Institute of Physics — Federal University of Goiás

Quantum Pequi Group

General Relativity

Lecture Notes

Lucas Chibebe Céleri

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Chapter 2

A Bit of Differential Geometry

As discussed in the previous chapter, space-time is a four-dimensional continuous set of events. Each event then needs four numbers to be characterized. Special Relativity assumes that this fact is true globally, meaning that there is a one-to-one map between the events and points in the $\mathbb{R}^4$ space. This is no longer true in general spacetimes described by General Relativity. If we think about the surface of a sphere, we will readily realize that it is impossible to build such a map. Therefore, we need a more precise definition of a manifold, which is a mathematical space that *locally* looks like the Euclidean space, just like a sphere. The main goal of this chapter is to formally introduce this concept, along with some others that are necessary in order to properly describe the spacetime within General Relativity. Some additional definitions are presented, for completeness, in Appendix A.

2.1 Manifolds

A manifold is just a continuous set of points that can exhibit complex global properties like curvature or torsion, but that *locally* looks like the Euclidean space. This means that in a small enough[1] neighborhood of any point on the manifold, Euclidean geometry applies. For instance, the surface of a sphere is a manifold and if we are on top of a very big sphere, like our planet for instance, our neighborhood will certainly look like the usual flat two-dimensional plane. This is the main mathematical structure behind the theory of relativity since the space-time is postulated to be a differentiable manifold[2].

[1]Small enough means that we cannot, by any means, detect any violation of Euclidean geometry within the precision of the measurements available to us.

[2]To be precise, it is a differentiable manifold that is also Hausdorff, paracompact and connected. See Appendix A for more details.

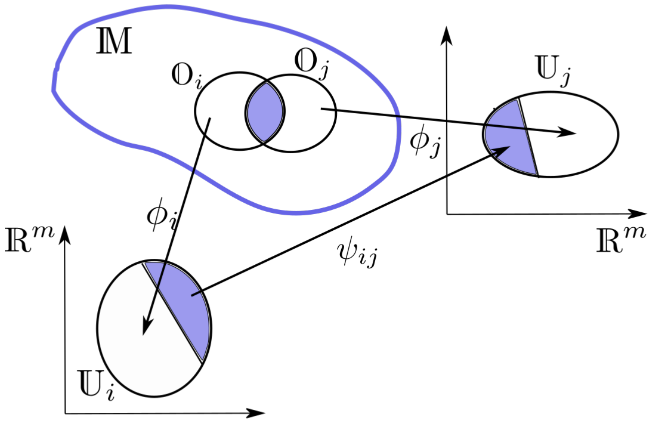

Figure 2.1: **Differentiable Manifold**. Homeomosphisms ϕ_i and ϕ_j mapping open subsets $\mathbb{O}_i$ and $\mathbb{O}_i$ on the manifold $\mathbb{M}$ into open subsets $\mathbb{U}_i$ and $\mathbb{U}_j$ on Euclidean spaces. Such maps specify two different coordinate systems. For points in the overlapping region $\mathbb{O}_i \cap \mathbb{O}_j$, the smooth map ψ_{ij} is called a coordinate transformation. This basic structure allows us to do calculus on manifolds as we normally do in $\mathbb{R}^n$.

Let us start with the formal definition of this object. A manifold can be viewed as a generalization of our concepts of curves and surfaces to objects with arbitrary dimensions. In the same way a curve and a two-dimensional surface are locally homeomorphic to $\mathbb{R}$ and $\mathbb{R}^2$, respectively, a manifold is a topological space which is locally homeomorphic to $\mathbb{R}^m$. This is a very important requirement, especially in the context of relativity since this local homeomorphism enables us to define local coordinate systems. If the manifold is not globally homeomorphic to $\mathbb{R}^m$, it will be impossible to cover it entirely with a single coordinate system. However, we can define several overlapping coordinate systems in order to cover $\mathbb{M}$. By imposing that the transformation between different coordinate systems is continuous, we can develop the usual calculus on manifolds. A precise definition, which is illustrated in Fig. 2.1, can be stated as follows.

Definition 1 (Differentiable Manifold) *$\mathbb{M}$ is an m-dimensional differentiable manifold if we have a set of pairs $\{(\mathbb{O}_i, \phi_i)\}_i$ of open subsets (local neighborhoods) $\mathbb{O}_i \subset \mathbb{M}$ and local homeomorphisms ϕ_i from $\mathbb{O}_i$ to open subsets $\mathbb{U}_i \subset \mathbb{R}^m$, such that the following conditions hold.*

1. *$\mathbb{M}$ is a topological space*[3]
2. *The set of $\mathbb{O}_i$ covers $\mathbb{M}$: $\cup_i \mathbb{O}_i = \mathbb{M}$.*
3. *Given two neighbourhoods $\mathbb{O}_i$ and $\mathbb{O}_j$ such that $\mathbb{O}_i \cap \mathbb{O}_j \neq \emptyset$ for $i \neq j$, the map $\psi_{ij} = \phi_i \circ \phi_j^{-1} : \phi_j(\mathbb{O}_i \cap \mathbb{O}_j) \mapsto \phi_i(\mathbb{O}_i \cap \mathbb{O}_j)$ is infinitely differentiable.*

The pair $(\mathbb{O}_i, \phi_i)$ is called a **coordinate chart** while the whole set $\{(\mathbb{O}_i, \phi_i)\}_i$ constitutes an **atlas**. If the atlas is complete, i.e. it is not contained in each other atlas, it is called a **differentiable structure**. Two atlases are said to be compatible if their union is also an atlas. In this sense, a differentiable structure on $\mathbb{M}$ is an equivalence class of

[3]See Appendix A for definitions.

compatible atlases. A topological space equipped with a differential structure is called a differentiable manifold.

The subset $\mathbb{O}_i$ is called the **coordinate neighborhood** and ϕ_i the **coordinate function**. Given a point $p \in \mathbb{M}$, the homeomorphism ϕ_i is represented by the set of coordinates $\{x^\mu(p)\} = \{x^0(p), ..., x^{m-1}(p)\}$. This is what we mean by the statement that the manifold locally looks like Euclidean space. In each coordinate neighbourhood $\mathbb{O}_i$, $\mathbb{M}$ looks like an open set of $\mathbb{R}^m$.

As we can see from Fig. 2.1, two coordinate systems can be assigned to the same point p if two of its neighborhoods $\mathbb{O}_i$ and $\mathbb{O}_j$ overlap. The axioms defining the manifold assure that the transition from one coordinate system to the other is smooth. In other words, the transition map, which is called a **coordinate transformation**, is of class C^∞. The map $\psi_{i,j}$ is explicitly given by the coordinate functions $x^\mu = x^\mu(y^\nu)$, where x^μ and y^ν are the coordinates assigned to p by ϕ_i and ϕ_j, respectively. Therefore, differentiability can be defined in the same way we do in usual calculus, thus justifying the name differential structure[4]. The coordinate transformation is differentiable if each function $x^\mu(y)$ is differentiable with respect to each y^ν.

In order to make these definitions more clear, let us consider some examples. The simplest case is the space $\mathbb{R}^m$ itself, which is a manifold that can be covered by a single coordinate system with the homeomorphism ϕ being the identity.

Moving to less trivial cases, we start with the usual two-dimensional surface of a sphere of unit radius, $\mathbb{S}^2$, which is a submanifold[5] of $\mathbb{R}^3$. We can simply choose the polar coordinates (θ, ϕ) in order to parametrize the surface of $\mathbb{S}^2$. Such coordinates are usually defined by the relations

$$\theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z} \quad \text{and} \quad \phi = \tan^{-1} \frac{y}{x}, \tag{2.1}$$

with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$, while (x, y, z) represent the usual Cartesian coordinates. However, we are also free to choose any other coordinate system, like the stereographic one (u, v), which is defined by the projection from the North pole to the equatorial plane by the equations

$$u = \frac{x}{1 - z} \quad \text{and} \quad v = \frac{y}{1 - z}. \tag{2.2}$$

[4]Actually, if the union of two atlases is again an atlas, they are said to be compatible. The differential structure is defined by the compatibility equivalence class

[5]$\mathbb{N} \subset \mathbb{M}$ is a smooth submanifold of $\mathbb{M}$ if every point of $\mathbb{N}$ lies in some chart $(\mathbb{O}_i, \phi_i)$ with $\phi_i(\mathbb{N} \cap \mathbb{O}_i) = \phi_i(\mathbb{O}_i) \cap \mathbb{R}^k$, with $0 < k \leq m$.

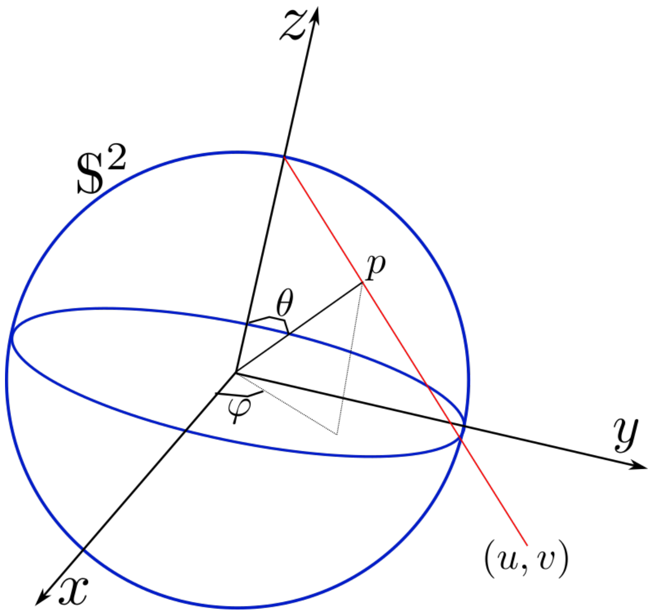

Figure 2.2: **Sphere**. Illustration of the polar (θ, φ) and stereographic (u, v) coordinates for the point $p \in \mathbb{S}^2$ over the $\mathbb{S}^2$ manifold. Such space frequently appears in many physical problems. For instance, the configuration space of a spherical pendulum is $\mathbb{S}^2$.

It is straightforward to see that both coordinate systems are related by the equations

$$u = \cot\frac{\theta}{2}\cos\phi \quad \text{and} \quad v = \cot\frac{\theta}{2}\sin\phi. \tag{2.3}$$

Figure 2.2 illustrates both coordinate systems.

Of course, there are many other coordinate systems that we can choose, and all of them are equally good. However, an important point here is that no coordinate system can be employed everywhere at once. In other words, there is no single coordinate system that is able to uniquely assign a set of coordinates to every point on $\mathbb{S}^2$. This can be illustrated from the stereographic coordinates at the pole or the polar coordinates at the equator ($\theta = \pi/2$). We cannot label the points on the sphere with a single coordinate system such that nearby points always have nearby coordinates. However, we can do this on parts of $\mathbb{S}^2$. We can construct two or more overlapping coordinate systems, such that each one of them covers some part of the manifold by uniquely labeling every one of its points and that nearby points have nearby coordinates (in at least one of them). Specifically, we can consider two stereographic coordinates, one defined by the projections from the North pole and the other one by projections from the South pole. By imposing that the transition from one coordinate system to the other is determined by functions of class C^∞, we have a differentiable manifold.

Another example is the unit circle, which is the submanifold of $\mathbb{R}^2$ defined by $\mathbb{S}^1 = \{(x, y)|x^2 + y^2 = 1; x, y \in \mathbb{R}\}$. Note that there is no way to globally parametrize the circle with a single coordinate function.

Now that we have a space with a differential structure defined on it, it is time to move forward and see how to do calculus on a manifold.

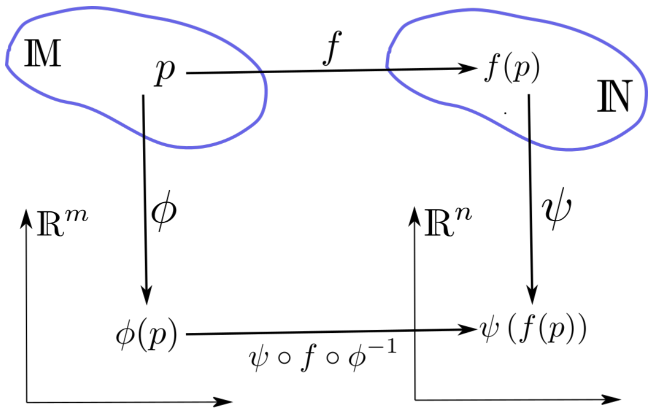

Figure 2.3: **Maps on manifolds**. Illustration of the map $f : \mathbb{M} \mapsto \mathbb{N}$. The maps ϕ and ψ are employed in order to provide a coordinate representations for f in the way discussed in the text. Note that the dimensions of the involved manifolds do not need to be the same.

2.2 Differentiable maps

As it is clear from the last section, the theory of manifolds is based on smoothness, so we can employ the usual calculus developed in $\mathbb{R}^n$. Let us start by defining a very important concept, differentiable maps between manifolds, which are transformations that preserve the structure.

Let $f : \mathbb{M} \mapsto \mathbb{N}$ be a map between an m-dimensional manifold $\mathbb{M}$ and an n-dimensional one $\mathbb{N}$. In this way, a point $p \in \mathbb{M}$ is mapped to the point $f(p) \in \mathbb{N}$, $f : p \mapsto f(p)$, as illustrated in Fig. 2.3. Now, in order to build a coordinate representation of such a map, we define the charts $(\mathbb{O}, \phi)$ and $(\mathbb{P}, \psi)$, in such a way that $p \in \mathbb{O}$ and $f(p) \in \mathbb{P}$. The coordinate representation for f can be written as

$$\psi \circ f \circ \phi^{-1} : \mathbb{R}^m \mapsto \mathbb{R}^n. \tag{2.4}$$

If $\phi(p) = \{x^\mu\}$ and $\psi(f(p)) = \{y^\nu\}$ we have $y = \psi \circ f \circ \phi^{-1}(x)$ and, when we know which coordinate systems are being used, we can write $y = f(x)$ or $y^\mu = f^\mu(x^\alpha)$, considering a certain abuse of notation. If these functions are of class C^∞, then the map f is said to be differentiable. Such a notion is independent of the coordinate system used.

A very important class of maps between manifolds can be defined as follows.

Definition 2 (Diffeomorphism) *Let $f : \mathbb{M} \mapsto \mathbb{N}$ be a homeomorphism*[6] *and ψ and ϕ coordinate functions. If $\psi \circ f \circ \phi^{-1}(x)$ is invertible, and both $y = \psi \circ f \circ \phi^{-1}(x)$ and $x = \phi \circ f^{-1} \circ \psi^{-1}(y)$ are C^∞, the map f is said to be a diffeomorphism and $\mathbb{M}$ and $\mathbb{N}$ are said to be diffeomorphic, denoted as $\mathbb{M} \cong \mathbb{N}$.*

Clearly, $\dim\mathbb{M} = \dim\mathbb{N}$ if $\mathbb{M} \cong \mathbb{N}$. This notion provides a classification o spaces into equivalence classes according to whether it is possible to smoothly deform one space into another. The set of diffeomorphisms $f : \mathbb{M} \mapsto \mathbb{M}$ is a group denoted by $\mathrm{Diff}(\mathbb{M})$.

[6]See Appendix A for definitions.

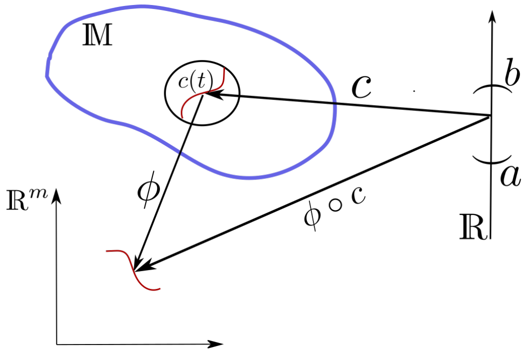

Figure 2.4: **Curves on manifolds.** Illustration of the curve c, parametrized by t, which is a map from an open set $(a, b) \subset \mathbb{R}$ of the real line to the manifold. Given a coordinate function ϕ, we can build the coordinate representation of the curve on $\mathbb{R}^m$.

A very special class of maps that will be employed to introduce the fundamental notion of vector space is called **curve**. Formally, an open curve on an m-dimensional manifold $\mathbb{M}$ is the map $c : (a, b) \mapsto \mathbb{M}$, where (a, b) is an open interval of the real line. We assume that the curve does not intersect with itself, as illustrated in Fig. 2.4. A generalization of such a concept is a closed curve, which is the map $c : \mathbb{S}^1 \mapsto \mathbb{M}$. By defining the chart $(\mathbb{O}, \phi)$, the curve $c(t)$ has the coordinate representation $x = \phi \circ c : \mathbb{R} \mapsto \mathbb{R}^m$.

Another important mapping is the function f on $\mathbb{M}$, which is a smooth map from $\mathbb{M}$ to $\mathbb{R}$. By choosing a chart $(\mathbb{O}, \phi)$, we can build the coordinate representation of f as $f \circ \phi^{-1} : \mathbb{R}^m \mapsto \mathbb{R}$, which is a real-valued function of m variables. We denote the set of smooth functions on $\mathbb{M}$ by $\mathbb{F}(\mathbb{M})$.

2.3 The tangent space

One of the ways we can define a tangent space to a point in a manifold is embedding such a manifold in a higher dimensional Euclidean space and selecting a specific linear subspace as the tangent one. However, it would be much more powerful if we could define all the necessary quantities in an intrinsic way, making reference to the manifold structure only. This is the way modern differential geometry deals with such problems. In this context, a tangent space is understood as an object that is tangent to a curve on the manifold.

The notion of a vector as an arrow connecting some given point to the origin does not work, in general, on a manifold. For instance, how to draw a straight arrow connecting two points on the surface of a sphere? Therefore, we need a more general definition of what a vector is. On a manifold, a vector is defined in terms of a **tangent vector** to a curve, which is the generalization of the tangent line to a curve in the usual two-dimensional plane. However, in an n-dimensional manifold, there are many curves that are indeed tangent to a given vector and, therefore, a tangent vector is an equivalence class of curves. Let us see how this idea works mathematically.

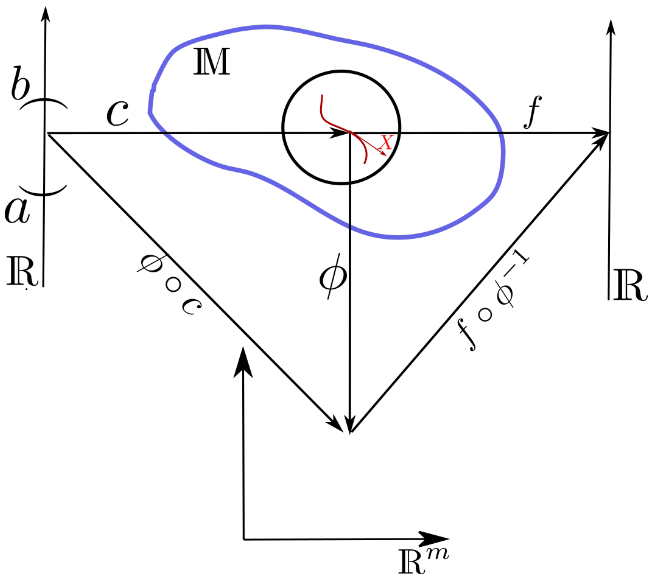

Figure 2.5: **Vector on manifolds.** A curve c along with a function f and a local coordinate system ϕ define the tangent vector X to the manifold $\mathbb{M}$ in the direction determined by the curve $c(t)$.

Let us consider a curve $c : (a, b) \mapsto \mathbb{M}$ and a function $f : \mathbb{M} \mapsto \mathbb{R}$, where (a, b) is an open interval in $\mathbb{R}$ containing the point $t = 0$ (t is an arbitrary parametrization of the curve). By choosing a local coordinate function ϕ, the rate of change of the function f at $t = 0$ along the curve c is given by

$$\left.\frac{\mathrm{d}f(c(t))}{\mathrm{d}t}\right|_{t=0} = \frac{\partial f}{\partial x^\mu} \left.\frac{\mathrm{d}x^\mu(c(t))}{\mathrm{d}t}\right|_{t=0}, \tag{2.5}$$

with

$$\frac{\partial f}{\partial x^\mu} \equiv \frac{\partial}{\partial x^\mu}\left[f \circ \phi^{-1}(x)\right]. \tag{2.6}$$

This means that the rate $(\mathrm{d}f/\mathrm{d}t)|_{t=0}$ is obtained by the application of the differential operator

$$X = X^\mu \left(\frac{\partial}{\partial x^\mu}\right) \quad \text{with} \quad X^\mu = \left.\frac{\mathrm{d}x^\mu(c(t))}{\mathrm{d}t}\right|_{t=0} \tag{2.7}$$

to the map f, denoted as $X[f]$. Therefore, we define X as the tangent vector to $\mathbb{M}$ at the point $p = c(0)$ along the direction given by the curve $c(t)$. Figure 2.5 illustrates this concept.

Since

$$\frac{\partial x^\mu}{\partial x^\nu} = \delta^\mu_\nu, \tag{2.8}$$

with δ^μ_ν being the Kronecker delta function, it follows directly

$$X[x^\mu] = \left.\frac{\mathrm{d}x^\mu(t)}{\mathrm{d}t}\right|_{t=0}. \tag{2.9}$$

If t is understood as time, this is simply the μ-th component of the velocity vector.

Based on the above definition, let us now define the equivalence class of curves on $\mathbb{M}$. Given two curves c_1 and c_2 such that $c_1(0) = c_2(0) = p$ and

$$\left.\frac{\mathrm{d}x^\mu(c_1(t))}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\mathrm{d}x^\mu(c_2(t))}{\mathrm{d}t}\right|_{t=0}, \tag{2.10}$$

then both curves define the same vector X at p, in which case we have $c_1 \sim c_2$. We therefore identify the tangent vector X with the equivalence class of curves

$$[c(t)] = \left\{ \tilde{c}(t) \,|\, \tilde{c}(0) = c(0) \quad \text{and} \quad \left.\frac{\mathrm{d}x^\mu(c_1(t))}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\mathrm{d}x^\mu(c_2(t))}{\mathrm{d}t}\right|_{t=0} \right\}, \tag{2.11}$$

rather than a curve itself. All the equivalence classes of curves at $p \in \mathbb{M}$, i. e. all the tangent vectors at p, form a vector space called the **tangent space** of $\mathbb{M}$ at p, denoted by $\mathrm{T}_p\mathbb{M}$. From Eq. (2.7) it is clear that $e_\mu \equiv \partial/\partial x^\mu$ is a basis vector. Evidently, $\dim\mathrm{T}_p\mathbb{M} = \dim\mathbb{M}$. The basis $\{e_\mu\}$ is called the coordinate basis. Note that a vector is not a line segment going from one point to the other on the manifold. Instead, it is defined at a single point. We have a tangent space associated with each point to the manifold. The collection of all these tangent spaces for all points on the manifold is called the **tangent bundle**, which is a manifold in its own right.

Although we have employed a coordinate system, it is clear from the above discussion that a vector X exists without this definition. We use coordinates only because it is convenient. The fact that the vector is coordinate independent allows us to find the transformation law of the components of the vector. Let $p \in \mathbb{O}_i \cap \mathbb{O}_j$ and $\{x^\mu\} = \phi_i(p)$ and $\{y^\nu\} = \phi_j(p)$ be the two coordinate functions. We then have two expressions for the vector $X \in \mathrm{T}_p\mathbb{M}$

$$X = X^\mu \frac{\partial}{\partial x^\mu} = \tilde{X}^\mu \frac{\partial}{\partial y^\mu}. \tag{2.12}$$

The second equality must hold since the vector is the same. This implies that the components of the vector in the two coordinate bases must be related by

$$\tilde{X}^\mu = X^\nu \frac{\partial y^\mu}{\partial x^\nu}. \tag{2.13}$$

This last relation follows from the application of the vector X to the coordinate functions y^ν. It is important to observe here that this transformation law is such that the vector itself is left invariant. So, the components of the vector change, not the vector.

If a vector is smoothly defined at each point of the manifold, we have a **vector field**. Therefore, X is a vector field if $X[f] \in \mathbb{F}(\mathbb{M})$ for any $f \in \mathbb{F}(\mathbb{M})$. The vector field defines, in this way, a map between smooth functions over the manifold.

Now, since we have defined the vector space $\mathrm{T}_p\mathbb{M}$, we can associate with it the dual vector space $\mathrm{T}^*_p\mathbb{M}$ (also called the cotangent space), whose elements are linear functions from $\mathrm{T}_p\mathbb{M}$ to $\mathbb{R}$, i.e. $\omega : \mathrm{T}_p\mathbb{M} \mapsto \mathbb{R}$ for $\omega \in \mathrm{T}^*_p\mathbb{M}$. ω is called the cotangent vector, the dual vector, or the one-form, with the simplest example being the differential $\mathrm{d}f$ of a function $f \in \mathbb{F}(\mathbb{M})$.

The action of $\mathrm{d}f \in \mathrm{T}^*_p\mathbb{M}$ on $V \in \mathrm{T}_p\mathbb{M}$ is defined as

$$\langle \mathrm{d}f, V\rangle = V[f] = V^\mu \frac{\partial f}{\partial x^\mu} \in \mathbb{R}. \tag{2.14}$$

We can clearly see that this expression is bilinear. Now, since $\mathrm{d}f = (\partial f/\partial x^\mu)\mathrm{d}x^\mu$, it is natural to take $\mathrm{d}x^\mu$ as the elements of the basis in $\mathrm{T}^*_p\mathbb{M}$, which is the dual basis since

$$\left\langle \mathrm{d}x^\nu, \frac{\partial}{\partial x^\mu}\right\rangle = \frac{\partial x^\nu}{\partial x^\mu} = \delta^\nu_\mu. \tag{2.15}$$

In this way, an arbitrary one form can be written as $\omega = \omega_\mu \mathrm{d}x^\mu$.

We are now in a position to define the **inner product** $\langle\, ,\rangle : \mathrm{T}^*_p\mathcal{M} \times \mathrm{T}_p\mathcal{M} \mapsto \mathbb{R}$ as

$$\langle \omega, V\rangle = \omega_\nu V^\mu \left\langle \mathrm{d}x^\nu, \frac{\partial}{\partial x^\mu}\right\rangle = \omega(V) = \omega_\mu V^\mu. \tag{2.16}$$

It is important to observe here that the inner product is defined in terms of the action of a dual vector on a vector, and not between two vectors.

Now, let us consider two coordinate systems $\{x^\mu\} = \phi_i(p)$ and $\{y^\nu\} = \phi_j(p)$ for the point $p \in \mathbb{O}_i \cap \mathbb{O}_j$. We thus have

$$\omega = \omega_\mu \mathrm{d}x^\mu = \tilde{\omega}_\nu \mathrm{d}y^\nu, \tag{2.17}$$

From the fact that $\mathrm{d}y^\nu = (\partial y^\nu/\partial x^\mu)\mathrm{d}x^\mu$ we can write down the transformation law for the components of the one-form ω

$$\tilde{\omega}_\nu = \omega_\mu \frac{\partial x^\mu}{\partial y^\nu}. \tag{2.18}$$

Again, the components of the one-form change in such a way that the one-form itself is left invariant.

2.4 Tensors

A tensor of type (q, r) is a multilinear object which maps q elements of $T_p^*\mathbb{M}$ and r elements of $T_p\mathbb{M}$ to a real number. The set of all tensors of type (q, r) at $p \in \mathbb{M}$ is denoted by $\mathbb{T}^q_{r,p}(\mathbb{M})$. We can write the tensor $T \in \mathbb{T}^q_{r,p}(\mathbb{M})$ in terms of the bases defined in the last section as

$$T = T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} \frac{\partial}{\partial x^{\mu_1}} \dots \frac{\partial}{\partial x^{\mu_1}} dx^{\nu_1} \dots dx^{\nu_r}. \tag{2.19}$$

Clearly, a tensor of type (q, r) is the multilinear map

$$T : \left[\times^q T_p^*\mathbb{M}\right] \left[\times^r T_p\mathbb{M}\right] \mapsto \mathbb{R}, \tag{2.20}$$

where the symbol $\times^q T_p^*\mathbb{M}$ means the Cartesian product of the space $T_p^*\mathbb{M}$ q times, with a similar definition for the vector space $T_p\mathbb{M}$. The action of a tensor on ω_i $(1 \leq i \leq r)$ and V_j $(1 \leq j \leq q)$ results in the number[7]

$$T(\omega_1, ..., \omega_r; V_1, ..., V_q) = T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} \omega_{1,\mu_1} \dots \omega_{q,\mu_q} V_1^{\nu_1} \dots V_r^{\nu_r}. \tag{2.21}$$

Given two coordinate systems x and x', the components of the tensor T change as

$$T^{\mu_1' \dots \mu_q'}{}_{\nu_1' \dots \nu_r'} = T^{\mu_1 \dots \mu_q}{}_{\nu_1 \dots \nu_r} \frac{dx^{\mu_1'}}{dx^{\mu_1}} \cdots \frac{dx^{\nu_1}}{dx^{\nu_1'}}, \tag{2.22}$$

which is the general transformation law for tensors.

Similarly, we did in the case of a vector field, we define a tensor field of type (q, r) by the smooth assignment of an element of $\mathbb{T}^q_{r,p}(\mathbb{M})$ to each point $p \in \mathbb{M}$. The set of the tensor fields of type (q, r) on $\mathbb{M}$ is denoted by $\mathbb{T}^q_r(\mathbb{M})$. For example, $\mathbb{T}^0_1(\mathbb{M})$ is the set of the dual vector fields, while $\mathbb{T}^1_0(\mathbb{M})$ is the tangent bundle.

A fundamental tensor for us is the metric tensor g_{ab}, which is a symmetric $(0, 2)$ tensor. By imposing that the determinant of the metric does not vanish, $\mathrm{Det}[g_{ab}] = g \neq 0$, we can properly define the inverse of the metric as

$$g^{\mu\nu} g_{\alpha\nu} = \delta^\mu_\alpha. \tag{2.23}$$

In the same way we used the Minkowski metric $\eta_{\mu\nu}$ to raise and lower indices in

[7]Note that the Latin indexes in these expressions are not labelling the components of the vectors, but they are labeling distinct vectors.

special relativity, in the general theory, we use the metric $g_{\mu\nu}$ to perform such tasks.

One of the many applications of the metric tensor is to provide the notion of the length of a path in spacetime, the line element

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu. \tag{2.24}$$

This equation makes sense because $\mathrm{d}x^\mu$ is really a basis dual vector. For instance, the Euclidean line element in the usual three-dimensional space is $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$, written in Cartesian coordinates. In this equation, and in most of these lectures, we employ the shorthand notation $\mathrm{d}x^2 \equiv \mathrm{d}x \otimes \mathrm{d}x$, except for the line element itself, $\mathrm{d}s^2$, which is just a new notation for the metric, not representing the square of any quantity.

The Newtonian spacetime is called Euclidean and has a positive metric, with signature[8] $(+,+,+)$. This is also true for the case of Riemannian geometry, where all eigenvalues of the metric are positive. However, the signature of the metric of special and general theory of relativity is $(+,-,-,-)$, which means that it is not positive semi-definite. This kind of metric is called **Lorentzian**.

An important observation here is that the partial derivative is not a true tensor, as can be seen by computing its transformation law. Physics needs derivatives, and if we want the laws of physics to be coordinate independent, we need to define the notion of the covariant derivative on manifolds.

[8]the number of positive and negative engenvalues.