Information Theory — Foundations and Applications

Classical dynamical systems

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Motivation

Definition of **heat** and **work** are problematic in quantum mechanics. **Information** theory can be unambiguously extended to quantum mechanics. The idea of defining thermodynamic quantities in terms of **efficiency** of information processing could pave a new way for a deeper understanding not only of thermodynamics, but also on the limitations for processing information. And, possibly, to a **relativistic** formulation of thermodynamics.

Goal

To describe a dynamical system as a communication problem and, based on this description, to establish a lower bound on dissipation.

Some aspects of information and physics[3]

Jayne's principle
Equilibrium probability distribution maximizes information transfer in the measurement process. Statistical physics can be derived from information theory[1].

Landauer's principle
Algorithmic complexity and energy cost of computation (physical implementation of a process) are deeply related. Logical irreversibility implies physical irreversibility[2].

[1] E. T. Jaynes, Phys. Rev. **106**, 620 (1957)
[2] R. Landauer, IBM J. Res. Dev. **5**, 183 (1961).
[3] J. Goold, M. Huber, A. Riera, L. del Rio and P. Skrzypczyk, J. Phys. A **49**, 143001 (2016).

Some aspects of information and physics

Phase space

Driven classical Hamiltonian systems[4].

$$\langle W_{\text{diss}} \rangle \equiv \langle W \rangle - \Delta F = \beta^{-1} D\left(\rho_t || \tilde{\rho}_t\right)$$

Predictive power

Driven and dissipative systems. Unwarranted retention of past information is fundamentally equivalent to energetic inefficiency[5].

$$\langle W_{\text{diss}}\left(x_t \to x_{t+1}\right) \rangle = \beta^{-1}\left[I_{\text{mem}}(t) - I_{\text{pre}}(t)\right]$$

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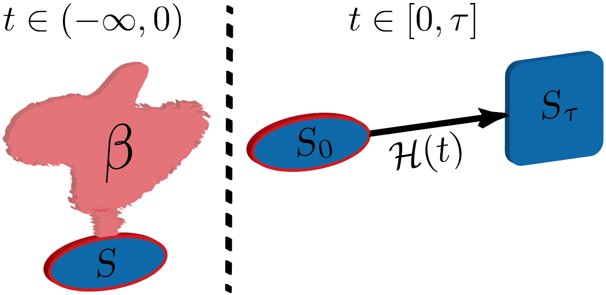
[4] R. Kawai, J. M. R. Parrondo and C. Van den Broeck, Phys. Rev. Lett. **98**, 080602 (2007).

[5] S. Still, D. A. Sivak, A. J. Bell and G. E. Crooks, Phys. Rev. Lett. **109**, 120604 (2012).

Our problem[6]

Is there any connection between randomness and dissipation? Yes! Based on a description of a dynamical system in terms of communication theory.



[6] M. Capela, M. Sanz, E. Solano and L. C. Céleri, Phys. Rev. E **98**, 052109 (2018)

Class of systems under consideration

Elements of the theory

- $\mathcal{H}\left(s_t;\lambda\right)$: System's Hamiltonian.
- λ: Set of external controlled parameters.
- $s_t=(q(t),\theta(t))$: Set of generalized coordinates and canonical conjugate momenta.
- Γ: The *finite-dimensional* phase-space.
- The system is initially in the state

$$\rho_0(s_0,\lambda_0)=\frac{e^{-\beta\mathcal{H}(s_0;\lambda_0)}}{Z(\lambda_0)} \qquad Z(\lambda)=\int_\Gamma ds\exp\{-\beta\mathcal{H}(s;\lambda)\}$$

Class of systems under consideration

Dynamics in phase-space

Is deterministic and governed by Hamilton equations

$$\dot{q}_i = \frac{\partial \mathcal{H}(s;\lambda)}{\partial \theta_i} \qquad \dot{\theta}_i = -\frac{\partial \mathcal{H}(s;\lambda)}{\partial q_i}$$

- Dynamical system: (Γ, p, ϕ^t). (Γ, p) is a probability space.
- $p : \Sigma \to [0,1]$ is the initial probability measure over the sigma-algebra Σ.
- Hamiltonian flow: $s_t = \phi^t(s_0)$. $\phi : \Gamma \to \Gamma$.
- The Shannon differential entropy (on the support of the probability density ρ_t) is defined as

$$S[\rho_t] = -\int_\Gamma ds \rho_t(s) \ln \rho_t(s)$$

Communication and dynamical systems

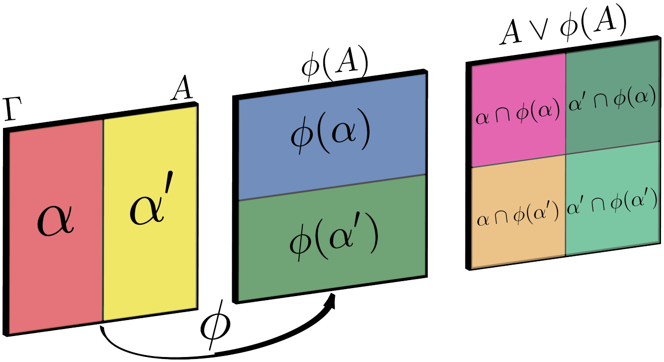
The source emits some discrete symbols (from some alphabet) to a receiver accordingly with a given probability distribution. The KSE quantifies how random such a process is. The goal here is to define this quantity for dynamical systems.

Partitioning the phase-space

Partition: A collection A of subsets of the phase-space Γ such that:

- $\forall\ \alpha, \alpha' \in A$, $\alpha \cap \alpha' = \emptyset$ if $\alpha \neq \alpha'$
- $\bigcup_{\alpha \in A} \alpha = \Gamma$

Refinements of partitions



Refinement

For partitions A and B we define the *refinement* $A \vee B = \{\alpha \cap \beta \,|\, \alpha \in A\,,\, \beta \in B\}$.

The symbols of a dynamical systems

Discrete time

- Let us consider that time is a discrete variable $t \in \mathbb{Z}$.
- Time evolution is generated by iterations of the map $\phi \equiv \phi^{t=1}$.
- The phase-space alphabet is constructed as
  - Initial phase-space partition: $A \to \phi(A) = \{\phi(\alpha) \,|\, \alpha \in A\}$
  - The alphabet is provided by the trajectories: $A, \phi(A), \phi^2(A), \ldots$

Kolmogorov-Sinai entropy is then defined for this alphabet.

The Kolmogorov-Sinai entropy

Entropy of a partition

$p(\alpha)$: probability of $(q, \theta) \in \alpha$, with $\alpha \in A$

$$S[A] = -\sum p(\alpha) \ln p(\alpha)$$

Randomness of the Dynamical System

$$h(\phi) := \sup_{A \in P} \lim_{t \to \infty} \frac{S\left[\bigvee_{n=0}^{t-1} \phi^{-n}(A)\right]}{t}, \qquad \bigvee_{n=0}^{t-1} \phi^{-n}(A) = A \vee \phi^{-1}(A) \vee \cdots \vee \phi^{-t+1}(A)$$

P is the set of all possible finite partitions of Γ.

Main result

Bird's eye view of the proof

- Every initial condition generates a trajectory $(\alpha_0, \alpha_1, \ldots, \alpha_t)$.
- From this we define the probability for every trajectory and then the conditional *coarse-grained* probability density

$$\rho^{cg}(s_t|\alpha_0, \ldots, \alpha_{t-1}) = \sum_{\alpha_t \in A} \frac{p(\alpha_t|\alpha_0, \ldots, \alpha_{t-1})}{v(\alpha_t)} \mathbb{1}_{\alpha_t}(s)$$

$\mathbb{1}_\alpha(s) = 1$ if $s \in \alpha$ and $\mathbb{1}_\alpha(s) = 0$ otherwise and $v(\alpha) = \int_\alpha ds$.

- Compute a lower bound on the phase-space average $\mathbb{E}S[\rho_t^{cg}]$.
- Rate in time of $\mathbb{E}S[\rho_t^{cg}] \to$ KSE and the rate of $D\left(\rho_t||\tilde{\rho}_t\right) \to$ rate of $\langle W_{\text{diss}} \rangle$.

Main result

Lower bound on dissipation rate

$$\beta\overline{\langle W_d\rangle} \geq \beta(\overline{\langle H\rangle} - \overline{F(\lambda_t)}) - \overline{I}_t(A) \qquad \overline{I_t}(A) = h(\phi) - \overline{c_t(A)} - \overline{d_t(A)}$$

$$c_t(A) = 1 - \sum_{\alpha_0,\ldots,\alpha_t \in A} p(\alpha_t|\alpha_0,\ldots,\alpha_{t-1}) \times \tilde{v}(\alpha_{t-1},\alpha_{t-2},\ldots,\alpha_0|\alpha_t)$$

$$d_t(A) := -\sum_{\alpha_t \in A} p[\phi^{-t}(\alpha_t)] \ln v[\alpha_t]$$

$F(\lambda_t) := \beta^{-1} \ln Z(\lambda_t)$ is the reference free energy at time t and the tilde represents backwards quantities.

Main result: Significance

$$\beta\overline{\langle W_d\rangle} \geq \beta(\overline{\langle H\rangle} - \overline{F(\lambda_t)}) - \overline{I}_t(A) \qquad \overline{I_t}(A) = h(\phi) - \overline{c_t(A)} - \overline{d_t(A)}$$

- Hidden information: Difference between the Shannon entropies before and after imposing the coarse-graining.
- $d_0(A)$ is the minimum hidden information: $S[p(\alpha)] - S[\rho_0] \geq d_0(A) \Rightarrow S[p(\alpha)] - d_0(A)$ is maximum information that is **not** hidden.
- $\overline{\mathbb{E}_{p(\alpha_0,\ldots,\alpha_{t-1})} S[p(\alpha_t|\alpha_0,\ldots,\alpha_{t-1})]} - \overline{S[\rho_t]} \equiv \overline{I_t^h}$ is the average hidden information.
- $\overline{c_t(A)} + \overline{d_t(A)}$ is the minimum average hidden information: $\overline{I_t^h} \geq \overline{c_t(A)} + \overline{d_t(A)}$.
- $\overline{I_t}(A)$ is the maximum average information that is **not** hidden (A is the generating partition): Information generated by the dynamics.

Take home messages

- New tools for studying the thermodynamics of out-of-equilibrium systems based on the understanding of dynamical systems in terms of communication theory.
- In summary, we build a connection between a dynamical quantity, KSE, and a macroscopic physical one, the dissipated work.
- Extension of our results to open systems? Non-Markovianity?[7]
- How about the quantum case? Extension of KSE for quantum stochastic processes[8] and its connections with quantum communication theory should be possible.

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[7] M. Campisi, P. Hänggi and P. Talkner, Rev. Mod. Phys. **83**, 771 (2011).
[8] G. Lindblad, Commun. Math. Phys. **65**, 281 (1979);
Pollock, Rodríguez-Rosario, Frauenheim, Paternostro and Modi, PRA **97**, 012127 (2018).

Thank you for your attention

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