Information Theory — Foundations and Applications

Quantum Typicality

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Introduction

Quantum typicality underpins the asymptotic theory of quantum information.

Most of the ideas from classical typicality can be applied here, but there are important differences, starting from the definition of a quantum information source.

Moreover, in order to determine if a classical sequence is typical we need to look at the sequence. In the quantum case, looking at the system will in general change the system. So, care must be taken here.

The main idea is to gentle ask the system if its state is typical or not, without returning any other information. So, if the state is typical, we expect to not disturb the system too much.

The quantum information source

> It is a device that randomly outputs quantum states (not necessarily distinguishable) according to some probability distribution.

Suppose that the source emits the state $|\psi_y\rangle$ according to some probability distribution $p_Y(y)$. The density operator ρ_A of the source is the expected emitted state

$$\rho_A = \mathbb{E}_Y\{|\psi_y\rangle\langle\psi_y|\} = \sum_y p_Y(y)\,|\psi_y\rangle\langle\psi_y|$$

It is important to observe that the quantum states $|\psi_y\rangle$ are not orthogonal. However, we know that the same quantum state can be written as

$$\rho_A = \sum_x p_X(x)\,|x\rangle\langle x|$$

with $|x\rangle$ and $p_X(x)$ are the eigenstates and the eigenvalues of ρ_A

The quantum information source

Since the density operator is an equivalence class of indistinguishable ensembles, both descriptions are equivalent

$$\{p_Y(y), |\psi_y\rangle\} \quad \text{and} \quad \{p_X(x), |x\rangle\}$$

However, the states $|x\rangle$ are distinguishable. This implies that

$$S(A)_{\rho_A} = H(X)$$

Now, suppose that the source emits a large number n of random states

$$\rho_{A^n} = \rho_{A_1} \otimes \rho_{A_2} \otimes \cdots \otimes \rho_{A_n} = (\rho_A)^{\otimes n}$$

This implies we are within the i.i.d setting in the quantum domain

The typical subspace

Due to orthonormality of $|x\rangle$, we can write

$$\rho_{A^n} = \sum_{x^n \in \mathcal{X}^n} p_{X^n}(x^n) \, |x^n\rangle\langle x^n|_{A^n}$$

with $p_{X^n}(x^n) = p_X(x_1) \cdots p_X(x_n)$ and $|x^n\rangle_{A^n} = |x_1\rangle_{A_1} \otimes \cdots \otimes |x_n\rangle_{A_1}$

Typical subspace

The δ-typical subspace $T^{\delta}_{A^n}$ is a subspace of the full Hilbert space

$$\mathcal{H}_{A^n} = \mathcal{H}_{A_1} \otimes \cdots \otimes \mathcal{H}_{A_n},$$

associated with many copies of a density operator. It is spanned by states $|x^n\rangle_{A^n}$ whose corresponding classical sequences x^n are δ-typical

$$T^{\delta}_{A^n} = \left\{ |x^n\rangle_{A^n} \, : \, x^n \in T^{X^n}_{\delta} \right\}$$

The typical projector

The definition of the typical subspace allows us to split the full Hilbert space into two parts, the typical and the atypical subspaces.

Typical projector

The typical projector for the typical subspace of the density operator ρ_A is defined as

$$\Pi_\delta^{A^n} = \sum_{x^n \in T_\delta^{X^n}} |x^n\rangle\langle x^n|_{A^n}$$

This, along with the complementary projector, defines the gentle measurement mentioned before, that is able to answer the question if the state is typical or not.

Typical subspace measurement

The following map is a quantum instrument that realizes the typical subspace measurement on $\sigma \in \mathcal{H}_{A^n}$

Measurement

$$\sigma \mapsto \left(\mathbb{1} - \Pi_\delta^{A^n}\right) \sigma \left(\mathbb{1} - \Pi_\delta^{A^n}\right) \otimes |0\rangle\langle 0| + \Pi_\delta^{A^n} \sigma \Pi_\delta^{A^n} \otimes |1\rangle\langle 1|$$

It associates a classical register with the outcome of the measurement: the value of the classical register is $|0\rangle$ for the support of the state that is not in the typical subspace, and it is equal to $|1\rangle$ for the support of the state that is in the typical subspace.

Quantum Shannon theory is theoretical

The implementation of a typical subspace measurement is currently far from the reality of what is experimentally accessible if we would like to have the measure concentration effects necessary for proving many of the results in quantum Shannon theory.

We would need a millions of qubits emitted from a quantum information source, and furthermore, we would require the ability to perform noiseless coherent operations over about these qubits in order to implement the typical subspace measurement.

Properties of the typical subspace

Unit Probability. The probability that the quantum state ρ_{A^n} is in the typical subspace $T^{\delta}_{A^n}$ approaches one as n becomes large

$$\text{Tr}\left\{\Pi_{\delta}^{A^n}\rho_{A^n}\right\} \geq 1-\epsilon \quad \forall\, \epsilon \in (0,1) \text{ and } \delta > 0$$

Exponentially small cardinality. The dimension of the δ-typical subspace is exponentially smaller than the dimension of the entire space of quantum states

$$(1-\epsilon)2^{n(S(A)-c\delta)} \leq \text{Tr}\left\{\Pi_{\delta}^{A^n}\right\} \leq 2^{n(S(A)+c\delta)}$$

with c being a constant that depends on weather we choose the notions of weak or strong typicality.

Properties of the typical subspace

Equipartition. The operator $\tilde{\rho}_{A^n} = \Pi_\delta^{A^n} \rho_{A^n} \Pi_\delta^{A^n} / \mathrm{Tr}\{\Pi_\delta^{A^n} \rho_{A^n} \Pi_\delta^{A^n}\}$ corresponds to a "slicing" of the density operator ρ_{A^n} where we slice out and keep only the part with support in the typical subspace. Therefore

$$2^{-n(S(A)+c\delta)} \Pi_\delta^{A^n} \leq \Pi_\delta^{A^n} \rho_{A^n} \Pi_\delta^{A^n} \leq 2^{-n(S(A)-c\delta)} \Pi_\delta^{A^n}$$

which is a statement about the eigenvalues of $\tilde{\rho}_{A^n}$ and $\Pi_\delta^{A^n}$. This is therefore, equivalent to the classical inequality

$$2^{-n(H(X)+c\delta)} \leq p_{X^n}(x^n) \leq 2^{-n(H(X)-c\delta)}$$

for $x^n \in T_\delta^{X^n}$, since $p_{X^n}(x^n)$ are the eigenvalues of $\tilde{\rho}_{A^n}$. Moreover

$$||\rho_{A^n} - \tilde{\rho}_{A^n}||_1 \leq 2\sqrt{\epsilon}$$

Multipartite typicality

Tow classical sequences x^n and y^n are jointly typical if

- $\overline{H}(x^n, y^n) \to H(X, Y)$
- $\overline{H}(x^n) \to H(X)$ and $\overline{H}(y^n) \to H(Y)$

The way that we determine whether a quantum state is typical is by performing a typical subspace measurement. If we perform a typical subspace measurement of the whole system followed by such a measurement on the marginals, the resulting state is not necessarily the same as if we performed the marginal measurements followed by the joint measurements. For this reason, the notion of weak joint typicality as given in in the classical case does not really exist in general for the quantum case.

Bipartite case

Let us consider the state

$$\rho_{AB} = \sum_{z \in \mathcal{Z}} p_Z(z) \, |\psi_z\rangle\langle\psi_z|_{AB}$$

The extension of this state is (using the same compact notation as before)

$$(\rho_{AB})^{\otimes n} = \sum_{z^n \in \mathcal{Z}^n} p_{Z^n}(z^n) \, |\psi_{z^n}\rangle\langle\psi_{z^n}|_{A^n B^n} \equiv \rho_{A^n B^n}$$

Typical subspace of a bipartite system

The δ-typical subspace $T^{\delta}_{A^n B^n}$ of ρ_{AB} is the space spanned by states $|\psi_{z^n}\rangle_{A^n B^n}$ whose corresponding classical sequence z^n is in the typical set $T^{Z^n}_{\delta}$

$$T^{\delta}_{A^n B^n} = \text{span}\left\{ |\psi_{z^n}\rangle_{A^n B^n} \; : \; z^n \in T^{Z^n}_{\delta} \right\}$$

Bipartite case

Typical projector

$$\Pi^{\delta}_{A^n B^n} = \sum_{z^n \in T^{Z^n}_{\delta}} |\psi_{z^n}\rangle\langle\psi_{z^n}|_{A^n B^n}$$

There is no difference between the typical subspace for a bipartite state and the typical subspace for a single-party state because the spectral decomposition gives a way for determining the typical subspace and the typical projector in both cases.

However, the commutation problem still exists and we cannot define this notion in general.

Classical states

Let us consider the following class of states

$$\rho_{AB} = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p_{X,Y}(x,y) \, |x\rangle\langle x|_A \otimes |y\rangle\langle y|_B$$

with $|x\rangle_A$ and $|y\rangle_B$ orthonormal basis for A and B, respectively. This is a state that only has classical correlations. Its extension can be written

$$\rho_{A^n B^n} = \sum_{x^n \in \mathcal{X}^n} \sum_{y^n \in \mathcal{Y}^n} p_{X^n,Y^n}(x^n, y^n) \, |x^n\rangle\langle x^n|_{A^n} \otimes |y^n\rangle\langle y^n|_{B^n}$$

which leads us directly to the definition of jointly typicality.

Classical states

Jointly typical subspace

$$T^{\delta}_{A^n B^n} = \text{span}\left\{ |x^n\rangle_{A^n} |y^n\rangle_{B^n} \; : \; x^n y^n \in T^{X^n Y^n}_{\delta} \right\}$$

Jointly typical projector

$$\Pi^{\delta}_{A^n B^n} = \sum_{x^n \in T^{X^n}_{\delta}} \sum_{y^n \in T^{Y^n}_{\delta}} |x^n\rangle\langle x^n|_{A^n} \otimes |y^n\rangle\langle y^n|_{B^n}$$

Conditional typicality

We start by defining the notion of the conditional quantum information source.

Let us consider the random variable $X = \{p_X(x), \mathcal{X}\}$ and the quantum system with an orthonormal set $|x\rangle_{x\in\mathcal{X}}$ to represent the realizations of X.

- We generate the realization x of the random variable X
- We follow by generating a random quantum state according to some conditional distribution

This procedure gives us a set of $|\mathcal{X}|$ quantum information sources, whose expected density operators are ρ_B^x

These are the conditional quantum information sources.

Conditional typicality

By correlating the classical state $|x\rangle_X$ with the quantum state ρ_B^x we obtain the classical–quantum ensemble

$$\{p_X(x), |x\rangle\langle x|_X \otimes \rho_B^x\}_{x\in\mathcal{X}}$$

From which we can build the expected density operator of the ensemble

$$\rho_{XB} = \sum_{x\in\mathcal{X}} p_X(x)\, |x\rangle\langle x|_X \otimes \rho_B^x$$

whose conditional entropy is

$$S(B|X) = \sum_{x\in\mathcal{X}} p_X(x) S(\rho_B^x)$$

Conditional typicality

By defining $\{|y_x\rangle_B\}_{y\in\mathcal{Y}}$ and $\{p_{Y|X}(y|x)\}_{y\in\mathcal{Y}}$ as the sets of eigenvectors and eigenvalues of ρ_B^x, respectively, the conditional entropy takes the form

$$S(B|X) = \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p_X(x)p_{Y|X}(y|x)\log\frac{1}{p_{Y|X}(y|x)}$$

Now we need to go to the asymptotic limit

$$\rho_{X^nB^n} = \sum_{x^n\in\mathcal{X}^n} p_{X^n}(x^n)\,|x^n\rangle\langle x^n|_{X^n}\otimes\rho_{B^n}^{x^n}$$

with $\rho_{B^n}^{x^n} = \rho_{B_1}^{x_1}\otimes\cdots\otimes\rho_{B_n}^{x_n}$, whose spectral decomposition is

$$\rho_{B^n}^{x^n} = \sum_{y^n\in\mathcal{Y}^n} p_{Y^n|X^n}(y^n|x^n)\,|y^n_{x^n}\rangle\langle y^n_{x^n}|_B$$

Conditional typicality

Weak Conditionally Typical Subspace

The conditionally typical subspace $T^{\delta}_{B^n|x^n}$ corresponds to a particular sequence x^n and an ensemble $\{p_X(x), \rho^x_B\}$

$$T^{\delta}_{B^n|x^n} = \text{span}\left\{|y^n_{x^n}\rangle \ : \ \left|\overline{H}(y^n|x^n) - S(X|B)\right| \leq \delta\right\}$$

Weak Conditionally Typical Projector

The projector onto the conditional typical subspace $T^{\delta}_{B^n|x^n}$ is

$$\Pi^{\delta}_{B^n|x^n} = \sum_{y^n_{x^n} \in T^{Y^n|x^n}_{\delta}} |y^n_{x^n}\rangle\langle y^n_{x^n}|_{B^n}$$

Properties of the Weak Conditionally Typical Subspace

We cannot say many things considering a particular sequence x^n, but we can state several properties regarding the average over the random variable X^n.

Unity probability. The expectation of the probability that we measure a random quantum state $\rho_{B^n}^{X^n}$ to be in a conditionally typical subspace $T_{B^n|X^n}^{\delta}$ approaches one as n becomes large

$$\mathbb{E}_{X^n}\left\{\text{Tr}\left\{\Pi_{B^n|X^n}^{\delta}\rho_{B^n}^{X^n}\right\}\right\} \geq 1-\epsilon$$

for all $\epsilon \in (0,1)$ and $\delta > 0$.

Properties of the Weak Conditionally Typical Subspace

Exponentially Smaller Dimension. The dimension of the δ-conditionally typical subspace is exponentially smaller than the dimension of the entire space of quantum states for most classical–quantum sources

$$\text{Tr}\left\{\Pi^{\delta}_{B^n|x^n}\right\} \leq 2^{n(H(B|X)+\delta)}$$

We also have

$$\mathbb{E}_{X^n}\left\{\text{Tr}\left\{\Pi^{\delta}_{B^n|X^n}\right\}\right\} \geq (1-\epsilon)2^{n(H(B|X)-\delta)}$$

for all $\epsilon \in (0,1)$ and $\delta > 0$.

Properties of the Weak Conditionally Typical Subspace

Finally, we can prove the equipartition property.

Equipartition. The density operator $\rho^x_{B^n}$ looks approximately maximally mixed when projected to the conditionally typical subspace

$$2^{-n(H(B|X)+\delta)}\Pi^{\delta}_{B^n|x^n} \leq \Pi^{\delta}_{B^n|x^n}\rho^x_{B^n}\Pi^{\delta}_{B^n|x^n} \leq \Pi^{\delta}_{B^n|x^n}2^{-n(H(B|X)-\delta)}$$

Thank you for your attention

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