Institute of Physics — Federal University of Goiás

Quantum Pequi Group

Special and General Relativity

Lecture Notes

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Chapter 2

The Foundations of Special Relativity

One of the most impressive consequences of Maxwell's equations is the existence of electromagnetic waves, that travel through space with velocity c, the speed of light. However, at that time, it was fully imprinted in the minds of scientists that waves need a medium for propagation. Therefore, the question is: what about light? The theory of the ether was then developed, and the search for its existence started.

Several experiments were developed in order to detect the ether. The most famous is an interferometric attempt. to measure the velocity of the earth with respect to this preferred reference frame. This is known as the Michelson-Morley experiment, which we now briefly describe.

The Michelson–Morley experiment was designed to test whether the speed of light depends on direction due to the Earth's motion through the hypothesized ether. If an *ether wind* of speed v existed, the round-trip times of light along two perpendicular arms of an interferometer should differ. Rotating the apparatus by 90° would exchange the roles of the arms, producing a shift in the interference fringes.

The apparatus was a Michelson interferometer whose schematic diagram is presented in Fig. 2.1. A monochromatic light beam is directed onto a half-silvered plate (beam splitter) at 45°, producing two coherent beams at its output. One beam travels along the x-arm of length L to mirror M_x, reflects, and returns to the splitter. The other travels along the y-arm, also of length L to mirror M_y, reflects, and returns. The two beams recombine and interfere at the detector. The entire apparatus is mounted on a rotatable platform so that the arms can be interchanged with respect to the ether wind $\vec{v}$.

Le us first concentrate on the parallel arm (aligned with $\vec{v}$). The time light takes to traverse the arm in the forward direction (same as $\vec{v}$) is $t_+ = L/(c-v)$, while on its way back to the beam splitter is $t_- = L/(c+v)$. Thus, the round-trip time is simply given

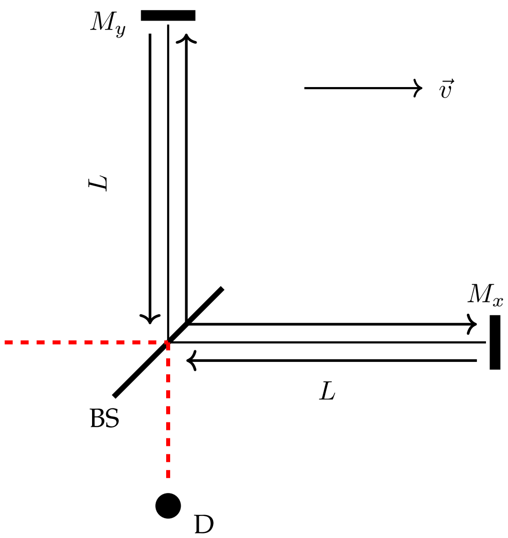

Figure 2.1: **Michelson-Morley interferometer**. A beam splitter BS sends light along two perpendicular arms toward mirrors M_x and M_y. Upon return, the beams recombine and produce interference fringes at the detector. Accordingly to the ether hypothesized, the speed of light is c with respect to the ether. We denote by $\vec{v}$ the velocity of the earth, and thus of the interferometer, with respect to the ether.

by

$$t_{\parallel} = \frac{L}{c-v} + \frac{L}{c+v} = \frac{2Lc}{c^2 - v^2} = \frac{2L}{c} \cdot \frac{1}{1 - v^2/c^2} \approx \frac{2L}{c}\left(1 + \frac{v^2}{c^2}\right),$$

where the approximation holds for $v \ll c$.

Let us now consider the transversal direction (with respect to $\vec{v}$). Here the light must be aimed at an angle so that its velocity relative to the ether is $\sqrt{c^2 - v^2}$. The one-way time is therefore $L/\sqrt{c^2 - v^2}$, resulting in the round-trip time

$$t_{\perp} = \frac{2L}{\sqrt{c^2 - v^2}} = \frac{2L}{c} \cdot \frac{1}{\sqrt{1 - v^2/c^2}} \approx \frac{2L}{c}\left(1 + \frac{1}{2}\frac{v^2}{c^2}\right).$$

The time difference between both arms is then

$$\Delta t = t_{\parallel} - t_{\perp} \approx \frac{2L}{c}\left(\frac{v^2}{c^2} - \frac{1}{2}\frac{v^2}{c^2}\right) = \frac{Lv^2}{c^3}.$$

This leads to an optical path difference $\Delta\ell = c\,\Delta t$. Upon rotating the apparatus by $90°$, the predicted optical path difference doubles[1]

$$\Delta\ell_{\text{rot}} = 2\frac{Lv^2}{c^2}.$$

Taking λ as the wavelength of light, the shift in the fringes that should be observed is

$$\Delta N = \frac{\Delta\ell_{\text{rot}}}{\lambda} = \frac{2Lv^2}{\lambda c^2}.$$

[1]Under rotation, the former parallel arm becomes perpendicular while the former perpendicular arm becomes parallel. The new time difference after rotation is $\Delta t' = t_{\perp} - t_{\parallel} = -\Delta t$. Therefore, the change in the measured optical path difference between the two orientations is $\Delta t_c = \Delta t - \Delta t' - -2\Delta t$.

For $v \approx 30\,\mathrm{Km/s}$, $L \approx 11\,\mathrm{m}$, and $\lambda \approx 5 \times 10^{-7}\,\mathrm{m}$, we obtain $\Delta N \approx 0.44$ fringes, which would be easily measurable. However, Michelson and Morley observed no systematic fringe shift of this magnitude (only random fluctuations). This null result contradicted the ether hypothesis.

In order to explain this failure, Lorentz and Fitzgerald developed the theory of aether based on two assumptions: 1) Longitudinal contraction of rigid bodies in motion; 2) Slowing down of clocks in motion. Both effects would occur with the same factor $\sqrt{1 - v^2/c^2}$. If the arm parallel to motion contracts by $\sqrt{1 - v^2/c^2}$, the parallel-arm travel time becomes

$$t_{\parallel}^{(\text{contracted})} = \frac{2L\sqrt{1 - v^2/c^2}}{c - v} + \frac{2L\sqrt{1 - v^2/c^2}}{c + v} = \frac{2L}{c} \cdot \frac{1}{\sqrt{1 - v^2/c^2}} = t_{\perp},$$

eliminating Δt to order v^2/c^2, consistent with the experimental results.

It is important to observe here that this is a real effect on the physical properties of the bodies due to the motion through the aether. As we will see later on, this is not the case in Special (or General) relativity, where measuring apparatuses just do their jobs, without being affected by motion (or gravity).

When developing his theory, Einstein[2] recognized the deep significance of the *principle of relativity*, putting it as a fundamental axiom of physics by stating the complete equivalence of all inertial reference frames. This principle explains the failure of all experiments trying to detect the ether, without the necessity of Lorentz's assumptions.

The main goal of the present chapter is to state Einstein's postulate of the Special Theory of Relativity and to derive from it the Lorentz transformations. However, before doing this, we need to precisely define what we mean by an inertial reference frame.

2.1 The postulates of Special Relativity

A reference frame is a conventional definition of rest with respect to which measurements can be performed and experiments can be described. If we rigidly attach a reference frame to the Earth, then all of its points will be at rest in this frame, while the stars and planets describe orbits around us. Among all possible reference frames, there is a very special class that plays a fundamental role both in Newtonian physics and in the Special Theory of Relativity.

[2]The same principle was considered one year latter by Poincare, but it was Einstein that provided its fully significance.

Definition 1 (Inertial Reference Frames) *An inertial reference frame is one in which the spatial relations, as determined by rigid scales at rest in the frame, are Euclidean and in which there exists a universal time in terms of which free particles remain at rest or continue to move at constant speed, along straight lines.*

Note that the definition explicitly states the geometry of the space, which must follow Euclidean axioms. In this way, free particles with constant velocity move through geodesics, which are straight lines. Velocity here is defined with respect to the universal time.

Special relativity is a physical model referred to a special class of infinitely extended gravity-free inertial reference frames. As discussed in the introduction to these lectures, the role of gravity is to redefine inertial frames as those in free-fall. Moreover, gravity is the curvature of spacetime and thus, affects all the physics. One of the implications is that gravity destroys Euclidicity and, therefore, inertial reference frames are just idealized frames that can only be approximated by real frames. This will be deeply discussed in later chapters, specially when presenting the equivalence principle. Special relativity deals with physics when this approximation holds.

In the first part of these lectures, all reference frames will be of this type. All observers will be considered to use one of such frames. In these frames, we assume homogeneity of spacetime and isotropy of space. This implies that experiments differing from translations in space and time (homogeneity) and/or rotations (isotropy) will have the same outcomes.

Newtonian mechanics is based on the Galileo relativity principle, which states that the laws of mechanics are the same in all inertial reference frames. However, as we discussed, Maxwell's Electrodynamics is not compatible with Newtonian spacetime unless we assume the existence of the ether as a preferred reference frame. Einstein took a very different path by stating his principle of relativity.

Postulate 1 (Principle of Relativity) *All the laws of physics are the same in all reference frames.*

Note the important difference between Galileo's and Einstein's relative principle. The former is a statement only about the laws of mechanics, while the latter deals with all the laws of physics, including Electrodynamics. This principle means that the outcomes of an experiment performed in one inertial frame will have the same outcomes in any other one if it is performed under identical conditions. This postulate, which is based on homogeneity and isotropy, seems to be very reasonable based on our daily observations. However, together with his second postulate, its consequences become clear.

Postulate 2 (Constancy of the speed of light) *There exists an inertial frame in which light signals in vacuum always travel rectilinearly with constant speed c in all directions, independently of the motion of the source.*

A closer look at Maxwell's theory reveals that this postulate is also reasonable, since it does not predict that light speed should have some dependence on the motion of the sources or on the direction. Moreover, empirical evidence for this postulate was already obtained from the null results of the experiments testing the direction-dependence of the speed of light.

Now, let us see what happens when this second postulate is taken along with the first one. If this statement is true in one inertial reference frame (second postulate), then it also must be true in any other reference frame (from the first postulate). This implies that light signals travel rectilinearly with constant speed in all directions, at all times, in all inertial frames. There is no way to reconcile this with the Newtonian notions of space and time. Einstein realized that these concepts are not fundamental and can be replaced by others. In his theory, he replaced the Newtonian functional of time and the Euclidean space metric by the metric of the four-dimensional spacetime.

Observe that the theory of ether is naturally eliminated here since every inertial frame has the properties the ether should have. Therefore, there is no reason to pick up any one of them as a preferred one.

It is important to observe here that, although we rely on Electrodynamics to state Einstein's postulate, there is nothing special in this theory. The second postulate is important only in order to determine the value of the constant c. However, if we disregard Electrodynamics, this constant could be taken from any other branch of physics. Special Relativity, as we understand it today, is a theory about the spacetime, and only at a second stage, a theory about other branches of physics.

Given that we have the postulates and the inertial reference frames, let us introduce the coordinates and look at some of the physical consequences of the theory.

2.2 Lorentz transformations

We call an **event** the point-like occurrence of anything, like the collision of two particles. Events are labeled by four numbers, the coordinates $x^\mu = (x^0, x^1, x^2, x^3)$, with $x^0 = ct$ and x^i, with $i = 1, 2, 3$ representing the spatial coordinates. Now, how can an observer assign coordinates to a given event? First, he must have a clock and a way to receive and emit light signals. Given the second postulate, distances can be measured by a method like radar, by sending a light signal to the event and by measuring the time of arrival of the signal reflected from the event. Note that the same signal can be employed to determine the time coordinate, which is the reflection time. This distance between the observer and the event is the value of this time interval multiplied by $c/2$. Secondly, the observer must have a way to measure angles, like a theodolite. From these measurements, the usual Cartesian coordinates (x, y, z) can be determined.

It is convenient to build coordinates for all the spacetime events at once. This can be done by placing clocks at all events. The spatial coordinates of these clocks can be determined by the observer at the origin as just defined. To synchronize the clocks, it suffices to select a certain time at the origin (t_0) and emit a light signal. Once this signal passes a clock, it is set to mark $t_0 + r/c$, where r is the distance of the clock from the origin.

We just saw how to assign coordinates to the events with respect to the standard time and the standard notion of distance. Let us call this standard coordinate system S. The question now is how to transform the coordinates of a given event to another reference frame, say S'.

For simplicity, let us assume that the set (t, x, y, z) of Cartesian coordinates in S and (t', x', y', z') in S' are chosen in the following way: i) S' moves in the direction of positive x-direction of S with constant velocity $\vec{v}$; ii) The two x-axes and their positive senses coincide; iii) The two spatial origins coincided when their clocks read 0.

In Newtonian mechanics, we have the following Galilean transformations:

$$t' = t, \quad x' = x - vt, \quad y' = y, \quad z' = z.$$

This leads to the usual velocity transformations by differentiating with respect to time t

$$u_{x'} = u_x - v, \quad u_{y'} = u_y \quad u_{z'} = u_z,$$

which clearly contradicts the postulates of Special Relativity, since if $u_x = c$, we must have $u_{x'} = c$.

Let us see how the postulates of Special Relativity modify these transformations. First, due to homogeneity, the new transformations must be linear. To see this, let us assume that the transformations are written in terms of some function as $(t', x', y', z') = F(t, x, y, z)$. No matter what the function F is, the transformations cannot depend on the coordinates themselves, only on their increment. If this is not true, the transformations would depend on where you started (some preferred origin), which is not allowed by the first postulate of Special Relativity. Moreover, if we have two displacements Δ_1 and Δ_2, we must have $F(\Delta_1 + \Delta_2) = F(\Delta_1) + F(\Delta_2)$ since two consecutive displacements must be the same as one single displacement by the total amount. This means that the transformations must be additive, thus implying linearity.

Let us now consider an event p and a neighboring event q whose coordinates differ from those of p by $(\mathrm{d}x, \mathrm{d}y, \mathrm{d}z)$ in S and $(\mathrm{d}x', \mathrm{d}y', \mathrm{d}z')$ in S'. Let us consider that at p a flash of light is emitted and event q is illuminated. According to the second postulate, we must have

$$c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 = 0.$$

Analogously, the observer in S' must find

$$c^2\mathrm{d}t'^2 - \mathrm{d}x'^2 - \mathrm{d}y'^2 - \mathrm{d}z'^2 = 0.$$

It is also true that any event near p whose coordinates satisfy either of these conditions will be illuminated by the flash of light, which implies that its coordinates must satisfy both.

Let us write the interval as $\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$, where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric. Now, from the previous discussion, we conclude that the null cone must be preserved, which means that $\mathrm{d}s^2 = 0$ in S must imply that $\mathrm{d}s'^2 = 0$ in S'. Therefore, the transformation Λ that takes us from S to S' must obey

$$\Lambda^T\eta\Lambda = \lambda(v)\eta,$$

with λ depending only on the velocity v[3]. Thus, for all intervals $\mathrm{d}s^2(\Lambda x^\mu) = \lambda(s)\mathrm{d}s^2(x^\mu)$.

Now we use reciprocity, which means that the transformation taking us from S to S' must be the reverse of that from S' to S, leading us to $\lambda(v)\lambda(-v) = 1$. But isotropy says that there is no preferred direction in space, implying that $\lambda(v) = \lambda(-v)$. Continuity of the transformation and the identity at $v = 0$ takes us to the conclusion that $\lambda(v) = 1$. Therefore, we get the important result that the spacetime interval is preserved by the transformations between two inertial reference frames

$$\mathrm{d}s^2 = \mathrm{d}s'^2.$$

From this, we can also conclude that the transformations themselves are homogeneous (no additive constant)

$$c^2t^2 - x^2 - y^2 - z^2 = c^2t'^2 - x'^2 - y'^2 - z'^2.$$

[3]Any nondegenerate symmetric bilinear form whose null set equals that of η must be a scalar multiple of η

This follows because the event $(0,0,0,0,)$ in S must correspond to the event $(0,0,0,0)$ in S'.

Under general coordinate transformations, we have

$$t' = \alpha t + \beta x + \delta y + \xi z,$$

for α, β, δ, and ξ constants. By hypothesis, the coordinate plane $y = 0$ coincides with $y' = 0$. This implies that $y' = \zeta y$, for some constant ζ. If we now reverse the roles of S and S' by reversing the directions of x and z axis, we must also have $y = \zeta y'$, thus implying that $\zeta = \pm 1$. However, since S must be mapped continuously into S' in the limit $v \to 0$, the case $\zeta = -1$ is discarded and we obtain $y' = y$. A similar analysis takes us to the conclusion that $z' = z$. Therefore, the invariance of the coordinates reduces to

$$c^2t^2 - x^2 = c^2t'^2 - x'^2.$$

However, $x = vt$ must imply that $x' = 0$, which implies that $x' = \delta(x - vt)$. Therefore, $t' = \alpha t + \beta x$.

Putting everything together we obtain

$$\delta = \alpha = \pm\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \quad \text{and} \quad \beta = -\frac{v}{c^2}\alpha.$$

This results in the following set of transformations

$$\begin{aligned} t' &= \gamma\left(t - \frac{v}{c^2}x\right) \\ x' &= \gamma\,(x - vt) \\ y' &= y \\ z' &= z, \end{aligned} \tag{2.1}$$

with

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}.$$

These are the Lorentz transformations. Note a fundamental difference between these relations and the Galilean transformations. In the former, time also transforms and mixes with the spatial coordinates.

If a law of physics is invariant under this set of transformations and under spatial rotations and spatial and temporal translations, it is the same in all inertial reference frames, thus satisfying the principle of relativity.

From the set of transformations (2.1) we directly see that if two events have the same t (they are simultaneous in S), they do not need to have the same t', implying that they are not simultaneous in S'. In other words, simultaneity is relative.

These transformations replace the Galilean ones in Newtonian mechanics. We can clearly see that the Galilean transformations are obtained from the Lorentz ones by taking the limit $c \to \infty$.

The set of all Lorentz transformations forms a group, the Lorentz group (see Appendix C for more details). A direct calculation from Eqs. (2.1) shows that the inverse of a Lorentz transformation is also a Lorentz transformation with parameter $-v$ instead of v. Also, the result of two Lorentz transformations with parameters v_1 and v_2 is also a Lorentz transformation with parameter $v = (v_1 + v_2)/(1 + v_1 v_2/c^2)$. Finally, the identity transformation follows from $v = 0$. Therefore, the set of transformations has all the properties of a group. Since rotations and translations form a group, it follows that the Poincaré transformations also form a group. One important consequence of this fact is that if S is related to all other inertial reference frames by a Lorentz transformation, then any two frames, S' and S'', are also related to each other due to the transitivity of the group.

Now, since finite coordinates in one frame must correspond to finite coordinates in any other one, the transformations imply that the speed v, the relative velocity between the frames, must be smaller than c. Let us see what happens if superluminal communication would be possible. Consider two events, p and q, and let p cause q (information is sent from p to q) at superluminal speed $u > c$ relative to S. Now, choose coordinates in S such that both events occur on the x-axis and time and space intervals obey $\Delta t > 0$ and $\Delta x > 0$. In another frame moving at speed v with respect to S, the time interval transforms as

$$\Delta t' = \gamma \left(\Delta t - \frac{v \Delta x}{c^2} \right) = \gamma \Delta t \left(1 - \frac{vu}{c^2} \right) ,$$

which implies $\Delta t' < 0$ if $c^2/u < v < c$. This means the causal relation between the events is reversed in S', in clear violation of causality. On the other hand, replacing u by $u \cos\theta$ in the above expression, we see that for all θ (directions) Δt and $\Delta t'$ have the same sign as $u \leq c$, thus ensuring causality invariance.

It is important to observe that this limit on the information speed forbids the existence of rigid bodies. If they exist, their points would communicate at infinite speed.

2.3 Spacetime diagrams

From here on, we consider $c = 1$ to simplify the equations. A spacetime diagram, or Minkowski diagram, is a two-dimensional plot of the (t, x) plane (one time and one space dimension) that captures causal structure and motion. The usual conventions can be described as follows: i) The vertical axis is t (time) and the horizontal axis is x (space); ii) Light rays from the origin are lines at $\pm 45^\circ$ (slope ± 1) when axes use the same scale; iii) Worldlines are curves $x(t)$ plotted in the (t, x) plane.

Let us see how the Lorentz transformations can be visualized in a spacetime diagram. Let us consider two reference frames, S and S', moving with velocity v with respect to each other. Figure 2.2 represents the axis of a reference system S' as viewed from the S perspective. First, we draw the axis for the S frame, identified by ct and x. Light cones are represented in the figure by dashed lines with slope ± 1 (remember that $c = 1$), $x = \pm t$. In S, the x-axis is the locus of events that happen at the same time, i.e. they are simultaneous. Moments in S have the equation $t = constant$, while histories of each fixed point in x have the equation $x = constant$. One of these curves is shown in the figure.

Now, moments in S' have the equation $t' = constant$, thus implying that they have the equation $t - vx = constant$ in S. These are lines with slope v in the S diagram. In particular, the x' axis ($t' = 0$) is the line $t = vx$, as shown in the figure. The wordlines of static points in S' have the equation $x' = constant$, which translates to the curves $x - vt = constant$ in the S diagram. The particular case of the t' axis ($x' = 0$) is the line $x = vt$, as shown in the figure.

2.4 Four vectors and the Minkowski spacetime

As discussed before, Special Relativity forces a revision of our notions of space and time: they must be treated together as a four-dimensional spacetime. The time functional and the Euclidean metric underlying Newtonian physics must be replaced by the spacetime metric η. The natural objects living in spacetime are four-vectors, which generalize the familiar three-vectors of Euclidean space. We present here just a brief description of these objects. A deeper treatment will be presented later on in these lectures.

Spacetime is a four-dimensional affine space, whose points are called **events**[4]. As usual, an event is specified by four numbers, the coordinates of the event relative to an

[4]This follows from homogeneity. Differences of events are vectors in a linear space $\mathbb{V} \cong \mathbb{R}^4$. These vectors transform among inertial frames by linear maps, the Lorentz transformations.

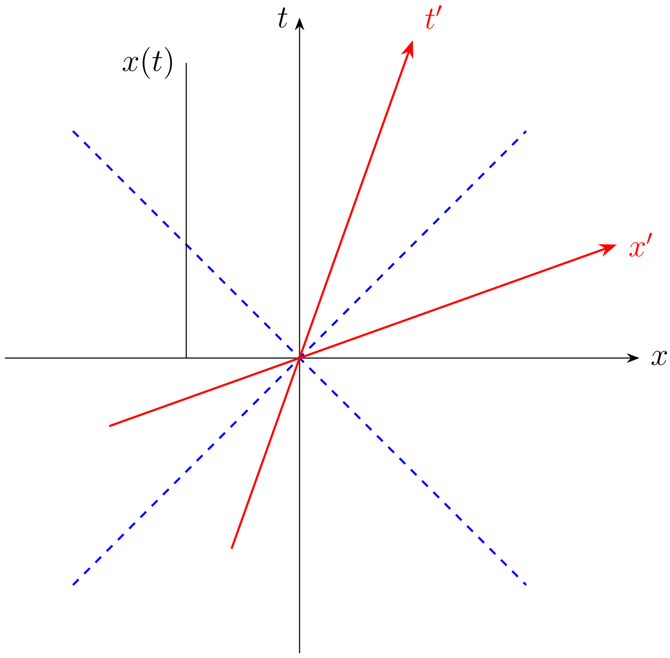

Figure 2.2: **Spacetime diagram**. Minkowski diagram showing two inertial frames S and S' related by a Lorentz transformation. The light cone $x = \pm t$ is represented by the dashed lines.

inertial frame: $x^\mu = (x^0, x^1, x^2, x^3)$. We usually write

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z,$$

so that the first component has the same units as the spatial components. Using ct avoids carrying explicit factors of c in many formulae. In this coordinate system, the metric takes the form $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. The interval between two events is written as $\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$. From the fact that the metric is not positive semi-definite, we identify three kinds of intervals:

- $\mathrm{d}s^2 < 0$: timelike separation (causally connectable events);
- $\mathrm{d}s^2 = 0$: null (light-like) separation;
- $\mathrm{d}s^2 > 0$: spacelike separation (causally disconnected).

A four–vector A whose components in a given reference frame are denoted by A^μ is an object that transforms under Lorentz transformations Λ as

$$A'^\mu = \Lambda^\mu{}_\nu \, A^\nu.$$

Equivalently, A is an element of the vector space $\mathbb{V} \cong \mathbb{R}^4$ equipped with the Minkowski metric. This law of transformation is actually a consequence of the fact that the vector must be the same in both reference systems, although its components may change. This will be more clear when we define tensors in Cap. 3. Important examples of four-vectors are the displacement (or position) $x^\mu = (ct, x, y, z)$, the four-velocity $u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$, where τ is the proper time, the four-momentum $p^\mu = mu^\mu = (E/c, \mathbf{p})$ and the electromagnetic four-potential $A^\mu = (\phi/c, \vec{A})$. At this point we did not distinguish between vectors (covariant) and dual vectors (contravariant) since this is not so important here. This will be discussed in detail in Cap. 3.

Given two four-vectors with components A^μ B^μ, we define the Lorentz-invariant scalar product as

$$\mathrm{A} \cdot \mathrm{B} \equiv \eta_{\mu\nu} A^\mu B^\nu = -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3.$$

Note that $\mathrm{A} \cdot \mathrm{B} = \mathrm{A}' \cdot \mathrm{B}'$ for any Lorentz transformation.

If we use the metric to raise and lower indices, the scalar product can be written in a compact form as $\mathrm{A} \cdot \mathrm{B} = A^\mu B_\mu = A_\mu B^\mu$, with $A_\mu = \eta_{\mu\nu} A^\nu$ and $A^\mu = \eta^{\mu\nu} A_\nu$ and analogously for B.

Let us consider now a timelike worldline $x^\mu(\lambda)$, which is a curve on the four-dimensional spacetime parametrized by λ. The proper time increment along the worldline is the invariant

$$\mathrm{d}\tau^2 = -\frac{1}{c^2} \eta_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}t^2 - \frac{1}{c^2} \mathrm{d}\vec{x}^2.$$

Proper time is the time measured by a clock traveling along the worldline. From this, we can define the four–velocity as

$$u^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \left(\frac{\mathrm{d}ct}{\mathrm{d}\tau}, \frac{\mathrm{d}x}{\mathrm{d}\tau}, \frac{\mathrm{d}y}{\mathrm{d}\tau}, \frac{\mathrm{d}z}{\mathrm{d}\tau} \right).$$

From the definition of the proper time, it follows immediately that

$$\mathrm{u} \cdot \mathrm{u} = u^\mu u_\mu = -c^2.$$

In particular, for a particle with ordinary (3D) velocity $\mathbf{v} = \mathrm{d}\mathbf{x}/\mathrm{d}t$ one gets the components

$$\nu^0 = \gamma c, \qquad \nu^i = \gamma v^i, \qquad \gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{2.2}$$

These are obtained by using $\mathrm{d}\tau = \mathrm{d}t/\gamma$.

The four-velocity is a unit timelike vector (with norm $-c^2$) that points tangent to the particle's worldline and encodes both the instantaneous speed and clock rate via its time component.

The four-momentum of a particle of mass m is given by $p^\mu = mu^\mu$. Therefore, $p^\mu p_\mu = -m^2c^2$ and, identifying $p^0 = E/c$ yield the familiar energy-momentum relation

$$E^2 = p^2c^2 + m^2c^4, \qquad \mathbf{p} = \gamma m\mathbf{v}, \quad E = \gamma mc^2.$$