Institute of Physics — Federal University of Goiás

Quantum Pequi Group

General Relativity

Lecture Notes

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Appendix A

Basic definitions

The main goal of this appendix is to provide some basic mathematical definitions supporting the main text. We present the definitions of vector, affine, and topological spaces, discussing some of their properties that are relevant for the understanding of the classical structure of spacetime. The idea of algebras, which are vector spaces with an additional structure, is also presented for completeness.

1.1 Vector Spaces

In order to introduce the concept of vector spaces, we need to define what is called a **field**, which is a non-empty set $\mathbb{F}$ together with two binary operations called addition ($+ : \mathbb{F} \times \mathbb{F} \mapsto \mathbb{F}$) and multiplication ($* : \mathbb{F} \times \mathbb{F} \mapsto \mathbb{F}$) that satisfy the following axioms for all $a, b \in \mathbb{F}$

- Associativity of addition and multiplication: $a+(b+c) = (a+b)+c$ and $a*(b*c) = (a * b) * c$
- Commutativity of addition and multiplication: $a + b = b + a$ and $a * b = b * a$
- Additive and multiplicative identity: There exist two different elements 0 and 1 in $\mathbb{F}$ such that $a + 0 = a$ and $1 * a = a$.
- Additive inverse: For every $a \in \mathbb{F}$, there exists the element $-a$, called the additive inverse of a, such that $a + (-a) = 0$
- Multiplicative inverse: For every $a \neq 0$ in $\mathbb{F}$, there exists an element, denoted by a^{-1}, called the multiplicative inverse of a, such that $a * a^{-1} = 1$
- Distributivity of multiplication over addition: $a * (b + c) = (a * b) + (a * c)$

The set $\mathbb{R}$ under the usual addition and multiplication of real numbers is an example of a field. The set of complex numbers $\mathbb{C}$ is $\mathbb{R}^2$, that is, the set of all vectors with two coordinates (x, y) with $x,\ y \in \mathbb{R}$. By defining addition and multiplication as $(x, y) + (u, v) = (x + u, y + v)$ and $(x, y) * (u, v) = (xu - yv, xv + yu)$, $\mathbb{C}$ is also a field. The set $\mathbb{Z}$ of integers is not a field since not all elements have a multiplicative inverse that belongs to the set.

Now, a **vector space** over a field $\mathbb{F}$ is the set $\mathbb{X}$ together with two operations: (i) Addition ($+ : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$) and ($ii$) scalar multiplication ($* : \mathbb{F} \times \mathbb{X} \to \mathbb{X}$). Such operations satisfy the following axioms for all $u, v, w \in \mathbb{X}$ and all $a, b \in \mathbb{F}$.

- Associativity: $x + (y + z) = (x + y) + z$
- Commutativity: $x + y = y + x$
- Identity element for addition: There exists an element $0 \in \mathbb{X}$ such that $x + 0 = x$ for all $x \in \mathbb{X}$.
- Inverse element for addition: For every $x \in \mathbb{X}$ there exists an element $-v \in \mathbb{X}$ such that $x + (-x) = 0$
- Compatibility: $a * (b * x) = (ab) * x$
- Identity element for scalar multiplication: $1 * x = x$.
- Distributivity of scalar multiplication with respect to vector addition: $a*(x+y) = a * x + a * y$
- Distributivity of scalar multiplication with respect to field addition: $(a + b) * x = a * x + b * x$.

In general, the elements of $\mathbb{X}$ are called **vectors**, while the ones belonging to $\mathbb{F}$ are called **scalars**. The simplest example of a vector space over a field $\mathbb{F}$ is the field itself. By performing addition and scalar multiplication pointwise, functions from any fixed set to $\mathbb{F}$ also form a vector space.

If $\mathbb{F}$ is a field, the Cartesian product $\mathbb{F}^n = \{(f_1, f_2, ..., f_n), f_j \in \mathbb{F}\}$ is a vector space over the field $\mathbb{F}$ with the addition operation defined as $(f_1, ..., f_n) + (g_1, ..., g_n) = (f_1 + g_1, ..., f_n + g_n)$ ($g_j \in \mathbb{F}$) and the product as $\alpha * (f_1, ..., f_n) = (\alpha * f_1, ..., \alpha * f_n)$ for all $\alpha \in \mathbb{F}$. The null vector is denoted by $(0, ..., 0)$.

The set $\mathbb{Z}$ is not a vector space since it is not closed under scalar multiplication. Also, the set of all polynomials of degree n is not a vector space since it is not closed under addition.

When considering geometry, we are often interested in properties that are invariant under the action of some symmetry group and then we can model a set of points (the space-time, for instance) as a vector space. However, this procedure has some disadvantages and we have a more appropriate construction, called **affine space**. First, in a vector space, the point 0, called the origin, plays a very important role, which is not a good thing when we are considering physical theories since we do not want preferred points. Moreover, vector spaces and affine spaces have very different geometrical properties. In the first case, bijective[1] linear maps keep the geometry invariant, while affine maps[2] (which form a much bigger set) are allowed in the last case. Moreover, affine spaces present a very interesting property for physics, since they allow us to handle geometry in an intrinsic manner that is independent of the coordinate system.

Let us define the affine space as the triple $\left(\mathbb{X}, \overrightarrow{\mathbb{X}}, +\right)$, with $\mathbb{X}$ being a set of points, $\overrightarrow{\mathbb{X}}$ a vector space and $(+ \,:\, \mathbb{X} \times \overrightarrow{\mathbb{X}} \mapsto \mathbb{X})$ a bilinear operation satisfying the following properties:

- $a + 0 = a$ for all $a \in \mathbb{X}$
- $(a + x) + y = a + (x + y)$ for all $a \in \mathbb{X}$ and all $x,\ y \in \overrightarrow{\mathbb{X}}$
- for any two points $a,\ b \in \mathbb{X}$, there is a unique $x \in \overrightarrow{\mathbb{X}}$ such that $a + x = b$.

In this definition, $\overrightarrow{\mathbb{X}}$ is called the set of free vectors, or free translations.

Given an $m \times n$ matrix A and a vector $b \in \mathbb{R}$, the set of solutions to the equation $Ax = b$, for $x \in \mathbb{R}^m$, is an affine space. Newtonian space-time is a four-dimensional affine space on which two additional structures are defined. A linear functional t called time and a Euclidean metric on each affine subspace defined by the vectors to which t assigns 0 (the simultaneity hypersurfaces).

Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. A map $l \,:\, \mathbb{F} \mapsto \mathbb{F}$ is a linear functional if

$$l(\alpha x + \beta y) = \alpha l(x) + \beta l(y), \tag{1.1}$$

for all $x, y \in \mathbb{V}$ and $\alpha, \beta \in \mathbb{F}$. The set of all linear functionals of $\mathbb{V}$ is the **dual space** of $\mathbb{V}$, denoted as $\mathbb{V}^*$.

The relation between $\mathbb{V}$ and $\mathbb{V}^*$ can be stated as follows. There exists at least one injective map $\phi \,:\, \mathbb{V} \mapsto \mathbb{V}^*$ in such a way that $\mathbb{V}$ is isomorphic[3] to its image under ϕ,

[1]The map $f \,:\, \mathbb{X} \mapsto \mathbb{Y}$ is injective if $x \neq x'$ implies $f(x) \neq f(x')$ for any $x, x' \in \mathbb{X}$. It is called surjective if for each $y \in \mathbb{Y}$ there exists $x \in \mathbb{X}$ such that $f(x) = y$. It is called bijective if it is both injective and surjective.

[2]A geometric transformation that preserves lines and parallelism, but not necessary angles and distances.

[3]Let us supposed that the sets $\mathbb{X}$ and $\mathbb{Y}$ are endowed with a certain algebraic structure (multiplication, for instance). If the map $f \,:\, \mathbb{X} \mapsto \mathbb{Y}$ preserves such structure, it is called an homomorphism. If f is bijective, is is called an isomorphism and the spaces are isomorphic to each other, $\mathbb{X} \cong \mathbb{Y}$.

$\mathbb{V} \cong \phi(\mathbb{V}) \subset \mathbb{V}^*$. If $\dim(\mathbb{V})$ is finite, then $\mathbb{V} \cong \mathbb{V}^*$. In the case $\dim(\mathbb{V})$ is infinite, there can be elements of $\mathbb{V}^*$ without such identification. In general, we have $\phi(\mathbb{V}) \subset \mathbb{V}^*$.

1.2 Topological spaces

Topological spaces are the most general mathematical spaces that allow us to define notions like limits and continuity. In general, topology is interested in the relations between points and regions and plays a fundamental role in general relativity.

Let us then define the **topological space**. Let $\mathbb{X}$ be a set and $\mathbb{T}$ a collection of subsets of $\mathbb{X}$ satisfying the following properties.

1. The union of an arbitrary collection of subsets, each of which is in $\mathbb{T}$, is in $\mathbb{T}$. If $\mathbb{O}_\alpha \in \mathbb{T}$ for all α, then $\cup_\alpha \mathbb{O}_\alpha \in \mathbb{T}$.

2. The intersection of a finite number of subsets of $\mathbb{T}$ is in $\mathbb{T}$. If $\{\mathbb{O}_i\}_{i=1}^n \in \mathbb{T}$, then $\cap_{i=1}^n \mathbb{O}_i \in \mathbb{T}$.

3. The entire set $\mathbb{X}$ and the empty set $\emptyset$ are in $\mathbb{T}$.

$\mathbb{X}$ is said to be a topological space, and $\mathbb{T}$ provides a topology to $\mathbb{X}$.

There is also a definition, due to Felix Hausdorff, in terms of neighborhoods of a point. Let $\mathbb{X}$ be a set. Let $\mathcal{N}$ be a function assigning to each $a \in \mathbb{X}$ a non-empty collection $\mathbb{N}(x)$ of subsets of $\mathbb{X}$. The elements of $\mathbb{N}(x)$ are called neighborhoods of x with respect to $\mathcal{N}$. The function $\mathcal{N}$ is called a neighborhood topology if the following axioms are satisfied:

1. If $\mathbb{N}$ is a neighborhood of x (i. e., $\mathbb{N} \in \mathbb{N}(x)$), then $x \in \mathbb{N}$. In other words, each point of the set $\mathbb{X}$ belongs to every one of its neighborhoods with respect to $\mathcal{N}$.

2. If $\mathbb{N}$ is a subset of $\mathbb{X}$ and includes a neighborhood of x, then $\mathbb{N}$ is a neighborhood of x. Every superset[4] of a neighborhood of a point $x \in \mathbb{X}$ is again a neighborhood of x.

3. The intersection of two neighborhoods of x is a neighborhood of x.

4. Any neighborhood $\mathbb{N}$ of x includes a neighborhood $\mathbb{M}$ of x such that $\mathbb{N}$ is a neighborhood of each point of $\mathbb{M}$

Then, $(\mathbb{X}, \mathcal{N})$ is called a topological space.

[4]A set $\mathbb{A}$ is a subset of a set $\mathbb{B}$ if all elements of $\mathbb{A}$ are also elements of $\mathbb{B}$; $\mathbb{B}$ is then a superset of $\mathbb{A}$.

A standard example of such a system of neighborhoods is for the real line $\mathbb{R}$, where a subset $\mathbb{N}$ of $\mathbb{R}$ is defined to be a neighborhood of a real number x if it includes an open interval containing x.

Let $\mathbb{X} = \{a, b, c\}$ and $\mathbb{T} = \{\mathbb{X}, \emptyset, \{a, b\}, \{b\}, \{b, c\}\}$. Then, the pair $(\mathbb{X}, \mathbb{T})$ is a topological space. Of course, there are many other topologies that can be chosen here by simply permuting a, b, and c. If $\mathbb{X}$ is any set, the collection of all subsets of $\mathbb{X}$ is called the discrete topology, while the set $\{\mathbb{X}, \emptyset\}$ is called the trivial topology. The set $\mathbb{R}$ along with all the open intervals (a, b) and their unions is a topological space.

If a metric[5] $d(x, y)$, with $x, y \in \mathbb{X}$, is defined on the set $\mathbb{X}$, its open sets are given by the open discs $\mathbb{O}_\epsilon = \{y \in \mathbb{X} \,|\, d(x, y) < \epsilon\}$ and all their possible unions. Such topology is called a metric topology determined by d and the topological space $(\mathbb{X}, \mathbb{T})$ is called a **metric space**.

From these notions we can define continuous maps between topological spaces. If $(\mathbb{X}, \mathbb{T}_x)$ and $(\mathbb{Y}, \mathbb{T}_y)$ are topological spaces, a map $f : \mathbb{X} \to \mathbb{Y}$ is said to be continuous if the inverse $f^{-1}[\mathbb{O}] = \{x \in \mathbb{X} \,|\, f(x) \in \mathbb{O}\}$ of every open set in $\mathbb{O} \subset \mathbb{Y}$ is an open set in $\mathbb{X}$.

A very important set of these open maps is the **homeomorphism**, which is continuous, one-to-one (distinct elements are mapped into distinct elements) and onto (every element of the domain is mapped into one of the elements of the codomain). The inverse map is also continuous. In this case, $(\mathbb{X}, \mathbb{T}_x)$ and $(\mathbb{Y}, \mathbb{T}_y)$ are said to be homeomorphic topological spaces, which means that they have identical topological properties. Homeomorphism is an equivalence relation.

From the physical point of view, we need to impose another condition on topological spaces in order for them to be able to describe space-times. The intuition behind this idea is that we need unique vector flows and, thus, unique limits (curves do not split into multiple curves). **Hausdorff** spaces present such property. A topological space is said to be Hausdorff if for each pair of distinct points $x,\ y \in \mathbb{X}$, $x \neq y$, one can find open sets $\mathbb{O}_x\,, \mathbb{O}_y \in \mathbb{T}$ such that $x \in \mathbb{O}_x$, $y \in \mathbb{O}_y$ and $\mathbb{O}_x \cap \mathbb{O}_y = \emptyset$. We can mention some important consequences of this definition. First, every finite set in a Hausdorff space is closed. Secondly, a sequence of points in a Hausdorff space converges to at most one point in the space. Moreover, the product of two Hausdorff spaces is also Hausdorff, and every subspace of a Hausdorff space is Hausdorff.

Another important property is **compactness**, which is the generalization of the Euclidean closed systems to topological spaces. Let $\mathbb{A}$ be a subset of $\mathbb{X}$ and $\{\mathbb{O}_\alpha\}$ a collection of open sets. If the union of these sets contains $\mathbb{A}$, $\{\mathbb{O}_\alpha\}$ is said to be an open cover of $\mathbb{A}$. A subcollection of the sets $\{\mathbb{O}_\alpha\}$ which also covers $\mathbb{A}$ is referred to as a subcover.

[5] $d : \mathbb{X} \times \mathbb{X} \mapsto \mathbb{R}$ under the conditions *i)* $d(x, y) = d(y, x)$, *ii)* $d(x, y) \geq 0$, with the equality holding only for $x = y$ and *iii)* $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in \mathbb{X}$.

If each of its covers has a finite subcover, the topological space is said to be compact. According to the Heine-Borel theorem for Euclidean spaces, compactness is equivalent to the set being closed and bounded.

Now, in order to study general relativity, we need a Lorentzian metric, which requires a Riemannian metric and the metric space must be **paracompact**. Let $(\mathbb{X}, \mathbb{T}_x)$ be a topological space and $\{\mathbb{O}_\alpha\}$ be an open cover of $\mathbb{X}$. An open cover $\{\mathbb{V}_\beta\}$ is said to be a refinement of $\{\mathbb{O}_\alpha\}$ if for each $\mathbb{V}_\beta$ there exists an $\mathbb{O}_\alpha$ such that $\mathbb{V}_\beta \subset \mathbb{O}_\alpha$. The cover $\{\mathbb{V}_\beta\}$ is said to be locally finite if each $x \in \mathbb{X}$ has an open neighborhood $\mathbb{W}$ such that only finitely many $\mathbb{V}_\beta$ satisfy $\mathbb{W} \cap \mathbb{V}_\beta \neq \emptyset$. A topological space is said to be paracompact if every open cover $\{\mathbb{O}_\alpha\}$ of $\mathbb{X}$ has a locally finite refinement $\{\mathbb{V}_\beta\}$. This last condition is necessary in order for the topological space to be homeomorphic to a metric space.

The space-time in general relativity is postulated as a differentiable manifold that is Hausdorff and paracompact. Moreover, it must be **connected**, which is a property that says that the topological space cannot be covered by the union of two or more disjoint non-empty open subsets.

1.3 Algebras

An algebra is a vector space $\mathbb{V}$ (over a field $\mathbb{F}$) along with a binary operation $\cdot$, the product of the algebra, such that the following properties are satisfied for all $a, b, c \in \mathbb{V}$ and $\alpha \in \mathbb{F}$.

- Distributivity with respect to the vectorial addition: $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$.
- Commutativity with respect to the scalar product.

Now we present some important examples of algebras that are commonly found while studying physics.

Lie algebra $\mathbb{L}$ — The product of $\mathbb{L}$ is denoted as $[a, b]$, with $a, b \in \mathbb{L}$, and it must satisfy the following properties: i) $[a, a] = 0$, which implies $[a, b] = -[b, a]$ and ii) the Jacobi identity $[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0$. The set $\mathbb{R}^3$ with the usual cross product is an example of a Lie algebra, as well as the set of all $n \times n$ matrices over the field $\mathbb{F}$, Mat$(\mathbb{F}, n)$, under the product $[A, B] = AB - BA$, with $A, B \in$ Mat$(\mathbb{F}, n)$.

Poisson algebra $\mathbb{P}$ — Is the vector space $\mathbb{P}$ (over the field $\mathbb{F}$) along with two products, $*$ and $\{,\}$ such that: i) $\mathbb{P}$ is associative with respect to $*$, ii) $\mathbb{P}$ is a Lie algebra with respect to $\{,\}$ and iii) for all $a, b, c \in \mathbb{P}$, the Leibniz identity holds $\{a, b * c\} = \{a, b\} * c + b * \{a, c\}$. Given two C^∞ functions $f(x, p) : \mathbb{R}^2 \mapsto \mathbb{R}$ and $g(x, p) : \mathbb{R}^2 \mapsto \mathbb{R}$, the Poisson brackets

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} \tag{1.2}$$

makes the set of all C^∞ a Lie algebra.

Jordan algebra $\mathbb{J}$ — The product of the algebra satisfies, for all $a, b \in \mathbb{J}$, the following properties: i) commutativity, $a \cdot b = b \cdot a$, and ii) the Jordan identity $(a \cdot a) \cdot (a \cdot b) = a \cdot ((a \cdot a) \cdot b)$. The set of all self-adjoint matrices of Mat$(\mathbb{C}, n)$ with the Jordan product

$$a \cdot b = \frac{1}{2}(a \cdot b + b \cdot a) \tag{1.3}$$

is a Jordan algebra.

Grassmann algebra $\Gamma(\mathbb{V})$ — Let $\mathbb{V}$ be a vector space over the field $\mathbb{F}$. A Grassmann algebra over $\mathcal{V}$ is an associative and unital[6] algebra over $\mathbb{F}$, with the product $\wedge$ satisfying the following properties: i) $\mathbb{V}$ is a subspace of $\Gamma(\mathbb{V})$ and ii) for all $v \in \mathbb{V}$, we have $v \wedge v = 0$.

Clifford algebra $Cl(\mathbb{V}, \omega)$ — Let $\mathbb{V}$ be a vector space over the field $\mathbb{F}$. Let ω be a symmetric bilinear form[7] over $\mathbb{V}$. A Clifford algebra over $\mathbb{V}$ and ω is an associative algebra with the unity e, such that the following properties hold: i) $\mathbb{V}$ is a subspace of $Cl(\mathbb{V}, \omega)$ and ii) for all $v \in \mathbb{V}$, $v^2 = \omega(v, v)e$. The set of Pauli matrices σ_x, σ_y and σ_z, with $\mathbb{V} = \mathbb{R}^3$ and $\omega(u, v) = \sum_{a,b} u_a v_b \delta_{a,b}$, where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ is a Clifford algebra. If $\mathbb{V}$ is the Minkowski spacetime and $\omega = \sum_{\mu\nu} u_\mu v_\nu \eta^{\mu\nu}$, the set of Dirac matrices γ^μ is a Clifford algebra. u_μ represents the components of the four-vector u (the same applying for v) while $\eta_{\mu\nu}$ is the usual Minkowski metric.

[6]The algebra is called unital if there is the neutral element $\mathbb{1}$ such that $\mathbb{1} \cdot a = a \cdot \mathbb{1} = a$ for all a in the algebra.

[7]See Chapter 2 for details on bilinear forms.