Institute of Physics — Federal University of Goiás

Quantum Pequi Group

General Relativity

Lecture Notes

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Chapter 1

Introduction

These lectures focus on the theory of General Relativity (GR), which is a field theory of gravity that also provides a description of the geometry of spacetime. This relationship represents a profound shift in our understanding of space and time and lies at the core of GR. The goal of this Chapter is to introduce the fundamental concepts that led to the development of the theory, offering a physical intuition behind it. The idea is not to be mathematically precise but to convey the main ideas underpinning the theory, explaining why GR is necessary and why it is the way it is.

1.1 A field theory

The first question we need to address is why we require a field theory for gravity. The answer lies in special relativity. Let's explore why.

Consider the Coulomb force between two stationary charges q_1 and q_2:

$$\vec{F} = \frac{q_1 q_2}{r^3}\vec{r},$$

where $\vec{r}$ is the vector pointing along the line connecting the two charges. This implies that the electric interaction acts at a distance, as a force. If we move q_2 suddenly, the charge q_1 would feel this change instantaneously. This is incompatible with special relativity, which asserts that interactions must propagate at a finite speed, no greater than the speed of light.

The resolution to this issue is recognizing that Coulomb's law is only valid in the static limit, where the charges are not in motion relative to each other. In more general scenarios, we must use Maxwell's theory, which is a field theory, providing field equations governing the electromagnetic interaction. In this framework, there is no action at a distance (there is no force), and the electromagnetic interaction becomes a local interaction that propagates at a finite speed (the speed of light) via the electromagnetic field.

To properly describe the electromagnetic interaction, we require three components: i) The Maxwell potential $\mathrm{A_a(x)}$ (the field); ii) The Lorentz force, which governs the motion of particles with mass m and charge e interacting with the field:

$$\ddot{\mathrm{x}}^a = \frac{e}{m}\mathrm{F^a{}_b\dot{x}^b},$$

with $\mathrm{F_{ab}} = \partial_\mathrm{a}\mathrm{A_b} - \partial_\mathrm{b}\mathrm{A_a}$ being Maxwell's tensor; iii) Maxwell's equation

$$\nabla_a \mathrm{F^{ab}} = 4\pi \mathrm{J^b},$$

with ∇_a representing the covariant derivative, and $\mathrm{J_a}$ being the four-current.

A similar situation arises with gravity. Newton describes the gravitational force between two masses m_1 and m_2 as:

$$\vec{F} = \frac{m_1 m_2}{r^3}\vec{r},$$

where $\vec{r}$ is the vector pointing along the line connecting the two masses. This also reveals the same issue, indicating that there must be a field theory that accounts for the degrees of freedom of the physical system carrying the gravitational interaction. This field should reduce to Newton's law in the static, non-relativistic limit. This field is the gravitational field.

Drawing a parallel with electrodynamics, General Relativity is also characterized by three components: i) The gravitational field $\mathrm{g_{ab}}$; ii) The geodesic equation:

$$\ddot{\mathrm{x}}^a = -\Gamma^a_{bc}\dot{\mathrm{x}}^b\dot{\mathrm{x}}^c,$$

which governs the motion of particles under the influence of the gravitational field; iii) Einstein's field equation:

$$\mathrm{R_{ab}} - \frac{1}{2}\mathrm{R g_{ab}} + \lambda \mathrm{g_{ab}} = 8\pi \mathrm{G T_{ab}}.$$

The specific definitions of these quantities will be provided in later chapters. In this sense, General Relativity is the field theory of the gravitational field.

Now we need to understand one of the most profound facts of GR, that this field theory also describes the geometry of the spacetime.

Before Isaac Newton introduced the concepts of absolute space and absolute time, philosophers typically thought of them as being intrinsically linked to the things of the world. Space was understood as a relational concept between various objects, such as rocks, people, animals, trees, and so on. In this sense, space did not exist independently but was defined by the relationships between things. Similarly, time was understood as fundamentally related to the changes we observe in the world, such as the cycle of day and night or the phases of the moon. Just as it would be meaningless to talk about space without referring to objects, it was also impossible to talk about time without changes taking place. Space and time, therefore, were not considered as independent entities but were intrinsically tied to the material world.

Newton transformed this view by positing that there is a three-dimensional Euclidean structure underlying the relational connections between objects. In this space, we can assign coordinates to label the points of the space. For time, he considered the real line $\mathbb{R}$ as its structure. In this way, both space and time gained a metric meaning that did not depend on any objects. They were now understood to have physical existence independent of the things in the world. This represented a profound shift in conceptual thinking. The metric structure could now be used to define the measurements of roads and clocks. This also changed the interpretation of Euclidean geometry, which was previously seen as concerned with the properties of objects but now was understood as the geometry of space itself.

When Einstein discovered Special Relativity, he recognized that it would be more appropriate to describe space and time as a four-dimensional entity, which he called spacetime. However, the structure of space and time did not change significantly compared to the Newtonian concepts. While there are differences, they do not concern the basic structure. Space and time are still considered physical, but they lack dynamics in Special Relativity. Space remains Euclidean, and time retains the metric structure of the real line.

General relativity revolutionized this view by proposing that space and time are not just physical entities but rather a dynamic physical field. This field is known as the gravitational field. In this framework, the readings of rulers and clocks are understood as manifestations of this dynamic field, rather than as the effect of gravity on these devices. Consequently, General Relativity replaces the static Minkowski metric of Special Relativity with a field that depends on the spacetime point. This means that the geometry of spacetime is no longer Minkowskian, and the geometry of space is not Euclidean.

In the context of Newtonian space and time and special relativistic spacetime, acceleration is considered absolute —acceleration with respect to the fixed geometry of spacetime! In this context, acceleration is a consequence of inertial forces. Let us think about Newton's bucket experiment[1], which demonstrates absolute space by showing that rotational motion is not merely relative. When the bucket first starts moving, the water remains still and its surface is flat, even though the bucket is rotating relative to the water. As the water speeds up, it takes on a concave shape, despite being at rest relative to the bucket at that point. The shape of the water's surface depends not on its motion relative to the bucket but on its rotation relative to absolute space. Crucially, the concavity persists even when the water and bucket rotate together, proving that rotation is not relative to the bucket but to an external frame: the absolute space. In other words, there must be a privileged frame of reference, the absolute space, against which rotation is measured. If only relative motion mattered, then the shape of the water's surface should depend only on its motion relative to the bucket, which it does not.

From the above discussion we realize the fundamental role of the geometric structure of the spacetime in order to define acceleration and, equivalently, to define what is an inertial frame. Recognizing that acceleration does not depend on the mass of the bodies involved, Einstein realized that during free fall, all objects should behave as though they are in an inertial reference frame. This insight led Einstein to conclude that the role of gravity is essentially to redefine inertial reference frames, the same role played by the spacetime in Special Relativity. Therefore, these two entities must be the same. *Causes assigned to effects of the same type must be, as much as possible, the same!* From this, Einstein inferred that Newtonian space and time, as well as the spacetime of special relativity, are specific configurations of the gravitational field. In general, spacetime must be curved. This is the essence of General Relativity, and it is going to be the main focus of these lectures.

For completeness, we review the main ideas behind the Newtonian concepts of space and time and the special relativity spacetime. A geometric approach to Newtonian mechanics is reviewed in Appendix C, while SR is briefly introduced in Appendix D.

[1]The experiment goes as follows: A bucket filled with water is suspended by a rope. The bucket is twisted and then released, causing it to spin. Initially, the water remains still while the bucket rotates. Over time, the water starts spinning along with the bucket due to friction, eventually matching its rotational speed. As this happens, the surface of the water forms a concave shape (a parabolic curve) due to centrifugal force. If the bucket is suddenly stopped, the water continues spinning, maintaining the concave shape temporarily.

1.2 The Newtonian space and time

Let us start by postulating the basic laws of Newtonian mechanics, known as Newton's laws.

- **The first law**: There exists **inertial reference frames** with respect to which every isolated particle remains at rest or in uniform motion in a straight line.

- **The second law**: In any inertial frame, the motion of a particle is governed by the following set of differential equations

$$\frac{\mathrm{d}\vec{p}}{\mathrm{d}t} = \vec{F}, \tag{1.1}$$

  with $\vec{p}$ being the linear momentum of the particle while $\vec{F}$ stands for the total force acting on the particle.

- **The third law**: To every action there corresponds an equal and opposite reaction

$$\vec{F}_{i,j} = -\vec{F}_{j,i}, \tag{1.2}$$

  where $\vec{F}_{i,j}$ is the force on particle j due to particle i. The forces are directed along the line joining both particles.

It is important to note that Newton's third law is not valid in general. It is violated, for instance, for moving charged particles due to the fact that the speed of propagation of the electromagnetic interaction is finite.

Such postulates are based on our experimental observations of the natural world. For instance, we know that space is three-dimensional and Euclidean, while time is one-dimensional. Moreover, we understand that there are special reference frames, called inertial frames, on which Newton's second law takes the same form, and that all other frames that are at rest or in rectilinear uniform motion with respect to one of these frames are also inertial frames. This is usually known as Galileo's principle of relativity. Finally, if we specify the position and velocity of a particle at a given instant of time, we should be able to tell the motion of the particle at any other time, thus justifying the second-order derivative appearing in Newton's equation of motion. This last fact is known as the principle of determinacy.

The Newtonian universe is a four-dimensional **affine space**[2] $\mathbb{A}^4$. $\mathbb{R}^4$ acts as the group of parallel displacements $a \to a + \vec{v}$ where a, $(a + \vec{v}) \in \mathbb{A}^4$ and $\vec{v} \in \mathbb{R}^4$. From

[2]See Appendix A for further details

this, we see that the difference of two points of $\mathbb{A}^4$ is a vector in $\mathbb{R}^4$, while the sum is not defined. Each element of $\mathbb{A}$ is called **event**.

Time is defined as a linear map $t : \mathbb{A}^4 \mapsto \mathbb{A}^1$. The kernel of this map —the set of vectors for which $t(a - b) = 0$— defines the simultaneous hypersurfaces, which are 3-dimensional subspaces of $\mathbb{A}^4$. On each of these spatial hypersurfaces, we can define a distance function $d : \mathbb{A}^3 \times \mathbb{A}^3 \mapsto \mathbb{R}$ such that, for all a, b, $c \in \mathbb{A}^3$, it is positive semidefinite $d(a, b) \geq 0$, with the equality holding if and only if $a = b$, symmetric, $d(a, b) = d(b, a)$, and fulfills the triangle inequality, $d(a, c) \geq d(a, b) + d(b, c)$. Observe here the strong connection between the notion of simultaneity —the kernel of the map t— and spatial distances.

A positive bilinear symmetric form $\langle x, y\rangle$, called a scalar product on $\mathbb{R}^4$, defines the Euclidean structure and allows us to define the distance function as

$$d(a, b) \equiv ||a - b|| = \sqrt{\langle a - b, a - b\rangle} \tag{1.3}$$

between points a and b of the corresponding simultaneity hyperspace. Since the difference of two events in $\mathbb{A}^4$ is a vector in $\mathbb{R}^4$, it is clear that distances on the spatial hypersurfaces are defined by the kernel of t. We call a **Galilean spacetime** (or Euclidean) the set $(\mathbb{A}^4, t, d)$.

The set of affine transformations that preserve time intervals and distances between simultaneous events forms the Galilean group[3], which is the symmetry group of Newtonian mechanics.

Since we defined the mathematical structure of space and time, we are now in a position to properly assign meaning to the laws of Newton that were stated earlier. We know that the laws of physics are expressed in terms of differential equations, which means that, in order to do physics, we should be able to employ calculus. We do this by introducing reference frames, which are ways to unambiguously label the points of $\mathbb{A}$. Reference frames are maps from the set $\mathbb{A}^4$ to the set $\mathbb{R}^4$, where we understand how calculus works. It is important to observe here that, in the case of Newtonian space and time, a single map is able to cover the entire set $\mathbb{A}^4$. However, this is not possible when gravity comes into play, but the definition of a reference frame is exactly the same, and we just need more than one of them to cover the entire spacetime. Moreover, we physicists usually deal with well-behaved functions, and we then demand that such maps are of class C^∞. Strictly speaking, it does not need to be C^∞, but it must be sufficiently smooth.

In order to introduce a reference frame, note that all Galilean spaces are isomor-

[3]See Appendix B for further details.

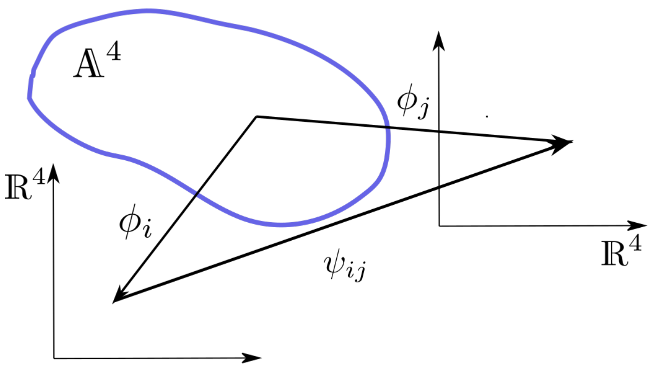

Figure 1.1: **Inertial reference frames.** ϕ_i and ϕ_j are two Galilean coordinate systems, defined by the isomorphisms $\phi_i : \mathbb{A} \mapsto \mathbb{R}^3 \times \mathbb{R}$ and $\phi_j : \mathbb{A} \mapsto \mathbb{R}^3 \times \mathbb{R}$, respectively. A Galilean transformation, or coordinate transformation, is defined by the map $\psi_{ij} \equiv \phi_j \circ \phi_i^{-1} : \mathbb{R}^3 \times \mathbb{R} \mapsto \mathbb{R}^3 \times \mathbb{R}$. Therefore, ψ_{ij} takes us from the set of coordinates defined by ϕ_i to the set of coordinates defined by ϕ_j.

phic[4] to each other and are also isomorphic to $\mathbb{R}^3 \times \mathbb{R}$. Therefore, we can use such isomorphism to define a coordinate system as the map

$$\phi : \mathbb{A}^4 \mapsto \mathbb{R}^3 \times \mathbb{R}, \tag{1.4}$$

which is called a Galilean coordinate system. In this way, we just labeled each one of the events in $\mathbb{A}^4$ with four real numbers, which are called the coordinates of the events. Now, a Galilean transformation takes us from one inertial coordinate system to some other. Mathematically, if a coordinate system ϕ_j moves with zero acceleration with respect to the coordinate system ϕ_i, the map $\psi_{i,j} \equiv \phi_i \circ \phi_j^{-1} : \mathbb{R}^3 \times \mathbb{R} \mapsto \mathbb{R}^3 \times \mathbb{R}$ is a Galilean transformation. We also demand that $\phi_i \circ \phi_j^{-1}$ to be C^∞. Both ϕ_i and ϕ_j give $\mathbb{A}^4$ the same Galilean structure. This idea is illustrated in Fig. 1.2.

It is important here to make a clear distinction between distances and time intervals —as measured by rulers and clocks— and the set of coordinates. The main goal of the map ϕ_i is to attribute a set of four numbers to each point of the set $\mathbb{A}^4$. In order to define a distance, we need to define the metric on $\mathbb{R}^n$. In principle, coordinates have no physical meaning. Moreover, note that the coordinates are not in the set $\mathbb{A}^4$, but in $\mathbb{R}^4$. Since $\mathbb{A}^4$ is isomorphic to $\mathbb{R}^4$, these observations make no important difference here. However, this will be fundamental in GR.

1.2.1 Dynamics

We start by defining the **motion** in $\mathbb{R}^3$, which is the image of the differentiable map $\vec{x} : \mathbb{I} \mapsto \mathbb{R}^3$, with $\mathbb{I} \subset \mathbb{R}$ being an open interval of the real line. Therefore, the motion is a **curve** in $\mathbb{R}^3$. It is also called a trajectory. The motion $\vec{x}$ defines a curve in $\mathbb{R}^3 \times \mathbb{R}$ called **world-line** γ. Figure 1.2 illustrates the concept of a motion along with the spacetime on which classical mechanics is built. $\mathbb{R}^3$ is called the **configuration space**. Each one of

[4]See Appendix A for further details.

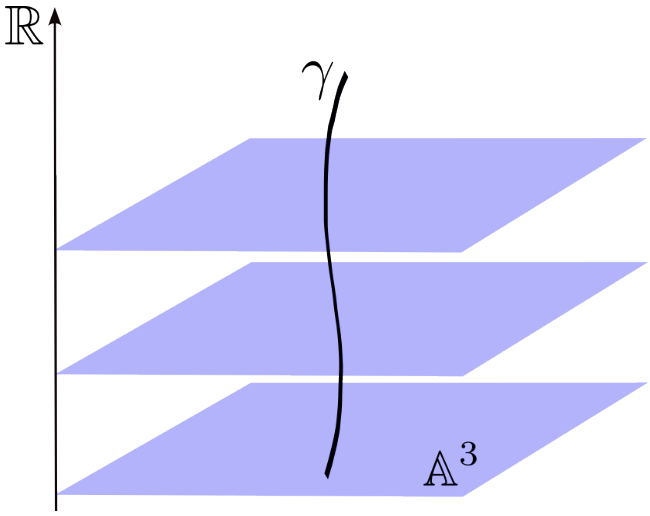

Figure 1.2: **World-line in Newtonian spacetime**. The trajectory $\vec{x} : \mathbb{I} \mapsto \mathbb{R}^3$ defines the motion of the system in the three dimensional space $\mathbb{R}^3$ (which is isomorphic to $\mathbb{A}^3$). The graph of this motion is a curve in $\mathbb{R}^3 \times \mathbb{R}$, called world-line γ, that intercepts each one of the simultaneity hypersurfaces $\mathbb{A}^3$ precisely once. The space is them an equivalent class of simultaneous hypersurfaces defined by the kernel of the time map t. The properties of $\vec{x}$ tell us that each one of these hypersurfaces is a Cauchy hypersurface.

these subspaces is a Cauchy hypersurface[5]. The velocity and the acceleration vectors are defined as the first and second time derivatives of the motion, respectively.

Now, according to Newton's principle of determinism, the initial position $\vec{x}_0 \in \mathbb{R}^3$ and velocity $\dot{\vec{x}}_0 \in \mathbb{R}^3$, at time t_0, uniquely determine the motion of the system at all times. In particular, they determine the acceleration, which implies that there is a function $\vec{F}\left(\vec{x}, \dot{\vec{x}}, t\right) : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \mapsto \mathbb{R}^3$ such that

$$\ddot{\vec{x}} = \vec{F}\left(\vec{x}, \dot{\vec{x}}, t\right), \tag{1.5}$$

According to the Galilean invariance principle, this last equation must be invariant under a Galilean transformation, that takes us from one reference frame to another. In particular, since this set of transformations includes time and space translations, we see that $\vec{F}$ must be independent of time and also must depend only on the relative coordinates and velocities of the particles composing the system. Such symmetry is a consequence of the homogeneity of space and time. Mathematically

$$\ddot{\vec{x}}_i = \vec{F}_i\left(\left\{\vec{x}_j - \vec{x}_k, \dot{\vec{x}}_j - \dot{\vec{x}}_k\right\}\right) \quad \text{for} \quad i, j, k = 1, 2, 3. \tag{1.6}$$

We interpret $\vec{F}$, the force, as the definition of the system under consideration.

The fact that Newtonian mechanics is invariant under a Galilean transformation thus implies that it is impossible to label the events with a preferred spatial position. That is why we need an affine space! Positions and velocities can only be defined relative to something else. In other words, given two events happening at different

[5]Intuitively, we can interpret a Cauchy hypersurface as the one defining an instant of time. This concept will be made clear after we introduce the Lorentzian manifold.

times, it has no meaning to say that they happened at the same position, unless we specify a reference object. The only thing that is absolute is the acceleration that, as mentioned earlier, is defined with respect to the static geometry of the space and time. This is the physical meaning of the Galilean transformation.

The isotropy of space implies that the force must fulfill the relation $\vec{F}\left(G\vec{x}, G\dot{\vec{x}}\right) = G\vec{F}\left(\vec{x}, \dot{\vec{x}}\right)$ for every orthogonal transformation G (a rotation in space). Since every Galilean transformation can be written as a combination of a translation (in time and in space), a uniform motion, and a rotation, we have considered all the possible symmetries.

Figure 1.2 also defines the causal structure of the Newtonian world. Let $p \in \mathbb{A}^4$ be the crossing point of the world-line and some hypersurface $\mathbb{A}^3$. This set defines the present of p, i.e. all the points that are simultaneous to p. This means that no observer can be at any other event in this set if it is at p. The past (future) of p are all the points in $\mathbb{A}^4$ below (above) p, with respect to the direction defined by time. Events that lie below this set can influence p, while events above can be influenced by p. Such causal structure is universal in the sense that it is the same for all observers. Time and space are absolute!

Let us see how this structure changes in Special Relativity.

1.3 Special Relativity

We saw that the Newtonian concept of spacetime (space and time taken together) is described by a continuous set of events and we can think about each one of these events as a point in space at a given instant of time. Moreover, we can unambiguously label every event of this set with four numbers, called the coordinates of the point. Additionally, given a specific point in space at a specific moment in time, there is an absolute meaning of simultaneity, defined by the set of points associated with the same instant of time. This last property was changed by Einstein in his special theory of relativity. In Newtonian physics, all the simultaneous events to a given one form a three-dimensional one, while in SR, it is much more than this. An observer can still define a three-dimensional hypersurface containing the events that occur at the same time as a given one. However, such hypersurface depends on the state of motion of the observer. Simultaneity is not absolute!

Such a structure leads to the fact that two distinct inertial observers will assign different coordinates to the events of the spacetime. This makes clear that such coordinates have no physical meaning. We must then look for quantities that are observer independent. In other words, we need to find functions of the coordinates that are observer independent. In Newtonian physics, we do have two of these quantities (and functions of them), the time interval between two events and the space interval between two simultaneous events. Given the above discussion regarding the nature of simultaneity, we can anticipate that none of these quantities will be invariant in SR.

Before introducing the corresponding quantities in SR, let us start with the physical principles behind this theory, which can be stated as follows.

- **Principle of relativity**: All the laws of physics are the same in all inertial frames.
- **Universality of the speed of light**: The speed of light in vacuum is the same for all observers, regardless of their state of motion.

The first principle is quite natural and it was already presented in physics since Galileo. Einstein realized the second principle due to the electromagnetic field equations. The solution to Maxwell's equations in vacuum is a propagating wave whose velocity is constant no matter the observer. Although simple, it presents profound consequences. For instance, in Newtonian physics, if a particle is in a certain point in space at a given instance of time, it can be at any other point in space in the subsequent instant, since there is no limit for its speed. In this way, simultaneity defines a three-dimensional hypersurface which is orthogonal to the time line. In SR, due to the finite speed of light, an observer at a given event cannot be everywhere in a subsequent one. Therefore, the notions of past, present and future must change.

At each event of spacetime, we can define a **light-cone**, which determines the locus of paths that point particles can follow, as illustrated in Fig. 1.3. At each event p, spacetime is split into three regions. The first one is the locus of the events that can be influenced by p. This is the future light-cone. The second one contains the events that can influence p and is called the past light-cone. Both of these regions form a three-dimensional set. All the events lying outside the light cone do not have any causal relation with p. The notion of the present of p is not defined.

Such structure is specified by the spacetime metric, which can be written as

$$\mathrm{d}s^2 = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2, \tag{1.7}$$

in the dual coordinate basis. The minus sign in this metric indicates that the line element $\mathrm{d}s^2$ can be negative. This Lorentzian metric determines three kinds of intervals between two points p and q: i) $\mathrm{d}s^2 > 0$ indicates that p and q hold no causal relation and the interval is called **space-like**; ii) Paths for which $\mathrm{d}s^2 = 0$ are not allowed for massive particles and they are called **null-like** (or light-like) interval; iii) $\mathrm{d}s^2 < 0$, determining a **time-like** interval, are the allowed paths for all massive particles. The path γ shown in

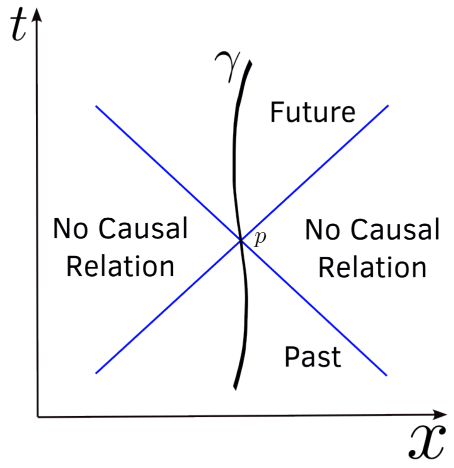

Figure 1.3: **Spacetime in special relativity**. At each event p we define the light-cone which determined the past, future and the set of events that holds not causal relation with that event. This is represented by the blue lines in the figure. The paths fallowed by material particles are constrained to the interior of the cone while light walks only on the surface of the cone, which are called null surfaces. Nothing can have a path whose inclination is bigger than the surface of the cone.

Fig. 1.3 is an example of a time-like curve. Remember that, when considering time-like intervals, the **proper time**[6] is defined as $\mathrm{d}\tau^2 = -\mathrm{d}s^2$.

Note the difference between this causal structure and the one in the Newtonian world. Considering p as the event defining the light-cone shown in Fig. 1.3, events that can be influenced by p lie in the future light-cone, while the past light-cone defines those events that can influence p. It is important to observe that no material particle can travel at the boundaries of this light-cone, since these are null-surfaces, which are regions only allowed for non-massive particles. The set of events that holds no causal relation to p defines a four-dimensional set, instead of a three-dimensional one defined in Newton's spacetime. It is still possible to define a three-dimensional set that constitutes those events forming the *present* of p. However, such a set will depend on the state of motion of the observer defining it. This is a fundamental difference between Newton and Einstein. Simultaneity is relative in SP, while it is absolute in the Newtonian world (see Appendix D for further details).

In Newtonian spacetime the space interval ($\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$) is conserved under rotations. This is a consequence of the fact that the notion of a three-dimensional space at a single instant of time is independent of coordinates. The notion of rotation in time is not defined. In Special Relativity, where the notion of simultaneity is observer dependent, this is not true anymore and we need to consider rotations of the entire four-dimensional set, called the **Minkowski spacetime**. This implies that we rotate time and space into each other. Under this sort of transformation, the spacetime interval (1.7) is invariant.

We can consider a curve parameterized by λ, with coordinates $x^\mu(\lambda)$ and, from the

[6]The proper time is the time measured by a clock carried by an observer moving along the path between the considered events.

line element 1.7 it is possible to compute the length of the path as

$$s = \int \sqrt{\eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} \mathrm{d}\lambda \quad \text{and} \quad \tau = \int \sqrt{-\eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} \mathrm{d}\lambda, \tag{1.8}$$

for space-like and time-like intervals, respectively. The precise notion of derivatives and coordinates will be discussed in the next chapter.

While the Newtonian spacetime interval is invariant under a Galilean transformation, the symmetry group of special relativity is the **Poincaré group**, which is the set of translations (in space and time) and the Lorentz transformations. Appendix B presents some details on the structure of this group.

Although Lorentz transformations were known before Einstein, the meaning of the transformation of time became clear only within SR. The physical content of this transformation is that identical clocks moving with respect to one another measure distinct times. Therefore, it is meaningless to say that two events happening in different locations occurred at the same time unless we specify with respect to what time is determined.

1.4 General Relativity

Both Newtonian and special relativistic spacetimes share the same basic structure. Newtonian spacetime can be considered as the low-velocity approximation of special relativistic spacetime. Both exist independently of any other physical entity, but neither possesses any inherent dynamics.

Einstein recognized that spacetime is a physical entity and, importantly, that it must have a dynamic nature. He argued that it is a physical field, which we now identify as the gravitational field. He replaced the static metric η_{ab} with a general one, g_{ab}, that varies depending on the point in spacetime. The metric g_{ab} represents a physical field whose behavior is governed by field equations (Einstein's equations) and interacts with matter. It defines the geometry of spacetime and is, at the same time, the gravitational field.

To understand why this is the case, remember that inertial forces arise due to the acceleration of a reference frame. As Newton pointed out, this acceleration must be defined with respect to the space itself. In other words, inertial forces result from acceleration with respect to the fixed geometry of spacetime. This holds true in Special Relativity as well. Essentially, Newtonian spacetime defines what is accelerating and what is not.

Einstein made a striking observation about gravity. Inside a free-falling laboratory, the laws of physics appear exactly as they would in an inertial reference frame. This follows from the fact that everything in the laboratory experiences gravity in exactly the same way. This insight is the core of the equivalence principle, a remarkable realization. Gravity's role is to redefine what constitutes an inertial reference frame —specifically, those that are in free fall[7].

As noted, the role of Newtonian spacetime is to define inertial reference systems, and this is true in Special Relativity as well. However, gravity also plays a role in determining inertial systems. Therefore, spacetime and gravity must be the same thing. Einstein concluded that both Newtonian and special relativistic spacetimes are simply specific configurations of the gravitational field. In more general cases, the geometry of spacetime is curved.

Based on the above discussions, we can rewrite the definition of an inertial frame as follows. The spatial relations as determined by rigid rods that remain at rest in the system are Euclidean and there is a universal time in terms of which massive particles remain at rest or in uniform motion on a straight line. The role of gravity is to break down the Euclidean character of space!

1.5 Prerequisites and literature

Although these lectures contain some basic mathematical definitions in the Appendices, some previous knowledge is assumed from the start. The reader should be familiar with differential and integral calculus, as well as linear algebra and geometry. A good knowledge of Classical Mechanics, especially its geometrical formulation, as well as of Special Relativity, is highly recommended.

Here are some very good books on the subject of these lectures, including the introductory content.

- V. I. Arnold, *Mathematical methods in classical mechanics* (Springer, 1997) —- This book presents classical mechanics from a rigorous mathematical point of view. In special, it presents the symplectic formulation of this theory, which is the geometric view of mechanics.

- M. Nakahara, *Geometry, topology and physics* (Taylor & Francis Group, 2003) — This book contains all the necessary mathematical definitions (and much more) that will be employed in these lectures. It is a very good reference book.

- W. Tung, *Group theory in physics* (World Scientific Publishing Company, 1985) — The book describes the basics of group theory, including the Lorentz and the

[7]Note that this holds true only locally, a point that will be clarified in later chapters.

Poincare groups. Although not necessary for understanding these lectures, it is mentioned here for those readers that want to go deeper into the theory.

- W. Rindler, *Introduction to Special Relativity* (Oxford University Press, 1991). Excellent book introducing the basic features of the special theory of relativity.

- J. B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Cambridge University Press, 2021); S. M. Carroll, *Spacetime and Geometry: An Introduction to General Relativity* (Cambridge University Press, 2019) — Excellent books on the general theory of relativity. Both present the theory from a physical point of view, gently introducing the necessary new math. These books are recommended for the first reading.

- R. M. Wald, *General relativity* (University of Chicago Press, 1984) — Very good book on the subject, presenting the theory under a rigorous mathematical notation.