal Hamiltonian, resulting in the eigenvalue

$$\epsilon_m^\tau = \mathcal{Z}\epsilon_m^0 \approx \left(1 + 2gx - \frac{p^2}{2m^2}\right)\epsilon_m^0. \tag{4.21}$$

From this, we can define the work as

$$W_{n,m} = \epsilon_n^\tau - \epsilon_m^0 = \left(\mathcal{Z}\epsilon_n^0 - \epsilon_m^0\right)\delta_{n,m}, \tag{4.22}$$

representing a shift in the energy eigenvalues.

Now, let us consider that our two-level quantum system is such that $\epsilon_0^0 = 0$ and $\epsilon_1^0 = \epsilon > 0$. Then, we can calculate the dissipated work defined by $W_{\text{diss}} = \langle W\rangle - \Delta F$, where $\langle W\rangle = \text{Tr}\{\mathfrak{h}(\tau)\rho_\tau\} - \text{Tr}\{\mathfrak{h}(0)\rho_0\}$ is the average work, $\rho_0 = e^{-\beta\mathfrak{h}(0)}/Z_0$ is the initial thermal state, and $\rho_\tau = e^{-\beta\mathfrak{h}(\tau)}/Z_\tau$ is the final thermal state. The average entropy production associated with the dissipated work can be defined as $\Sigma = \beta W_{\text{diss}}$, which gives us

$$\Sigma = (\mathcal{Z} - 1)\beta\epsilon + \ln\left(\frac{1 - e^{-\mathcal{Z}\beta\epsilon}}{1 - e^{-\beta\epsilon}}\right). \tag{4.23}$$

It is important to observe here that the entropy production Σ can be negative, positive, or zero, depending on whether $\mathcal{Z}$ is less than, greater than, or equal to one, respectively. Of course, the sign of Σ has the same sign as the dissipated work W_{diss}, which depends on whether the quantum system is doing work on the spacetime or if spacetime is doing work on the quantum system. Moreover, if we consider the system at rest together with the laboratory frame, we obtain $\mathcal{Z} = 1$ and, consequently, $\Sigma = \beta W_{\text{diss}} = 0$. For this case, we do not expect any entropy to be produced since there will be no measurable time dilation.

On the other hand, if $\mathcal{Z} > 1$, then $\Sigma = \beta W_{\text{diss}} > 0$. In our example, this occurs in the limit of large m, where the kinematic degrees of freedom become negligible, resulting in $\mathcal{Z} = 1+2gx$. This is similar to the well-known fact that a photon must expend energy to overcome the gravitational (metric) potential.

The last possibility happens for trajectories such that $\mathcal{Z} < 1$, implying that $\Sigma = \beta W_{\text{diss}} < 0$. Note that, in this case, both $\langle W\rangle_{\alpha,\gamma}$ and $\Delta F_{\alpha,\gamma}$ are negative quantities. However, the energy flux can be defined as positive regardless of the direction, whether it's from the system to the field or from the field to the system. As a result, we arrive at the same conclusion when accounting for the redshift scenario.

4.5 The expanding universe

The second example that we discuss is a quantum mechanical harmonic oscillator (QHO) in an expanding universe as depicted in Fig. 4.3. This example was selected to demonstrate that our protocol applies to the center-of-mass degrees of freedom while neglecting the internal ones.

An isotropic and homogeneous expanding universe with zero spatial curvature is described by the following metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathfrak{a}^2(t)\Big(\mathrm{d}X^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2\Big), \tag{4.24}$$

in the Friedmann-Robertson-Walker (FRW) coordinates, where $\mathfrak{a}(t)$ is the scale factor. The worldline of our laboratory frame is the geodesic γ described by $\tau = \lambda$ and $X = Y = Z = 0$. This implies that the acceleration of our laboratory frame is zero. The relation between the FRW coordinates $\{t, X, Y, Z\}$ and the Fermi normal coordinates $\{\tau, x, y, z\}$ is given by [9]

$$t = \tau - \frac{\dot{\mathfrak{a}}}{2\mathfrak{a}} r^2 + \mathcal{O}(r^4), \quad X^i = \frac{x^i}{\mathfrak{a}}\Big(1 + \frac{\dot{\mathfrak{a}}^2}{3\mathfrak{a}^2} r^2\Big) + \mathcal{O}(r^4), \tag{4.25}$$

where $r \equiv \delta_{ij} x^i x^j$ and $\dot{\mathfrak{a}} = \mathrm{d}\mathfrak{a}/\mathrm{d}t$. The transformation above allows us to rewrite the metric in terms of the Fermi normal coordinates, i.e.,

$$\mathrm{d}s^2 = -\Big(1 - \frac{\ddot{\mathfrak{a}}}{\mathfrak{a}} r^2\Big)\mathrm{d}\tau^2 + \Big[\delta_{ij} - \frac{\dot{\mathfrak{a}}^2}{\mathfrak{a}^2}\Big(\frac{r^2\delta_{ij} - x_i x_j}{3}\Big)\Big]\mathrm{d}x^i \mathrm{d}x^j, \tag{4.26}$$

which implies that the components of the curvature in the Fermi normal coordinates are given by

$$R_{\tau x \tau x} = R_{\tau y \tau y} = R_{\tau z \tau z} = -\frac{\ddot{\mathfrak{a}}}{\mathfrak{a}}, \tag{4.27}$$

$$R_{xyxy} = R_{xzxz} = R_{yzyz} = \frac{\dot{\mathfrak{a}}^2}{\mathfrak{a}^2}. \tag{4.28}$$

Given that the stage is set, our system is a one-dimensional quantum harmonic oscillator, where the energy eigenvalues are given by $\epsilon_n^0 = (n + 1/2)\omega$ for the unperturbed Hamiltonian $H_0 = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2$, which is in thermal equilibrium at a point $\mathfrak{p} = \gamma(0)$ of the worldline of the laboratory frame. At this point, a projective measurement is performed in the eigenbasis $|\epsilon_m^0\rangle$ of H_0. The probability of measuring the eigenvalue ϵ_m^0 is $p_m = e^{-\beta\epsilon_m^0}/Z_0$, where $Z_0 = 2/\sinh\big(\frac{1}{2}\beta\omega\big)$.

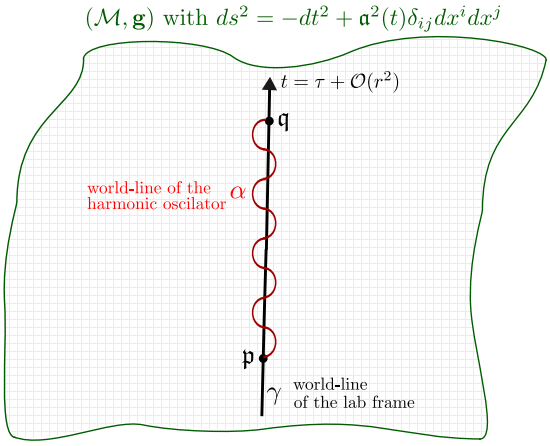

Figure 4.3: Harmonic oscillator in an expanding universe.

The next step involves allowing the quantum system to evolve under the following Hamiltonian

$$H(\tau) = H_0 + \frac{1}{2}mR_{\tau x \tau x}(\tau)x^2 = H_0 - \frac{m\ddot{\mathfrak{a}}}{2\mathfrak{a}}x^2. \tag{4.29}$$

From the equation above, we can see that the curvature of the spacetime makes the frequencies time-dependent, i.e.,

$$\omega(\tau) = \sqrt{\omega_0^2 - \frac{\ddot{\mathfrak{a}}}{\mathfrak{a}}}. \tag{4.30}$$

Therefore, the energy of this oscillator is not conserved, which leads to non-zero transition probabilities between the initial and final energy states, as we shall see.

In the interaction picture, the evolution of our system can be described by

$$i\partial_\tau \left|\psi\right\rangle_I = V_I(\tau)\left|\psi\right\rangle_I, \tag{4.31}$$

with $V_I(\tau) = e^{iH_0\tau}V(\tau)e^{-iH_0\tau}$, $V(\tau) = \frac{1}{2}mR_{\tau x \tau x}(\tau)x^2$, and $\left|\psi\right\rangle_I = e^{iH_0\tau}\left|\psi\right\rangle$. By expanding $\left|\psi\right\rangle_I = \sum_n c_n(\tau)\left|\epsilon_n^0\right\rangle$, we have

$$c_n(\tau) = -i\sum_m \int_0^\tau \left\langle \epsilon_n^0 \right| V(\tau') \left| \epsilon_m^0 \right\rangle e^{i(n-m)\tau'} c_m(\tau')\mathrm{d}\tau'. \tag{4.32}$$

Since the initial state of our system is $\left|\epsilon_m^0\right\rangle$, we have

$$c_n(\tau) = -\frac{im}{2}\left\langle \epsilon_n^0 \right| x^2 \left| \epsilon_m^0 \right\rangle f(\tau), \tag{4.33}$$

Since $x = \sqrt{\frac{1}{2m\omega}}(A + A^\dagger)$ with A and $A^\dagger$ being the annihilation and the creation operators, we have

$$\left\langle \epsilon_n^0 \right| x^2 \left| \epsilon_m^0 \right\rangle = \frac{1}{2m\omega}\Big(\sqrt{m(m-1)}\delta_{n,m-2} + (2m+1)\delta_{n,m} + \sqrt{(m+1)(m+2)}\delta_{n,m+2}\Big), \tag{4.34}$$

and

$$f(\tau) = \int_0^\tau R_{\tau x \tau x}(\tau') e^{i(k-l)\omega_0 \tau'} \mathrm{d}\tau' = -\int_0^\tau \frac{\ddot{\mathfrak{a}}}{\mathfrak{a}} e^{i(k-l)\omega_0 \tau'} \mathrm{d}\tau'. \tag{4.35}$$

From the equations provided, it is evident that the transition probability, $p^\tau_{n|m} = |c_{n\neq m}(\tau)|^2$, is influenced by the curvature of the expanding universe, which leads to the generation of entropy, as it is directly associated with the changes in population distributions.

In the universe dominated only by a positive cosmological constant Λ, i. e., the de Sitter spacetime, the Einstein field equation can be rewritten $G_{\mu\nu} = -\Lambda g_{\mu\nu}$, which corresponds to the energy-momentum tensor of a perfect fluid such that $p_\Lambda = -\rho_\Lambda = -\frac{\Lambda}{8\pi}$, where p_Λ is the isotropic pressure and ρ_Λ is the positive energy density. In this case, the scale factor is given by $\mathfrak{a}(t) = e^{\mathbb{H}t}$, where $\mathbb{H} = \frac{\dot{\mathfrak{a}}}{\mathfrak{a}} = \sqrt{\Lambda/3}$ is the Hubble parameter and we have $\frac{\ddot{\mathfrak{a}}}{\mathfrak{a}} = \mathbb{H}^2$. Then, the transition probability for $n \neq m$ is given by

$$p^\tau_{n|m} = 4\Big(\frac{m\mathbb{H}^2}{2}\Big)^2 \frac{\left|\langle \epsilon_n^0 | x^2 | \epsilon_m^0 \rangle\right|^2}{|\epsilon_n^0 - \epsilon_m^0|^2} \sin^2\Big(\frac{(n-m)\omega t}{2}\Big). \tag{4.36}$$

Thus, we observe that, in this scenario, the transition probability explicitly depends on the Hubble parameter or the cosmological constant, leading to the generation of entropy. For example, if we initially consider the system in its ground state, the only permissible transition is to the second excited state. A direct calculation reveals that Eq. (4.36) gives us $p^\tau_{2|0} = (\mathbb{H}/\sqrt{2}\omega_0)^4 \sin\omega_0 t$. Given that $\mathbb{H} \approx 10^{-61} t_p^{-1}$, where $t_p = 5.391 \times 10^{-44} s$ is the Planck time, and $\omega_0 \approx 10^{-30} t_p^{-1}$ ($\omega_0 \approx 10^{13} s^{-1}$) for typical molecular vibrational modes, the ratio $\mathbb{H}/\omega_0$ is of order 10^{-31}. This indicates that while the transition probability is very small, it is nonetheless non-zero.

4.6 Concluding remarks

This chapter discussed the detailed fluctuation theorem for a localized quantum system living in a general curved spacetime, which reveals how the spacetime curvature can produce entropy.

In order to better understand the role of entropy production due to the curvature of spacetime, let us resort to the gravito-electromagnetic analogy discussed, for instance, in Refs. [10, 41] and define the gravito-electric potential as $\phi(\tau) \equiv -\frac{1}{2}R_{\tau i\tau j}(\tau)x^i x^j$, such that the gravito-electric field (up to linear order in x^i) is given by $E_i(\tau) = R_{\tau i\tau j}(\tau)x^j$. The contribution of the gravito-magnetic potential in the gravito-electric field is second order and therefore will not be considered in our analysis. Thus, we can describe the term $\frac{m}{2}R_{\tau i\tau j}(\tau)x^i x^j$ that appears in Eq. (4.11) as $mE_i(\tau)x^i$, while the term $\frac{1}{2}R_{\tau i\tau j}(\tau)x^i x^j H_{int}$ that appears in $\mathcal{Z}(\tau)H_{\text{int}}$ can be written as $H_{\text{int}}E_i(\tau)x^i$. It is noteworthy the similarity of these two terms with the electric dipole interaction, with both m and H_{int} playing the role of the charge of the gravitational field, which is reasonable since (internal) energy also gravitates in general relativity. Therefore, we can interpret the terms $mE_i(\tau)x^i$ and $H_{\text{int}}E_i(\tau)x^i$ as the gravitational analogue of a charged quantum system interacting with a time-dependent electric field.

Our main result implies that entropy production is not an invariant quantity defined solely by the system. Rather, it depends on the observer who measures it, since it depends on the worldline of the laboratory in an arbitrary spacetime. This is a robust result that goes in the same direction as those discussed in Refs. [50, 30] regarding the subtleties of defining entropy in a curved spacetime. Specifically, two different families of observers will not agree on the entropy production in general. It is worth remembering that, for comparison, each family of observers has to realize the same protocol, since the measurements in the energy basis are locally performed.

Additionally, our findings establish a deep and fundamental link between the time-orientability of the laboratory frame's worldline γ and the production of entropy and, therefore, with the thermodynamic arrow of time. This is the precise meaning of the observer-dependent nature of entropy production. Such orientability is needed in order to obtain the Hamiltonian (4.9), which governs the evolution of the quantum system and thereby defines a notion of time flow and thermal equilibrium reference states, as discussed in Refs. [8, 40, 39]. For instance, in the quantum harmonic oscillator in an expanding universe, the curvature drives the quantum system out of equilibrium due to the last term in Eq. (4.29), causing the change in the populations. Therefore, our result shows that the arrow of time is rooted in the causal structure of spacetime.

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Appendices

Appendix A

Basic definitions

The main goal of this appendix is to provide some basic mathematical definitions supporting the main text. We present the definitions of vector, affine, and topological spaces, discussing some of their properties that are relevant for the understanding of the classical structure of spacetime. The idea of algebras, which are vector spaces with an additional structure, is also presented for completeness.

1.1 Vector Spaces

In order to introduce the concept of vector spaces, we need to define what is called a **field**, which is a non-empty set $\mathbb{F}$ together with two binary operations called addition ($+ : \mathbb{F} \times \mathbb{F} \mapsto \mathbb{F}$) and multiplication ($* : \mathbb{F} \times \mathbb{F} \mapsto \mathbb{F}$) that satisfy the following axioms for all $a, b \in \mathbb{F}$

- Associativity of addition and multiplication: $a+(b+c) = (a+b)+c$ and $a*(b*c) = (a*b)*c$
- Commutativity of addition and multiplication: $a + b = b + a$ and $a * b = b * a$
- Additive and multiplicative identity: There exist two different elements 0 and 1 in $\mathbb{F}$ such that $a + 0 = a$ and $1 * a = a$.
- Additive inverse: For every $a \in \mathbb{F}$, there exists the element $-a$, called the additive inverse of a, such that $a + (-a) = 0$
- Multiplicative inverse: For every $a \neq 0$ in $\mathbb{F}$, there exists an element, denoted by a^{-1}, called the multiplicative inverse of a, such that $a * a^{-1} = 1$
- Distributivity of multiplication over addition: $a * (b + c) = (a * b) + (a * c)$

The set $\mathbb{R}$ under the usual addition and multiplication of real numbers is an example of a field. The set of complex numbers $\mathbb{C}$ is $\mathbb{R}^2$, that is, the set of all vectors with two coordinates (x, y) with $x,\ y \in \mathbb{R}$. By defining addition and multiplication as $(x, y) + (u, v) = (x + u, y + v)$ and $(x, y) * (u, v) = (xu - yv, xv + yu)$, $\mathbb{C}$ is also a field. The set $\mathbb{Z}$ of integers is not a field since not all elements have a multiplicative inverse that belongs to the set.

Now, a **vector space** over a field $\mathbb{F}$ is the set $\mathbb{X}$ together with two operations: (i) Addition $(+ : \mathbb{X} \times \mathbb{X} \to \mathbb{X})$ and (ii) scalar multiplication $(* : \mathbb{F} \times \mathbb{X} \to \mathbb{X})$. Such operations satisfy the following axioms for all $u, v, w \in \mathbb{X}$ and all $a, b \in \mathbb{F}$.

- Associativity: $x + (y + z) = (x + y) + z$
- Commutativity: $x + y = y + x$
- Identity element for addition: There exists an element $0 \in \mathbb{X}$ such that $x + 0 = x$ for all $x \in \mathbb{X}$.
- Inverse element for addition: For every $x \in \mathbb{X}$ there exists an element $-v \in \mathbb{X}$ such that $x + (-x) = 0$
- Compatibility: $a * (b * x) = (ab) * x$
- Identity element for scalar multiplication: $1 * x = x$.
- Distributivity of scalar multiplication with respect to vector addition: $a*(x+y) = a * x + a * y$
- Distributivity of scalar multiplication with respect to field addition: $(a + b) * x = a * x + b * x$.

In general, the elements of $\mathbb{X}$ are called **vectors**, while the ones belonging to $\mathbb{F}$ are called **scalars**. The simplest example of a vector space over a field $\mathbb{F}$ is the field itself. By performing addition and scalar multiplication pointwise, functions from any fixed set to $\mathbb{F}$ also form a vector space.

If $\mathbb{F}$ is a field, the Cartesian product $\mathbb{F}^n = \{(f_1, f_2, ..., f_n), f_j \in \mathbb{F}\}$ is a vector space over the field $\mathbb{F}$ with the addition operation defined as $(f_1, ..., f_n) + (g_1, ..., g_n) = (f_1 + g_1, ..., f_n + g_n)$ $(g_j \in \mathbb{F})$ and the product as $\alpha * (f_1, ..., f_n) = (\alpha * f_1, ..., \alpha * f_n)$ for all $\alpha \in \mathbb{F}$. The null vector is denoted by $(0, ..., 0)$.

The set $\mathbb{Z}$ is not a vector space since it is not closed under scalar multiplication. Also, the set of all polynomials of degree n is not a vector space since it is not closed under addition.

When considering geometry, we are often interested in properties that are invariant under the action of some symmetry group and then we can model a set of points

(the space-time, for instance) as a vector space. However, this procedure has some disadvantages and we have a more appropriate construction, called **affine space**. First, in a vector space, the point 0, called the origin, plays a very important role, which is not a good thing when we are considering physical theories since we do not want preferred points. Moreover, vector spaces and affine spaces have very different geometrical properties. In the first case, bijective[1] linear maps keep the geometry invariant, while affine maps[2] (which form a much bigger set) are allowed in the last case. Moreover, affine spaces present a very interesting property for physics, since they allow us to handle geometry in an intrinsic manner that is independent of the coordinate system.

Let us define the affine space as the triple $\left(\mathbb{X}, \overrightarrow{\mathbb{X}}, +\right)$, with $\mathbb{X}$ being a set of points, $\overrightarrow{\mathbb{X}}$ a vector space and $(+ : \mathbb{X} \times \overrightarrow{\mathbb{X}} \mapsto \mathbb{X})$ a bilinear operation satisfying the following properties:

- $a + 0 = a$ for all $a \in \mathbb{X}$
- $(a + x) + y = a + (x + y)$ for all $a \in \mathbb{X}$ and all $x,\, y \in \overrightarrow{\mathbb{X}}$
- for any two points $a,\, b \in \mathbb{X}$, there is a unique $x \in \overrightarrow{\mathbb{X}}$ such that $a + x = b$.

In this definition, $\overrightarrow{\mathbb{X}}$ is called the set of free vectors, or free translations.

Given an $m \times n$ matrix A and a vector $b \in \mathbb{R}$, the set of solutions to the equation $Ax = b$, for $x \in \mathbb{R}^m$, is an affine space. Newtonian space-time is a four-dimensional affine space on which two additional structures are defined. A linear functional t called time and a Euclidean metric on each affine subspace defined by the vectors to which t assigns 0 (the simultaneity hypersurfaces).

Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. A map $l : \mathbb{F} \mapsto \mathbb{F}$ is a linear functional if

$$l(\alpha x + \beta y) = \alpha l(x) + \beta l(y), \tag{1.1}$$

for all $x, y \in \mathbb{V}$ and $\alpha, \beta \in \mathbb{F}$. The set of all linear functionals of $\mathbb{V}$ is the **dual space** of $\mathbb{V}$, denoted as $\mathbb{V}^*$.

The relation between $\mathbb{V}$ and $\mathbb{V}^*$ can be stated as follows. There exists at least one injective map $\phi : \mathbb{V} \mapsto \mathbb{V}^*$ in such a way that $\mathbb{V}$ is isomorphic[3] to its image under ϕ,

[1] The map $f : \mathbb{X} \mapsto \mathbb{Y}$ is injective if $x \neq x'$ implies $f(x) \neq f(x')$ for any $x, x' \in \mathbb{X}$. It is called surjective if for each $y \in \mathbb{Y}$ there exists $x \in \mathbb{X}$ such that $f(x) = y$. It is called bijective if it is both injective and surjective.

[2] A geometric transformation that preserves lines and parallelism, but not necessary angles and distances.

[3] Let us supposed that the sets $\mathbb{X}$ and $\mathbb{Y}$ are endowed with a certain algebraic structure (multiplication, for instance). If the map $f : \mathbb{X} \mapsto \mathbb{Y}$ preserves such structure, it is called an homomorphism. If f is bijective, is is called an isomorphism and the spaces are isomorphic to each other, $\mathbb{X} \cong \mathbb{Y}$.

$\mathbb{V} \cong \phi(\mathbb{V}) \subset \mathbb{V}^*$. If $\dim(\mathbb{V})$ is finite, then $\mathbb{V} \cong \mathbb{V}^*$. In the case $\dim(\mathbb{V})$ is infinite, there can be elements of $\mathbb{V}^*$ without such identification. In general, we have $\phi(\mathbb{V}) \subset \mathbb{V}^*$.

1.2 Topological spaces

Topological spaces are the most general mathematical spaces that allow us to define notions like limits and continuity. In general, topology is interested in the relations between points and regions and plays a fundamental role in general relativity.

Let us then define the **topological space**. Let $\mathbb{X}$ be a set and $\mathbb{T}$ a collection of subsets of $\mathbb{X}$ satisfying the following properties.

1. The union of an arbitrary collection of subsets, each of which is in $\mathbb{T}$, is in $\mathbb{T}$. If $\mathbb{O}_\alpha \in \mathbb{T}$ for all α, then $\cup_\alpha \mathbb{O}_\alpha \in \mathbb{T}$.

2. The intersection of a finite number of subsets of $\mathbb{T}$ is in $\mathbb{T}$. If $\{\mathbb{O}_i\}_{i=1}^n \in \mathbb{T}$, then $\cap_{i=1}^n \mathbb{O}_i \in \mathbb{T}$.

3. The entire set $\mathbb{X}$ and the empty set $\emptyset$ are in $\mathbb{T}$.

$\mathbb{X}$ is said to be a topological space, and $\mathbb{T}$ provides a topology to $\mathbb{X}$.

There is also a definition, due to Felix Hausdorff, in terms of neighborhoods of a point. Let $\mathbb{X}$ be a set. Let $\mathcal{N}$ be a function assigning to each $a \in \mathbb{X}$ a non-empty collection $\mathbb{N}(x)$ of subsets of $\mathbb{X}$. The elements of $\mathbb{N}(x)$ are called neighborhoods of x with respect to $\mathcal{N}$. The function $\mathcal{N}$ is called a neighborhood topology if the following axioms are satisfied:

1. If $\mathbb{N}$ is a neighborhood of x (i. e., $\mathbb{N} \in \mathbb{N}(x)$), then $x \in \mathbb{N}$. In other words, each point of the set $\mathbb{X}$ belongs to every one of its neighborhoods with respect to $\mathcal{N}$.

2. If $\mathbb{N}$ is a subset of $\mathbb{X}$ and includes a neighborhood of x, then $\mathbb{N}$ is a neighborhood of x. Every superset[4] of a neighborhood of a point $x \in \mathbb{X}$ is again a neighborhood of x.

3. The intersection of two neighborhoods of x is a neighborhood of x.

4. Any neighborhood $\mathbb{N}$ of x includes a neighborhood $\mathbb{M}$ of x such that $\mathbb{N}$ is a neighborhood of each point of $\mathbb{M}$

Then, $(\mathbb{X}, \mathcal{N})$ is called a topological space.

[4] A set $\mathbb{A}$ is a subset of a set $\mathbb{B}$ if all elements of $\mathbb{A}$ are also elements of $\mathbb{B}$; $\mathbb{B}$ is then a superset of $\mathbb{A}$.

A standard example of such a system of neighborhoods is for the real line $\mathbb{R}$, where a subset $\mathbb{N}$ of $\mathbb{R}$ is defined to be a neighborhood of a real number x if it includes an open interval containing x.

Let $\mathbb{X} = \{a, b, c\}$ and $\mathbb{T} = \{\mathbb{X}, \emptyset, \{a, b\}, \{b\}, \{b, c\}\}$. Then, the pair $(\mathbb{X}, \mathbb{T})$ is a topological space. Of course, there are many other topologies that can be chosen here by simply permuting a, b, and c. If $\mathbb{X}$ is any set, the collection of all subsets of $\mathbb{X}$ is called the discrete topology, while the set $\{\mathbb{X}, \emptyset\}$ is called the trivial topology. The set $\mathbb{R}$ along with all the open intervals (a, b) and their unions is a topological space.

If a metric[5] $d(x, y)$, with $x, y \in \mathbb{X}$, is defined on the set $\mathbb{X}$, its open sets are given by the open discs $\mathbb{O}_\epsilon = \{y \in \mathbb{X} \,|\, d(x, y) < \epsilon\}$ and all their possible unions. Such topology is called a metric topology determined by d and the topological space $(\mathbb{X}, \mathbb{T})$ is called a **metric space**.

From these notions we can define continuous maps between topological spaces. If $(\mathbb{X}, \mathbb{T}_x)$ and $(\mathbb{Y}, \mathbb{T}_y)$ are topological spaces, a map $f : \mathbb{X} \to \mathbb{Y}$ is said to be continuous if the inverse image (preimage) $f^{-1}[\mathbb{O}] = \{x \in \mathbb{X} \,|\, f(x) \in \mathbb{O}\}$ of every open set in $\mathbb{O} \subset \mathbb{Y}$ is an open set in $\mathbb{X}$.

A very important set of these open maps is the **homeomorphism**, which is continuous, one-to-one (distinct elements are mapped into distinct elements) and onto (every element of the domain is mapped into one of the elements of the codomain). The inverse map is also continuous. In this case, $(\mathbb{X}, \mathbb{T}_x)$ and $(\mathbb{Y}, \mathbb{T}_y)$ are said to be homeomorphic topological spaces, which means that they have identical topological properties. Homeomorphism is an equivalence relation.

From the physical point of view, we need to impose another condition on topological spaces in order for them to be able to describe space-times. The intuition behind this idea is that we need unique vector flows and, thus, unique limits (curves do not split into multiple curves). **Hausdorff** spaces present such property. A topological space is said to be Hausdorff if for each pair of distinct points $x,\ y \in \mathbb{X}$, $x \neq y$, one can find open sets $\mathbb{O}_x\,, \mathbb{O}_y \in \mathbb{T}$ such that $x \in \mathbb{O}_x$, $y \in \mathbb{O}_y$ and $\mathbb{O}_x \cap \mathbb{O}_y = \emptyset$. We can mention some important consequences of this definition. First, every finite set in a Hausdorff space is closed. Secondly, a sequence of points in a Hausdorff space converges to at most one point in the space. Moreover, the product of two Hausdorff spaces is also Hausdorff, and every subspace of a Hausdorff space is Hausdorff.

Another important property is **compactness**, which is the generalization of the Euclidean closed systems to topological spaces. Let $\mathbb{A}$ be a subset of $\mathbb{X}$ and $\{\mathbb{O}_\alpha\}$ a collection of open sets. If the union of these sets contains $\mathbb{A}$, $\{\mathbb{O}_\alpha\}$ is said to be an open cover of $\mathbb{A}$. A subcollection of the sets $\{\mathbb{O}_\alpha\}$ which also covers $\mathbb{A}$ is referred to as a subcover.

[5] $d : \mathbb{X} \times \mathbb{X} \mapsto \mathbb{R}$ under the conditions *i)* $d(x, y) = d(y, x)$, *ii)* $d(x, y) \geq 0$, with the equality holding only for $x = y$ and *iii)* $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in \mathbb{X}$.

If each of its covers has a finite subcover, the topological space is said to be compact. According to the Heine-Borel theorem for Euclidean spaces, compactness is equivalent to the set being closed and bounded.

Now, in order to study general relativity, we need a Lorentzian metric, which requires a Riemannian metric and the metric space must be **paracompact**. Let $(\mathbb{X}, \mathbb{T}_x)$ be a topological space and $\{\mathbb{O}_\alpha\}$ be an open cover of $\mathbb{X}$. An open cover $\{\mathbb{V}_\beta\}$ is said to be a refinement of $\{\mathbb{O}_\alpha\}$ if for each $\mathbb{V}_\beta$ there exists an $\mathbb{O}_\alpha$ such that $\mathbb{V}_\beta \subset \mathbb{O}_\alpha$. The cover $\{\mathbb{V}_\beta\}$ is said to be locally finite if each $x \in \mathbb{X}$ has an open neighborhood $\mathbb{W}$ such that only finitely many $\mathbb{V}_\beta$ satisfy $\mathbb{W} \cap \mathbb{V}_\beta \neq \emptyset$. A topological space is said to be paracompact if every open cover $\{\mathbb{O}_\alpha\}$ of $\mathbb{X}$ has a locally finite refinement $\{\mathbb{V}_\beta\}$. This last condition is necessary in order for the topological space to be homeomorphic to a metric space.

The space-time in general relativity is postulated as a differentiable manifold that is Hausdorff and paracompact. Moreover, it must be **connected**, which is a property that says that the topological space cannot be covered by the union of two or more disjoint non-empty open subsets.

1.3 Algebras

An algebra is a vector space $\mathbb{V}$ (over a field $\mathbb{F}$) along with a binary operation $\cdot$, the product of the algebra, such that the following properties are satisfied for all $a, b, c \in \mathbb{V}$ and $\alpha \in \mathbb{F}$.

- Distributivity with respect to the vectorial addition: $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$.
- Commutativity with respect to the scalar product.

Now we present some important examples of algebras that are commonly found while studying physics.

Lie algebra $\mathbb{L}$ — The product of $\mathbb{L}$ is denoted as $[a, b]$, with $a, b \in \mathbb{L}$, and it must satisfy the following properties: $i)$ $[a, a] = 0$, which implies $[a, b] = -[b, a]$ and $ii)$ the Jacobi identity $[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0$. The set $\mathbb{R}^3$ with the usual cross product is an example of a Lie algebra, as well as the set of all $n \times n$ matrices over the field $\mathbb{F}$, Mat$(\mathbb{F}, n)$, under the product $[A, B] = AB - BA$, with $A, B \in$ Mat$(\mathbb{F}, n)$.

Poisson algebra $\mathbb{P}$ — Is the vector space $\mathbb{P}$ (over the field $\mathbb{F}$) along with two products, $*$ and $\{,\}$ such that: $i)$ $\mathbb{P}$ is associative with respect to $*$, $ii)$ $\mathbb{P}$ is a Lie algebra with respect to $\{,\}$ and $iii)$ for all $a, b, c \in \mathbb{P}$, the Leibniz identity holds $\{a, b * c\} = \{a, b\} * c + b * \{a, c\}$. Given two C^∞ functions $f(x, p) : \mathbb{R}^2 \mapsto \mathbb{R}$ and $g(x, p) : \mathbb{R}^2 \mapsto \mathbb{R}$,

the Poisson brackets

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} \tag{1.2}$$

makes the set of all C^∞ a Lie algebra.

Jordan algebra $\mathbb{J}$ — The product of the algebra satisfies, for all $a, b \in \mathbb{J}$, the following properties: i) commutativity, $a \cdot b = b \cdot a$, and ii) the Jordan identity $(a \cdot a) \cdot (a \cdot b) = a \cdot ((a \cdot a) \cdot b)$. The set of all self-adjoint matrices of Mat$(\mathbb{C}, n)$ with the Jordan product

$$a \cdot b = \frac{1}{2}(a \cdot b + b \cdot a) \tag{1.3}$$

is a Jordan algebra.

Grassmann algebra $\Gamma(\mathbb{V})$ — Let $\mathbb{V}$ be a vector space over the field $\mathbb{F}$. A Grassmann algebra over $\mathcal{V}$ is an associative and unital[6] algebra over $\mathbb{F}$, with the product $\wedge$ satisfying the following properties: i) $\mathbb{V}$ is a subspace of $\Gamma(\mathbb{V})$ and ii) for all $v \in \mathbb{V}$, we have $v \wedge v = 0$.

Clifford algebra $Cl(\mathbb{V}, \omega)$ — Let $\mathbb{V}$ be a vector space over the field $\mathbb{F}$. Let ω be a symmetric bilinear form[7] over $\mathbb{V}$. A Clifford algebra over $\mathbb{V}$ and ω is an associative algebra with the unity e, such that the following properties hold: i) $\mathbb{V}$ is a subspace of $Cl(\mathbb{V}, \omega)$ and ii) for all $v \in \mathbb{V}$, $v^2 = \omega(v, v)e$. The set of Pauli matrices σ_x, σ_y and σ_z, with $\mathbb{V} = \mathbb{R}^3$ and $\omega(u, v) = \sum_{a,b} u_a v_b \delta_{a,b}$, where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ is a Clifford algebra. If $\mathbb{V}$ is the Minkowski spacetime and $\omega = \sum_{\mu\nu} u_\mu v_\nu \eta^{\mu\nu}$, the set of Dirac matrices γ^μ is a Clifford algebra. u_μ represents the components of the four-vector u (the same applying for v) while $\eta_{\mu\nu}$ is the usual Minkowski metric.

[6] The algebra is called unital if there is the neutral element $\mathbb{1}$ such that $\mathbb{1} \cdot a = a \cdot \mathbb{1} = a$ for all a in the algebra.

[7] See Chapter 3 for details on bilinear forms.

Appendix B

A bit of calculus on $\mathbb{R}^n$

This Appendix has the simple goal of defining vectors as directional derivatives in $\mathbb{R}^n$, motivating in this way such a definition in a general manifold presented in Chapter 3. Let us then start by stating a few definitions.

Let $\{x^\mu\} = \{x^0, x^1, \cdots, x^{n-1}\}$ be some coordinate system in $\mathbb{R}^n$ and $p \in \mathbb{U} \subset \mathbb{R}^n$ a point in the open set $\mathbb{U}$. A function $f : \mathbb{U} \mapsto \mathbb{R}$ is C^k at p if the partial derivatives

$$\frac{\partial f}{\partial x^{i_1} \cdots \partial x^{i_j}},$$

exist and are continuous[1] at p for all $j \leq k$. A vector valued function $\mathbf{f} : \mathbb{U} \mapsto \mathbb{R}^m$ ($m \leq n$) is said to be C^k at $p \in \mathbb{U}$ if all of its component functions $\{f^0, f^1, \cdots, f^{m-1}\}$ are of C^k at p.

A neighborhood of a point p in $\mathbb{R}^n$ is an open set containing p. A function f is analytic at p if in some neighborhood of p it is equal to its Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-p)^n}{n!} \left.\frac{\partial^n f}{\partial x^n}\right|_{x=p}.$$

It is clear that an analytic function is necessary C^∞. We are now in a position to define tangent vectors.

Let $\mathbb{T}_p(\mathbb{R}^n)$ be the **tangent space** of $\mathbb{R}^n$ at point p. A vector $\mathrm{v} \in \mathbb{T}_p(\mathbb{R}^n)$ can be defined as

$$\mathrm{v} = \sum_{\mu=1}^{n} v^\mu e_\mu \equiv v^\mu e_\mu,$$

[1] A function $f : \mathbb{U} \mapsto \mathbb{R}$ is said to be continuous at a point $p \in \mathbb{U}$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in \mathbb{U}$: $|x - p| < \delta$ implies $|f(x) - f(p)| < \epsilon$.

with e_μ representing some coordinate basis vector.

Now, let $f \in C^\infty$ in a neighborhood of p, $\mathrm{v} \in \mathbb{T}_p(\mathbb{R}^n)$ and $c(t) = (p^0 + tv^0, \cdots, p^{n-1} + tv^{n-1})$ a curve through p parametrized by t. $p^\mu + tv^\mu$ is the μ-th coordinate of the curve. The **directional derivative** of f in the direction of v at p is defined as

$$\mathrm{D_v}f = \lim_{t\to\infty} \frac{f(c(t)) - f(p)}{t} = \left.\frac{\mathrm{d}f(c(t))}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\mathrm{d}c^\mu(t)}{\mathrm{d}t}\right|_{t=0} \left.\frac{\partial f(c(t))}{\partial x^\mu}\right|_{\mathrm{x}=p} = v^\mu \left.\frac{\partial f(c(t))}{\partial x^\mu}\right|_{\mathrm{x}=p}.$$

Note that $\mathrm{D_v}f$ is a number, not a function. $\mathrm{D_v} : f \in C^\infty \mapsto \mathrm{D_v}f \in \mathbb{R}$ is the directional derivative operator. Any linear map $\mathrm{D} : C^\infty \mapsto \mathbb{R}$ satisfying the Leibniz rule is called a derivation. We denote by $\mathrm{D}_p\mathbb{R}^n$ the set of all the derivations at p.

Now, since $\mathrm{D}_p\mathbb{R}^n$ is a vector space, the map

$$\phi : \mathbb{T}_p\mathbb{R}^n \mapsto \mathrm{D}_p\mathbb{R}^n$$

is an isomorphism. This shows that we can identify the tangent vectors at p with the derivations at p. Under this isomorphism, the standard basis e_μ for $\mathbb{T}_p\mathbb{R}^n$ corresponds to the basis

$$\left\{ \left.\frac{\partial}{\partial x^\mu}\right|_p \right\}_{\mu=0}^{n-1},$$

which allows us to write any tangent vector as

$$\mathrm{v} = v^\mu \left.\frac{\partial}{\partial x^\mu}\right|_p.$$

From this, we can define the **vector field** as the function that assigns to every point $p \in \mathbb{U} \subset \mathbb{R}^2$ a tangent vector $\mathrm{X}_p \in \mathrm{T_pR^n}$

$$\mathrm{X}_p = \alpha^\mu(p) \left.\frac{\partial}{\partial x^\mu}\right|_p \qquad \alpha^\mu \in \mathbb{R}.$$

If X is a vector field on $\mathbb{U}$ and f is a C^∞ on $\mathbb{U}$, we can write

$$(\mathrm{X}f)_p \equiv \mathrm{X}_p f = \alpha^\mu(p) \left.\frac{\partial f}{\partial x^\mu}\right|_p,$$

which shows that $\mathrm{X}f$ is a C^∞ function on $\mathbb{U}$. A basic principle in the theory of manifolds is that every manifold can be locally approximated by a tangent space at a given point. This is the way linear algebra enters into manifold theory.

To close this Appendix, we define the differential form. The **cotangent space** to $\mathbb{R}^n$ at p is defined to be the dual space of $\mathbb{T}_p\mathbb{R}^n$, denoted by $\mathbb{T}_p^*\mathbb{R}^n$ and whose elements are called covectors, or linear functionals on $\mathbb{T}_p\mathbb{R}^n$.

A **covector field** on an open subset $\mathbb{U} \subset \mathbb{R}^n$ is a function ω that assigns to every point $p \in \mathbb{U}$ a covector $\omega_p \in \mathbb{T}_p^*\mathbb{R}^n$

$$\omega : \mathbb{U} \mapsto \cup_{p\in\mathbb{U}} \mathbb{T}_p^*\mathbb{R}^n \qquad p \mapsto \omega_p \in \mathbb{T}_p^*\mathbb{R}^n.$$

ω is also called a differential 1-form, or 1-form for short.

For any $f \in C^\infty$ we can construct a 1-form called the differential of f, denoted by $\mathrm{d}f$ as follows. For $p \in \mathbb{U}$ and $\mathrm{X}_p \in \mathbb{T}_p\mathbb{U}$ define

$$(\mathrm{d}f)_p \, \mathrm{X}_p - \mathrm{X}_p f.$$

Now, we choose $\{x^\mu\}$ as a coordinate system on $\mathbb{R}^n$ and, at each point $p \in \mathbb{R}^n$, $\{\mathrm{d}x^\mu\}$ a basis for the cotangent space $\mathbb{T}_p^*\mathbb{R}^n$, dual to the basis e_μ for the tangent space $\mathbb{T}_p\mathbb{R}^n$

$$(\mathrm{d}x^\mu)_p \left(\left. \frac{\partial}{\partial x^\nu} \right|_p \right) = \left. \frac{\partial x^\mu}{\partial x^\nu} \right|_p = \delta^\mu_\nu,$$

with δ^μ_ν being the Kronecker delta symbol. It follows that each covector can be written as

$$\omega_p = \alpha_\mu(p) \mathrm{d}x^\mu|_p \equiv \alpha_\mu \mathrm{d}x^\mu.$$

Appendix C

Riemann Normal Coordinates

Let $(\mathbb{M}, g)$ be a smooth n-dimensional Riemannian (or pseudo-Riemannian) manifold. For any point p of the spacetime, the **exponential map** at p,

$$\exp_p : \mathbb{T}_p\mathbb{M} \to \mathbb{M},$$

is defined as follows: for any vector $\mathrm{v} \in \mathbb{T}_\mathrm{p}\mathbb{M}$, $\exp_p(\mathrm{v})$ is the point on the spacetime manifold reached at unit affine parameter[1] along the unique geodesic γ_v satisfying

$$\gamma_v(0) = p, \qquad \dot{\gamma}_v(0) = v, \qquad \nabla_{\dot{\gamma}}\dot{\gamma} = 0. \tag{3.1}$$

The Riemann normal coordinates (RNC) centered at p are defined as follows:

- Choose an orthonormal basis $\{e_i\}$ for the tangent space $\mathbb{T}_p\mathbb{M}$.
- For each tangent vector $\mathrm{v} = x^i e_i \in \mathbb{T}_p\mathbb{M}$, define a point $q \in \mathbb{M}$ by:

$$q = \exp_p(\mathrm{v}).$$

- The coordinates of q are given by x^i, and vary over an open neighborhood around 0 in $\mathbb{T}_p\mathbb{M}$.

In these coordinates:

- The point p corresponds to the origin, $x^i = 0$,
- The metric satisfies $g_{ij}(0) = \delta_{ij}$ (or η_{ij} in Lorentzian signature),

[1] An affine parameterization for a curve is a parameterization by a parameter τ such that the parametric equations for the curve satisfy the geodesic equation.

- The first derivatives of the metric vanish at the origin: $\partial_k g_{ij}(0) = 0$.

As a consequence of the last condition, we must have $\Gamma^i_{jk}(0) = 0$. These coordinates are extremely useful in local geometric analysis and in deriving curvature-related quantities such as the volume of small geodesic balls.

We can now write any metric around the origin of this coordinate system.

3.1 Expansion of the Metric Tensor

Let us compute the Taylor expansion of the metric tensor near p in RNC. Since the first derivatives vanish, the expansion begins at second order.

$$g_{ij}(x) = g_{ij}(0) + \frac{1}{2}\partial_k\partial_l g_{ij}(0)\, x^k x^l + \mathcal{O}(|x|^3). \tag{3.2}$$

To express $\partial_k\partial_l g_{ij}(0)$ in terms of curvature, we use the relation between the second derivatives of the metric and the Riemann curvature tensor.

The Riemann curvature tensor

$$R^i{}_{jkl} = \partial_k\Gamma^i_{jl} - \partial_l\Gamma^i_{jk} + \Gamma^i_{km}\Gamma^m_{jl} - \Gamma^i_{lm}\Gamma^m_{jk},$$

at the origin, where $\Gamma^i_{jk}(0) = 0$, takes the form

$$R^i{}_{jkl}(0) = \partial_k\Gamma^i_{jl}(0) - \partial_l\Gamma^i_{jk}(0).$$

Now, from the expression for Γ^i_{jk} we obtain

$$\partial_k\Gamma^i_{jl} = \frac{1}{2}g^{im}\left(\partial_k\partial_j g_{lm} + \partial_k\partial_l g_{jm} - \partial_k\partial_m g_{jl}\right).$$

which implies

$$\partial_k\partial_l g_{ij}(0) = -\frac{1}{3}\left(R_{ikjl} + R_{jkil}\right). \tag{3.3}$$

Because of the symmetries of the Riemann tensor, and because $g_{ij} = g_{ji}$, this ensures that $g_{ij}(x)$ remains symmetric.

Putting everything together, we obtain

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3}R_{ikjl}(0)x^k x^l + \mathcal{O}(|x|^3).$$

This is the second-order expansion of the metric tensor in Riemann normal coordinates.

3.2 The volume of a ball

Now, the set of tangent vectors $\mathrm{v} \in \mathbb{T}_{\mathrm{p}}\mathbb{M}$ can be classified as timelike, null, or spacelike depending on the sign of $\mathrm{g}(\mathrm{v},\mathrm{v})$. By restricting our attention to **spacelike geodesics**, we can define a ball as

$$B^{(\text{spacelike})}(p,\varepsilon) = \left\{ \exp_p(\mathrm{v}) \in \mathbb{M} \;\middle|\; \mathrm{v} \in \mathbb{T}_{\mathrm{p}}\mathbb{M},\ \mathrm{g}_{\mathrm{p}}(\mathrm{v},\mathrm{v}) = +\sigma^2 < \varepsilon^2 \right\},$$

where $\sigma = \sqrt{\mathrm{g}_{\mathrm{p}}(\mathrm{v},\mathrm{v})}$ is the proper length in the tangent space.

This construction defines a local spacelike hypersurface centered at p, analogous to an instantaneously comoving Euclidean 3-ball in the local rest frame of an observer at p. The observer's rest space is the 3-dimensional subspace orthogonal to a unit timelike vector u (their 4-velocity) satisfying $\mathrm{g}_{\mathrm{p}}(\mathrm{u},\mathrm{u}) = \mathrm{u}^\mu \mathrm{u}_\mu = -1$. Let us then compute the volume of this ball.

Consider the 3-dimensional spacelike hypersurface Σ orthogonal to u^μ at p, equipped with the induced Riemannian metric h_{ij}. This induced metric defines a Levi-Civita connection and a Ricci scalar $R^{(3)}$. Let $B(p,\varepsilon) \subset \Sigma$ denote the geodesic ball of radius ε centered at p, defined using the exponential map on Σ. In local Riemann normal coordinates x^i centered at $p \in \Sigma$, the metric takes the form

$$h_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{ikjl}(p) x^k x^l + \mathcal{O}(x^3),$$

while the volume element takes the form

$$\sqrt{\det h(x)} = 1 - \frac{1}{6} R_{kl}(p) x^k x^l + \mathcal{O}(x^3),$$

where $R_{kl} = R^i{}_{kil}$ is the Ricci tensor of h_{ij}, and we have used the identity

$$\det(h) = 1 - \frac{1}{3} R_{ij} x^i x^j + \mathcal{O}(x^3) \quad \Rightarrow \quad \sqrt{\det h} = 1 - \frac{1}{6} R_{ij} x^i x^j + \mathcal{O}(x^3).$$

We compute the volume of the ball $B(p,\varepsilon)$ by integrating over the Euclidean ball in

$\mathbb{R}^3$

$$\begin{aligned} V(\varepsilon) &= \int_{B(0,\varepsilon)} \sqrt{\det h(x)}\mathrm{d}^3x = \int_{B(0,\varepsilon)} \left(1 - \frac{1}{6}R_{ij}x^i x^j + \mathcal{O}(x^3)\right)\mathrm{d}^3x \\ &= \frac{4}{3}\pi\varepsilon^3\left[1 - \frac{R}{30}\varepsilon^2\right], \end{aligned} \tag{3.4}$$

where we have employed space isotropy $R_{ij}x^i x^j = Rr^2/3$. This follows since, in an isotropic space, R_{ij} must be proportional to the metric δ_{ij}, $R_{ij} = (R/3)\delta_{ij}$ and $r^2 = \delta_{ij}x^i x^j$.

Therefore, the curvature of spacetime causes small deviations in the spatial volume of a freely falling local neighborhood. These deviations are governed, to leading order, by the Ricci scalar curvature of the induced 3-metric. In other words, the Ricci scalar curvature measures how much the volume of a sphere deviates from that in an Euclidean space.

This result also holds *locally* in the Lorentzian case, with the understanding that: i) The volume is computed on a spacelike hypersurface orthogonal to a timelike observer; ii) The curvature quantities are evaluated with respect to the induced Riemannian 3-metric on that slice; iii) The Ricci scalar R remains a contraction of the Ricci tensor.

Thus, for an observer in free fall, the volume of a small spacelike ball of geodesics initially comoving with them is given by

$$V(\varepsilon) = \frac{4}{3}\pi\varepsilon^3\left[1 - \frac{R(p)}{30}\varepsilon^2 + \mathcal{O}(\varepsilon^4)\right] \quad \text{(in 3 spatial dimensions)}.$$

The deviation from flat volume reflects the presence of spacetime curvature. This effect is interpreted physically as follows:

- If $R > 0$, spacelike geodesics initially moving apart (in the local rest frame) begin to reconverge. The geodesic ball has **less** volume than in flat space. This signals a net tidal compression of spatial volume.
- If $R < 0$, the geodesic ball has **greater** volume than in flat space. This reflects a local tidal stretching effect.

Thus, the Ricci scalar governs the *average* volume expansion (or contraction) rate of an initially small and isotropic distribution of geodesics. It directly encodes the strength of isotropic tidal forces at a point and is responsible for compressing or stretching volumes of matter or light rays in spacetime. In other words, the Ricci scalar, via its effect on geodesic ball volumes, measures how curvature focuses or disperses matter in space.

This insight is particularly important in gravitational collapse, cosmology, and gravitational lensing, where convergence or divergence of spacelike (and null) congruences encodes physically observable distortions of matter or light.