Institute of Physics — Federal University of Goiás

Quantum Pequi Group

Special and General Relativity

Lecture Notes

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Appendix A

Basic definitions

The main goal of this appendix is to provide some basic mathematical definitions supporting the main text. We present the definitions of vector, affine, and topological spaces, discussing some of their properties that are relevant for the understanding of the classical structure of spacetime. The idea of algebras, which are vector spaces with an additional structure, is also presented for completeness.

1.1 Vector Spaces

In order to introduce the concept of vector spaces, we need to define what is called a **field**, which is a non-empty set $\mathbb{F}$ together with two binary operations called addition ($+ : \mathbb{F} \times \mathbb{F} \mapsto \mathbb{F}$) and multiplication ($* : \mathbb{F} \times \mathbb{F} \mapsto \mathbb{F}$) that satisfy the following axioms for all $a, b \in \mathbb{F}$

- Associativity of addition and multiplication: $a+(b+c) = (a+b)+c$ and $a*(b*c) = (a*b)*c$
- Commutativity of addition and multiplication: $a + b = b + a$ and $a * b = b * a$
- Additive and multiplicative identity: There exist two different elements 0 and 1 in $\mathbb{F}$ such that $a + 0 = a$ and $1 * a = a$.
- Additive inverse: For every $a \in \mathbb{F}$, there exists the element $-a$, called the additive inverse of a, such that $a + (-a) = 0$
- Multiplicative inverse: For every $a \neq 0$ in $\mathbb{F}$, there exists an element, denoted by a^{-1}, called the multiplicative inverse of a, such that $a * a^{-1} = 1$
- Distributivity of multiplication over addition: $a * (b + c) = (a * b) + (a * c)$

The set $\mathbb{R}$ under the usual addition and multiplication of real numbers is an example of a field. The set of complex numbers $\mathbb{C}$ is $\mathbb{R}^2$, that is, the set of all vectors with two coordinates (x, y) with $x,\ y \in \mathbb{R}$. By defining addition and multiplication as $(x, y) + (u, v) = (x + u, y + v)$ and $(x, y) * (u, v) = (xu - yv, xv + yu)$, $\mathbb{C}$ is also a field. The set $\mathbb{Z}$ of integers is not a field since not all elements have a multiplicative inverse that belongs to the set.

Now, a **vector space** over a field $\mathbb{F}$ is the set $\mathbb{X}$ together with two operations: (i) Addition ($+ : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$) and (ii) scalar multiplication ($* : \mathbb{F} \times \mathbb{X} \to \mathbb{X}$). Such operations satisfy the following axioms for all $u, v, w \in \mathbb{X}$ and all $a, b \in \mathbb{F}$.

- Associativity: $x + (y + z) = (x + y) + z$
- Commutativity: $x + y = y + x$
- Identity element for addition: There exists an element $0 \in \mathbb{X}$ such that $x + 0 = x$ for all $x \in \mathbb{X}$.
- Inverse element for addition: For every $x \in \mathbb{X}$ there exists an element $-v \in \mathbb{X}$ such that $x + (-x) = 0$
- Compatibility: $a * (b * x) = (ab) * x$
- Identity element for scalar multiplication: $1 * x = x$.
- Distributivity of scalar multiplication with respect to vector addition: $a*(x+y) = a * x + a * y$
- Distributivity of scalar multiplication with respect to field addition: $(a + b) * x = a * x + b * x$.

In general, the elements of $\mathbb{X}$ are called **vectors**, while the ones belonging to $\mathbb{F}$ are called **scalars**. The simplest example of a vector space over a field $\mathbb{F}$ is the field itself. By performing addition and scalar multiplication pointwise, functions from any fixed set to $\mathbb{F}$ also form a vector space.

If $\mathbb{F}$ is a field, the Cartesian product $\mathbb{F}^n = \{(f_1, f_2, ..., f_n), f_j \in \mathbb{F}\}$ is a vector space over the field $\mathbb{F}$ with the addition operation defined as $(f_1, ..., f_n) + (g_1, ..., g_n) = (f_1 + g_1, ..., f_n + g_n)$ $(g_j \in \mathbb{F})$ and the product as $\alpha * (f_1, ..., f_n) = (\alpha * f_1, ..., \alpha * f_n)$ for all $\alpha \in \mathbb{F}$. The null vector is denoted by $(0, ..., 0)$.

The set $\mathbb{Z}$ is not a vector space since it is not closed under scalar multiplication. Also, the set of all polynomials of degree n is not a vector space since it is not closed under addition.

When considering geometry, we are often interested in properties that are invariant under the action of some symmetry group and then we can model a set of points (the space-time, for instance) as a vector space. However, this procedure has some disadvantages and we have a more appropriate construction, called **affine space**. First, in a vector space, the point 0, called the origin, plays a very important role, which is not a good thing when we are considering physical theories since we do not want preferred points. Moreover, vector spaces and affine spaces have very different geometrical properties. In the first case, bijective[1] linear maps keep the geometry invariant, while affine maps[2] (which form a much bigger set) are allowed in the last case. Moreover, affine spaces present a very interesting property for physics, since they allow us to handle geometry in an intrinsic manner that is independent of the coordinate system.

Let us define the affine space as the triple $\left(\mathbb{X}, \overrightarrow{\mathbb{X}}, +\right)$, with $\mathbb{X}$ being a set of points, $\overrightarrow{\mathbb{X}}$ a vector space and $(+ \,:\, \mathbb{X} \times \overrightarrow{\mathbb{X}} \mapsto \mathbb{X})$ a bilinear operation satisfying the following properties:

- $a + 0 = a$ for all $a \in \mathbb{X}$
- $(a + x) + y = a + (x + y)$ for all $a \in \mathbb{X}$ and all $x,\ y \in \overrightarrow{\mathbb{X}}$
- for any two points $a,\ b \in \mathbb{X}$, there is a unique $x \in \overrightarrow{\mathbb{X}}$ such that $a + x = b$.

In this definition, $\overrightarrow{\mathbb{X}}$ is called the set of free vectors, or free translations.

Given an $m \times n$ matrix A and a vector $b \in \mathbb{R}$, the set of solutions to the equation $Ax = b$, for $x \in \mathbb{R}^m$, is an affine space. Newtonian space-time is a four-dimensional affine space on which two additional structures are defined. A linear functional t called time and a Euclidean metric on each affine subspace defined by the vectors to which t assigns 0 (the simultaneity hypersurfaces).

Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. A map $l \,:\, \mathbb{F} \mapsto \mathbb{F}$ is a linear functional if

$$l(\alpha x + \beta y) = \alpha l(x) + \beta l(y), \tag{1.1}$$

for all $x, y \in \mathbb{V}$ and $\alpha, \beta \in \mathbb{F}$. The set of all linear functionals of $\mathbb{V}$ is the **dual space** of $\mathbb{V}$, denoted as $\mathbb{V}^*$.

The relation between $\mathbb{V}$ and $\mathbb{V}^*$ can be stated as follows. There exists at least one injective map $\phi \,:\, \mathbb{V} \mapsto \mathbb{V}^*$ in such a way that $\mathbb{V}$ is isomorphic[3] to its image under ϕ,

[1]The map $f \,:\, \mathbb{X} \mapsto \mathbb{Y}$ is injective if $x \neq x'$ implies $f(x) \neq f(x')$ for any $x, x' \in \mathbb{X}$. It is called surjective if for each $y \in \mathbb{Y}$ there exists $x \in \mathbb{X}$ such that $f(x) = y$. It is called bijective if it is both injective and surjective.

[2]A geometric transformation that preserves lines and parallelism, but not necessary angles and distances.

[3]Let us supposed that the sets $\mathbb{X}$ and $\mathbb{Y}$ are endowed with a certain algebraic structure (multiplication, for instance). If the map $f \,:\, \mathbb{X} \mapsto \mathbb{Y}$ preserves such structure, it is called an homomorphism. If f is bijective, is is called an isomorphism and the spaces are isomorphic to each other, $\mathbb{X} \cong \mathbb{Y}$.

$\mathbb{V} \cong \phi(\mathbb{V}) \subset \mathbb{V}^*$. If $\dim(\mathbb{V})$ is finite, then $\mathbb{V} \cong \mathbb{V}^*$. In the case $\dim(\mathbb{V})$ is infinite, there can be elements of $\mathbb{V}^*$ without such identification. In general, we have $\phi(\mathbb{V}) \subset \mathbb{V}^*$.

1.2 Topological spaces

Topological spaces are the most general mathematical spaces that allow us to define notions like limits and continuity. In general, topology is interested in the relations between points and regions and plays a fundamental role in general relativity.

Let us then define the **topological space**. Let $\mathbb{X}$ be a set and $\mathbb{T}$ a collection of subsets of $\mathbb{X}$ satisfying the following properties.

1. The union of an arbitrary collection of subsets, each of which is in $\mathbb{T}$, is in $\mathbb{T}$. If $\mathbb{O}_\alpha \in \mathbb{T}$ for all α, then $\cup_\alpha \mathbb{O}_\alpha \in \mathbb{T}$.

2. The intersection of a finite number of subsets of $\mathbb{T}$ is in $\mathbb{T}$. If $\{\mathbb{O}_i\}_{i=1}^n \in \mathbb{T}$, then $\cap_{i=1}^n \mathbb{O}_i \in \mathbb{T}$.

3. The entire set $\mathbb{X}$ and the empty set $\emptyset$ are in $\mathbb{T}$.

$\mathbb{X}$ is said to be a topological space, and $\mathbb{T}$ provides a topology to $\mathbb{X}$.

There is also a definition, due to Felix Hausdorff, in terms of neighborhoods of a point. Let $\mathbb{X}$ be a set. Let $\mathcal{N}$ be a function assigning to each $a \in \mathbb{X}$ a non-empty collection $\mathbb{N}(x)$ of subsets of $\mathbb{X}$. The elements of $\mathbb{N}(x)$ are called neighborhoods of x with respect to $\mathcal{N}$. The function $\mathcal{N}$ is called a neighborhood topology if the following axioms are satisfied:

1. If $\mathbb{N}$ is a neighborhood of x (i. e., $\mathbb{N} \in \mathbb{N}(x)$), then $x \in \mathbb{N}$. In other words, each point of the set $\mathbb{X}$ belongs to every one of its neighborhoods with respect to $\mathcal{N}$.

2. If $\mathbb{N}$ is a subset of $\mathbb{X}$ and includes a neighborhood of x, then $\mathbb{N}$ is a neighborhood of x. Every superset[4] of a neighborhood of a point $x \in \mathbb{X}$ is again a neighborhood of x.

3. The intersection of two neighborhoods of x is a neighborhood of x.

4. Any neighborhood $\mathbb{N}$ of x includes a neighborhood $\mathbb{M}$ of x such that $\mathbb{N}$ is a neighborhood of each point of $\mathbb{M}$

Then, $(\mathbb{X}, \mathcal{N})$ is called a topological space.

[4]A set $\mathbb{A}$ is a subset of a set $\mathbb{B}$ if all elements of $\mathbb{A}$ are also elements of $\mathbb{B}$; $\mathbb{B}$ is then a superset of $\mathbb{A}$.

A standard example of such a system of neighborhoods is for the real line $\mathbb{R}$, where a subset $\mathbb{N}$ of $\mathbb{R}$ is defined to be a neighborhood of a real number x if it includes an open interval containing x.

Let $\mathbb{X} = \{a, b, c\}$ and $\mathbb{T} = \{\mathbb{X}, \emptyset, \{a, b\}, \{b\}, \{b, c\}\}$. Then, the pair $(\mathbb{X}, \mathbb{T})$ is a topological space. Of course, there are many other topologies that can be chosen here by simply permuting a, b, and c. If $\mathbb{X}$ is any set, the collection of all subsets of $\mathbb{X}$ is called the discrete topology, while the set $\{\mathbb{X}, \emptyset\}$ is called the trivial topology. The set $\mathbb{R}$ along with all the open intervals (a, b) and their unions is a topological space.

If a metric[5] $d(x, y)$, with $x, y \in \mathbb{X}$, is defined on the set $\mathbb{X}$, its open sets are given by the open discs $\mathbb{O}_\epsilon = \{y \in \mathbb{X} \,|\, d(x, y) < \epsilon\}$ and all their possible unions. Such topology is called a metric topology determined by d and the topological space $(\mathbb{X}, \mathbb{T})$ is called a **metric space**.

From these notions we can define continuous maps between topological spaces. If $(\mathbb{X}, \mathbb{T}_x)$ and $(\mathbb{Y}, \mathbb{T}_y)$ are topological spaces, a map $f : \mathbb{X} \to \mathbb{Y}$ is said to be continuous if the inverse image (preimage) $f^{-1}[\mathbb{O}] = \{x \in \mathbb{X} \,|\, f(x) \in \mathbb{O}\}$ of every open set in $\mathbb{O} \subset \mathbb{Y}$ is an open set in $\mathbb{X}$.

A very important set of these open maps is the **homeomorphism**, which is continuous, one-to-one (distinct elements are mapped into distinct elements) and onto (every element of the domain is mapped into one of the elements of the codomain). The inverse map is also continuous. In this case, $(\mathbb{X}, \mathbb{T}_x)$ and $(\mathbb{Y}, \mathbb{T}_y)$ are said to be homeomorphic topological spaces, which means that they have identical topological properties. Homeomorphism is an equivalence relation.

From the physical point of view, we need to impose another condition on topological spaces in order for them to be able to describe space-times. The intuition behind this idea is that we need unique vector flows and, thus, unique limits (curves do not split into multiple curves). **Hausdorff** spaces present such property. A topological space is said to be Hausdorff if for each pair of distinct points $x,\ y \in \mathbb{X}$, $x \neq y$, one can find open sets $\mathbb{O}_x\,, \mathbb{O}_y \in \mathbb{T}$ such that $x \in \mathbb{O}_x$, $y \in \mathbb{O}_y$ and $\mathbb{O}_x \cap \mathbb{O}_y = \emptyset$. We can mention some important consequences of this definition. First, every finite set in a Hausdorff space is closed. Secondly, a sequence of points in a Hausdorff space converges to at most one point in the space. Moreover, the product of two Hausdorff spaces is also Hausdorff, and every subspace of a Hausdorff space is Hausdorff.

Another important property is **compactness**, which is the generalization of the Euclidean closed systems to topological spaces. Let $\mathbb{A}$ be a subset of $\mathbb{X}$ and $\{\mathbb{O}_\alpha\}$ a collection of open sets. If the union of these sets contains $\mathbb{A}$, $\{\mathbb{O}_\alpha\}$ is said to be an open cover of $\mathbb{A}$. A subcollection of the sets $\{\mathbb{O}_\alpha\}$ which also covers $\mathbb{A}$ is referred to as a subcover.

[5] $d : \mathbb{X} \times \mathbb{X} \mapsto \mathbb{R}$ under the conditions *i)* $d(x, y) = d(y, x)$, *ii)* $d(x, y) \geq 0$, with the equality holding only for $x = y$ and *iii)* $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in \mathbb{X}$.

If each of its covers has a finite subcover, the topological space is said to be compact. According to the Heine-Borel theorem for Euclidean spaces, compactness is equivalent to the set being closed and bounded.

Now, in order to study general relativity, we need a Lorentzian metric, which requires a Riemannian metric and the metric space must be **paracompact**. Let $(\mathbb{X}, \mathbb{T}_x)$ be a topological space and $\{\mathbb{O}_\alpha\}$ be an open cover of $\mathbb{X}$. An open cover $\{\mathbb{V}_\beta\}$ is said to be a refinement of $\{\mathbb{O}_\alpha\}$ if for each $\mathbb{V}_\beta$ there exists an $\mathbb{O}_\alpha$ such that $\mathbb{V}_\beta \subset \mathbb{O}_\alpha$. The cover $\{\mathbb{V}_\beta\}$ is said to be locally finite if each $x \in \mathbb{X}$ has an open neighborhood $\mathbb{W}$ such that only finitely many $\mathbb{V}_\beta$ satisfy $\mathbb{W} \cap \mathbb{V}_\beta \neq \emptyset$. A topological space is said to be paracompact if every open cover $\{\mathbb{O}_\alpha\}$ of $\mathbb{X}$ has a locally finite refinement $\{\mathbb{V}_\beta\}$. This last condition is necessary in order for the topological space to be homeomorphic to a metric space.

The space-time in general relativity is postulated as a differentiable manifold that is Hausdorff and paracompact. Moreover, it must be **connected**, which is a property that says that the topological space cannot be covered by the union of two or more disjoint non-empty open subsets.

1.3 Algebras

An algebra is a vector space $\mathbb{V}$ (over a field $\mathbb{F}$) along with a binary operation $\cdot$, the product of the algebra, such that the following properties are satisfied for all $a, b, c \in \mathbb{V}$ and $\alpha \in \mathbb{F}$.

- Distributivity with respect to the vectorial addition: $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$.
- Commutativity with respect to the scalar product.

Now we present some important examples of algebras that are commonly found while studying physics.

Lie algebra $\mathbb{L}$ — The product of $\mathbb{L}$ is denoted as $[a, b]$, with $a, b \in \mathbb{L}$, and it must satisfy the following properties: $i)$ $[a, a] = 0$, which implies $[a, b] = -[b, a]$ and $ii)$ the Jacobi identity $[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0$. The set $\mathbb{R}^3$ with the usual cross product is an example of a Lie algebra, as well as the set of all $n \times n$ matrices over the field $\mathbb{F}$, Mat$(\mathbb{F}, n)$, under the product $[A, B] = AB - BA$, with $A, B \in$ Mat$(\mathbb{F}, n)$.

Poisson algebra $\mathbb{P}$ — Is the vector space $\mathbb{P}$ (over the field $\mathbb{F}$) along with two products, $*$ and $\{,\}$ such that: $i)$ $\mathbb{P}$ is associative with respect to $*$, $ii)$ $\mathbb{P}$ is a Lie algebra with respect to $\{,\}$ and $iii)$ for all $a, b, c \in \mathbb{P}$, the Leibniz identity holds $\{a, b * c\} = \{a, b\} * c + b * \{a, c\}$. Given two C^∞ functions $f(x, p) : \mathbb{R}^2 \mapsto \mathbb{R}$ and $g(x, p) : \mathbb{R}^2 \mapsto \mathbb{R}$, the Poisson brackets

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial p} \tag{1.2}$$

makes the set of all C^{∞} a Lie algebra.

Jordan algebra $\mathbb{J}$ — The product of the algebra satisfies, for all $a, b \in \mathbb{J}$, the following properties: i) commutativity, $a \cdot b = b \cdot a$, and ii) the Jordan identity $(a \cdot a) \cdot (a \cdot b) = a \cdot ((a \cdot a) \cdot b)$. The set of all self-adjoint matrices of Mat$(\mathbb{C}, n)$ with the Jordan product

$$a \cdot b = \frac{1}{2}(a \cdot b + b \cdot a) \tag{1.3}$$

is a Jordan algebra.

Grassmann algebra $\Gamma(\mathbb{V})$ — Let $\mathbb{V}$ be a vector space over the field $\mathbb{F}$. A Grassmann algebra over $\mathcal{V}$ is an associative and unital[6] algebra over $\mathbb{F}$, with the product $\wedge$ satisfying the following properties: i) $\mathbb{V}$ is a subspace of $\Gamma(\mathbb{V})$ and ii) for all $v \in \mathbb{V}$, we have $v \wedge v = 0$.

Clifford algebra $Cl(\mathbb{V}, \omega)$ — Let $\mathbb{V}$ be a vector space over the field $\mathbb{F}$. Let ω be a symmetric bilinear form[7] over $\mathbb{V}$. A Clifford algebra over $\mathbb{V}$ and ω is an associative algebra with the unity e, such that the following properties hold: i) $\mathbb{V}$ is a subspace of $Cl(\mathbb{V}, \omega)$ and ii) for all $v \in \mathbb{V}$, $v^2 = \omega(v, v)e$. The set of Pauli matrices σ_x, σ_y and σ_z, with $\mathbb{V} = \mathbb{R}^3$ and $\omega(u, v) = \sum_{a,b} u_a v_b \delta_{a,b}$, where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ is a Clifford algebra. If $\mathbb{V}$ is the Minkowski spacetime and $\omega = \sum_{\mu\nu} u_\mu v_\nu \eta^{\mu\nu}$, the set of Dirac matrices γ^μ is a Clifford algebra. u_μ represents the components of the four-vector u (the same applying for v) while $\eta_{\mu\nu}$ is the usual Minkowski metric.

[6]The algebra is called unital if there is the neutral element $\mathbb{1}$ such that $\mathbb{1} \cdot a = a \cdot \mathbb{1} = a$ for all a in the algebra.

[7]See Chapter 3 for details on bilinear forms.

Appendix B

A bit of calculus on $\mathbb{R}^n$

This Appendix has the simple goal of defining vectors as directional derivatives in $\mathbb{R}^n$, motivating in this way such a definition in a general manifold presented in Chapter 3. Let us then start by stating a few definitions.

Let $\{x^\mu\} = \{x^0, x^1, \cdots, x^{n-1}\}$ be some coordinate system in $\mathbb{R}^n$ and $p \in \mathbb{U} \subset \mathbb{R}^n$ a point in the open set $\mathbb{U}$. A function $f : \mathbb{U} \mapsto \mathbb{R}$ is C^k at p if the partial derivatives

$$\frac{\partial f}{\partial x^{i_1} \cdots \partial x^{i_j}},$$

exist and are continuous[1] at p for all $j \leq k$. A vector valued function $\mathbf{f} : \mathbb{U} \mapsto \mathbb{R}^m$ ($m \leq n$) is said to be C^k at $p \in \mathbb{U}$ if all of its component functions $\{f^0, f^1, \cdots, f^{m-1}\}$ are of C^k at p.

A neighborhood of a point p in $\mathbb{R}^n$ is an open set containing p. A function f is analytic at p if in some neighborhood of p it is equal to its Taylor series

$$f(x) = \sum_{n=0}^{\infty} \frac{(x-p)^n}{n!} \left.\frac{\partial^n f}{\partial x^n}\right|_{x=p}.$$

It is clear that an analytic function is necessary C^∞. We are now in a position to define tangent vectors.

Let $\mathbb{T}_p(\mathbb{R}^n)$ be the **tangent space** of $\mathbb{R}^n$ at point p. A vector $\mathrm{v} \in \mathbb{T}_p(\mathbb{R}^n)$ can be defined as

$$\mathrm{v} = \sum_{\mu=1}^{n} v^\mu e_\mu \equiv v^\mu e_\mu,$$

[1] A function $f : \mathbb{U} \mapsto \mathbb{R}$ is said to be continuous at a point $p \in \mathbb{U}$ if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x \in \mathbb{U}$: $|x - p| < \delta$ implies $|f(x) - f(p)| < \epsilon$.

with e_μ representing some coordinate basis vector.

Now, let $f \in C^\infty$ in a neighborhood of p, $\mathrm{v} \in \mathbb{T}_p(\mathbb{R}^n)$ and $c(t) = (p^0 + tv^0, \cdots, p^{n-1} + tv^{n-1})$ a curve through p parametrized by t. $p^\mu + tv^\mu$ is the μ-th coordinate of the curve. The **directional derivative** of f in the direction of v at p is defined as

$$\mathrm{D}_{\mathrm{v}} f = \lim_{t \to \infty} \frac{f(c(t)) - f(p)}{t} = \frac{\mathrm{d} f(c(t))}{\mathrm{d}t}\bigg|_{t=0} = \frac{\mathrm{d}c^\mu(t)}{\mathrm{d}t}\bigg|_{t=0} \frac{\partial f(c(t))}{\partial x^\mu}\bigg|_{\mathrm{x}=p} = v^\mu \, \frac{\partial f(c(t))}{\partial x^\mu}\bigg|_{\mathrm{x}=p}.$$

Note that $\mathrm{D}_{\mathrm{v}} f$ is a number, not a function. $\mathrm{D}_{\mathrm{v}} : f \in C^\infty \mapsto \mathrm{D}_{\mathrm{v}} f \in \mathbb{R}$ is the directional derivative operator. Any linear map $\mathrm{D} : C^\infty \mapsto \mathbb{R}$ satisfying the Leibniz rule is called a derivation. We denote by $\mathrm{D}_p\mathbb{R}^n$ the set of all the derivations at p.

Now, since $\mathrm{D}_p\mathbb{R}^n$ is a vector space, the map

$$\phi : \mathbb{T}_p\mathbb{R}^n \mapsto \mathrm{D}_p\mathbb{R}^n$$

is an isomorphism. This shows that we can identify the tangent vectors at p with the derivations at p. Under this isomorphism, the standard basis e_μ for $\mathbb{T}_p\mathbb{R}^n$ corresponds to the basis

$$\left\{ \frac{\partial}{\partial x^\mu}\bigg|_p \right\}_{\mu=0}^{n-1},$$

which allows us to write any tangent vector as

$$\mathrm{v} = v^\mu \, \frac{\partial}{\partial x^\mu}\bigg|_p.$$

From this, we can define the **vector field** as the function that assigns to every point $p \in \mathbb{U} \subset \mathbb{R}^2$ a tangent vector $\mathrm{X}_p \in \mathrm{T_p R^n}$

$$\mathrm{X}_p = \alpha^\mu(p) \, \frac{\partial}{\partial x^\mu}\bigg|_p \qquad \alpha^\mu \in \mathbb{R}.$$

If X is a vector field on $\mathbb{U}$ and f is a C^∞ on $\mathbb{U}$, we can write

$$(\mathrm{X}f)_p \equiv \mathrm{X}_p f = \alpha^\mu(p) \, \frac{\partial f}{\partial x^\mu}\bigg|_p,$$

which shows that $\mathrm{X}f$ is a C^∞ function on $\mathbb{U}$. A basic principle in the theory of manifolds is that every manifold can be locally approximated by a tangent space at a given point. This is the way linear algebra enters into manifold theory.

To close this Appendix, we define the differential form. The **cotangent space** to $\mathbb{R}^n$ at p is defined to be the dual space of $\mathbb{T}_p\mathbb{R}^n$, denoted by $\mathbb{T}_p^*\mathbb{R}^n$ and whose elements are called covectors, or linear functionals on $\mathbb{T}_p\mathbb{R}^n$.

A **covector field** on an open subset $\mathbb{U} \subset \mathbb{R}^n$ is a function ω that assigns to every point $p \in \mathbb{U}$ a covector $\omega_p \in \mathbb{T}_p^*\mathbb{R}^n$

$$\omega : \mathbb{U} \mapsto \cup_{p\in\mathbb{U}} \mathbb{T}_p^*\mathbb{R}^n \qquad p \mapsto \omega_p \in \mathbb{T}_p^*\mathbb{R}^n.$$

ω is also called a differential 1-form, or 1-form for short.

For any $f \in C^\infty$ we can construct a 1-form called the differential of f, denoted by $\mathrm{d}f$ as follows. For $p \in \mathbb{U}$ and $\mathrm{X}_p \in \mathbb{T}_p\mathbb{U}$ define

$$(\mathrm{d}f)_p \mathrm{X}_p - \mathrm{X}_p f.$$

Now, we choose $\{x^\mu\}$ as a coordinate system on $\mathbb{R}^n$ and, at each point $p \in \mathbb{R}^n$, $\{\mathrm{d}x^\mu\}$ a basis for the cotangent space $\mathbb{T}_p^*\mathbb{R}^n$, dual to the basis e_μ for the tangent space $\mathbb{T}_p\mathbb{R}^n$

$$(\mathrm{d}x^\mu)_p \left(\left.\frac{\partial}{\partial x^\nu}\right|_p \right) = \left.\frac{\partial x^\mu}{\partial x^\nu}\right|_p = \delta^\mu_\nu,$$

with δ^μ_ν being the Kronecker delta symbol. It follows that each covector can be written as

$$\omega_p = \alpha_\mu(p)\mathrm{d}x^\mu|_p \equiv \alpha_\mu \mathrm{d}x^\mu.$$

Appendix C

Introduction to group theory

This Appendix presents a basic introduction to group theory and group representation theory, which provides the mathematical language to describe symmetries. We also describe the structure of some very special groups that play fundamental roles in General Relativity.

Let us start by defining what a group is. The set $\mathbb{G}$ is called a group if there is a binary operation $\cdot : \mathbb{G} \times \mathbb{G} \mapsto \mathbb{G}$ called group multiplication that satisfies the following properties:

- Associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in \mathbb{G}$.
- There exists the identity element $e \in \mathbb{G}$ such that: $a \cdot e = a$ for all $a \in \mathbb{G}$.
- For very element $a \in \mathbb{G}$ there exists the inverse element $b \neq a$, denoted as $b = a^{-1}$, such that $a \cdot b = b \cdot a = e$.

For simplicity, we refer to the group simply as $\mathbb{G}$, omitting the notation $(\mathbb{G}, \cdot)$, but making it clear that the group structure is formed by the tuple of a non-empty set and the group product.

The simplest group consists only of one element, the identity. The number one under the usual multiplication of real numbers is an example of such a group. Another simple example is the set $\{e, a\}$. The group multiplication is defined by the relations: $ea = ae = e$, $ee = e$ and $aa = e$. The set $\{+1, -1\}$ with respect to the usual multiplication is such a group. In the end of this Appendix, we describe the structure of some other groups and discuss their relations to symmetry in Physics.

Although we gave only finite and discrete examples of groups, many of the groups in physics are continuous. These groups carry a parameter, or set of parameters, that are continuous. The group of rotation in three dimensions and the translations in space are examples of continuous groups.

$\mathbb{G}$ is called **Abelian** if the group multiplication is commutative

$$a \cdot b = b \cdot a \qquad \forall\, a, b \in \mathbb{G}.$$

A subset $\mathbb{H} \subset \mathbb{G}$ which forms a group under the same multiplication rule as $\mathbb{G}$ is said to be a **subgroup**.

3.1 Lie groups

The notion of a Lie group lies at the crossroads of algebra and geometry. On one hand, it is a group in the usual sense, that is, a set equipped with a multiplication rule satisfying associativity, identity, and inverses. On the other hand, it is also a smooth manifold, a geometric object locally modeled on Euclidean space. The key feature of a Lie group is that these two structures are compatible: the group operations are smooth maps. This interplay allows us to use analytic and geometric tools to study algebraic structures, and conversely, to bring the power of group theory into geometry and physics.

Definition 4 *A* Lie group *is a set G that is simultaneously a group and a smooth manifold, such that the multiplication map*

$$m : \mathbb{G} \times \mathbb{G} \to \mathbb{G}, \qquad (g, h) \mapsto gh,$$

and the inversion map

$$i : \mathbb{G} \to \mathbb{G}, \qquad g \mapsto g^{-1},$$

are both smooth.

This simple definition encodes a deep compatibility. The group multiplication is not just algebraically defined but varies smoothly with the elements, and the inverse of an element depends smoothly on the element itself. This smoothness allows us to apply differential calculus to study group-theoretic questions.

The most familiar example of a Lie group is the additive group of real numbers. The underlying manifold is simply the real line, and the group operation is addition. Both addition and taking negatives are smooth operations, so $(\mathbb{R}, +)$ is a one-dimensional Lie group.

Another classical example is the circle, which we may think of as the set of complex numbers of modulus one. Multiplication of complex numbers restricts to a group operation on the circle, and the resulting group is compact, connected, and abelian. Its manifold structure is one-dimensional, and its group structure encodes the idea of rotations in the plane.

Matrix groups provide an extremely important family of Lie groups. The general linear group $GL(n, \mathbb{R})$ consists of all invertible real $n \times n$ matrices. The set of such matrices is open in $\mathbb{R}^{n^2}$, hence a smooth manifold. Multiplication of matrices and taking inverses are polynomial operations in the entries, hence smooth. Special subgroups such as the orthogonal group $O(n)$, the special orthogonal group $SO(n)$, the unitary group $U(n)$, and the special unitary group $SU(n)$ are all Lie groups of great importance. Each of these captures different symmetry properties: rotations preserving inner products, volume-preserving transformations, or symmetries arising in quantum mechanics.

Every Lie group has associated to it a Lie algebra, which is the tangent space at the identity element endowed with a bilinear operation called the Lie bracket. Intuitively, while the group encodes global symmetries, the Lie algebra describes the infinitesimal symmetries near the identity. Remarkably, much of the structure of the Lie group can be understood through its Lie algebra, which is often easier to study.

For instance, the Lie algebra of $GL(n, \mathbb{R})$ is simply the space of all real $n \times n$ matrices with the commutator bracket. The Lie algebra of $SO(3)$, the group of rotations in three dimensions, can be identified with the space of skew-symmetric 3×3 matrices.

Lie groups appear naturally whenever one studies continuous symmetries. In mathematics, they describe symmetry groups of geometric structures and provide a framework for representation theory, in which one studies linear actions of groups on vector spaces. They are also central to number theory, through their connection to automorphic forms and arithmetic geometry.

In physics, their role is even more pronounced. The rotation groups $SO(2)$ and $SO(3)$ embody the symmetries of classical mechanics. The Lorentz group, $SO(3, 1)$, governs the invariance of spacetime in special relativity. In quantum mechanics, the unitary and special unitary groups appear in describing internal symmetries of quantum systems. Modern particle physics is fundamentally built on gauge theories, where the gauge groups are Lie groups such as $SU(3) \times SU(2) \times U(1)$. These groups dictate the interactions of fundamental particles and the very structure of the Standard Model.

3.2 Group Representation Theory

Symmetry is one of the most powerful guiding principles in mathematics and physics. Groups abstract the concept of symmetry, while *representations* of groups allow these abstract objects to act concretely on vector spaces. Representation theory thus bridges algebra and geometry, providing both computational tools and deep conceptual insight. In physics, group representations are central for classifying states, understanding conservation laws, and organizing fundamental particles.

A **representation** of a group $\mathbb{G}$ on a vector space $\mathbb{V}$ over a field (usually $\mathbb{C}$ or $\mathbb{R}$) is a homomorphism

$$\rho : \mathbb{G} \to GL(\mathbb{V}),$$

where $GL(\mathbb{V})$ is the group of invertible linear maps from $\mathbb{V}$ to itself. Thus, for each $g \in \mathbb{G}$, $\rho(g)$ is an invertible linear operator on $\mathbb{V}$, and $\rho(gh) = \rho(g)\rho(h)$.

The vector space $\mathbb{V}$ together with the homomorphism ρ is called a $\mathbb{G}$-*module* or a *representation space*.

For example, the cyclic group $\mathbb{C}_3$ has a one-dimensional representation given by

$$\rho_k(r) = e^{2\pi i k/3}, \quad k = 0, 1, 2,$$

which assigns a complex phase to each rotation.

An important concept in the representation theory is the **irreducible representation**. A subspace $\mathbb{W} \subseteq \mathbb{V}$ is *invariant* under ρ if

$$\rho(g)w \in \mathbb{W} \quad \forall g \in \mathbb{G},\ w \in \mathbb{W}.$$

A representation is irreducible if it has no proper nontrivial invariant subspaces. Otherwise, it is reducible.

A key result in representation theory is Schur's Lemma:

Theorem 3 (Schur's Lemma) *If $\rho_1 : \mathbb{G} \to GL(\mathbb{V}_1)$ and $\rho_2 : \mathbb{G} \to GL(\mathbb{V}_2)$ are irreducible representations of a group $\mathbb{G}$, and $T : \mathbb{V}_1 \to \mathbb{V}_2$ is a linear map such that*

$$T\rho_1(g) = \rho_2(g)T \quad \forall g \in \mathbb{G},$$

then either $T = 0$ or T is an isomorphism. If $\mathbb{V}_1 = \mathbb{V}_2$ and $\rho_1 = \rho_2$, then T is proportional to the identity.

Schur's lemma shows that irreducible representations are highly constrained and serve as the fundamental building blocks of all representations.

For finite groups over $\mathbb{C}$, every representation can be decomposed into a direct sum of irreducible representations. This is Maschke's Theorem, which allows the classification of representations to be reduced to the study of irreducible components.

In abelian groups, all irreducible representations are one-dimensional. This simplifies their study significantly. In non-abelian groups, irreducible representations may have dimensions greater than one, leading to richer structures.

Lie groups are smooth manifolds with group structure. Their representations are analyzed via their Lie algebras, which capture infinitesimal generators of transformations.

For example, $SO(3)$ has Lie algebra $\mathfrak{so}(3)$ spanned by infinitesimal rotation generators J_x, J_y, J_z satisfying

$$[J_x, J_y] = iJ_z, \quad [J_y, J_z] = iJ_x, \quad [J_z, J_x] = iJ_y.$$

Finite-dimensional irreducible representations of $\mathfrak{so}(3)$ are labeled by $\ell = 0, 1, 2, \ldots$, and have dimension $2\ell + 1$. This is the mathematical underpinning of angular momentum theory in quantum mechanics.

In quantum mechanics, representations classify possible state spaces consistent with symmetries. Angular momentum is described by irreducible representations of $SU(2)$. In particle physics, irreducible representations of $SU(3)$ describe quarks and hadrons. In crystallography, point groups and their representations classify vibrational modes of crystals.

3.3 Haar measure

In this section, we enter the context of Haar integration. In fact, the conception of a measure associated with groups is a generalization of the Lebesgue measure.

Theorem 4 *(Haar) Let $\mathbb{G}$ be a compact group. Then, there exists a unique positive and normalized measure d which is left and right invariant under the action of the group $\mathbb{G}$. Therefore, for any integrable function, we have:*

$$\int_{\mathbb{G}} \mathrm{d}(g)\, f(g) = \int_{\mathbb{G}} \mathrm{d}(g)\, f(hg) = \int_{\mathbb{G}} \mathrm{d}(g)\, f(gh) = \int_{\mathbb{G}} \mathrm{d}(g)\, f(g^{-1}) \tag{3.1}$$

for all $h \in \mathbb{G}$.

The normalization of Haar measure is:

$$\int_{\mathbb{G}} d\mu(g) = 1. \tag{3.2}$$

3.4 The Galilean group

The *Galilean group* is the group of transformations connecting inertial reference frames in classical (non-relativistic) mechanics. It encodes the symmetries of Newtonian space and time: the laws of mechanics are invariant under these transformations. Understanding this group is fundamental for classical physics, and it also provides a prototype for studying more general symmetry groups, such as the Poincaré group in relativity.

Consider two inertial reference frames S and S' with coordinates $(\mathbf{x}, t)$ and $(\mathbf{x}', t')$, respectively. A Galilean transformation is a map $(\mathbf{x}, t) \mapsto (\mathbf{x}', t')$ of the form

$$\mathbf{x}' = R\mathbf{x} + \mathbf{v}t + \mathbf{a}, \tag{3.3}$$

$$t' = t + b, \tag{3.4}$$

where $R \in SO(3)$ is a rotation matrix (spatial rotation), $\vec{v} \in \mathbb{R}^3$ is a constant velocity (boost), $\vec{a} \in \mathbb{R}^3$ is a spatial translation vector, and $b \in \mathbb{R}$ is a time translation.

The set of all such transformations, equipped with composition, forms the *Galilean group*.

The Galilean group is a 10-parameter Lie group in three spatial dimensions. Three parameters for rotations ($SO(3)$), three parameters for spatial translations, three parameters for boosts, and one parameter for time translation.

Infinitesimal transformations lead to the *Lie algebra of the Galilean group*. Its generators correspond to fundamental symmetries: The components of the momentum, P_i, are the generators of spatial translations, while the Hamiltonian H generates time translations. Rotations are generated by the components of the angular momentum J_i and the generators of boosts are denoted by K_i. The algebra is defined by the commutation relations

$$\begin{aligned}
[J_i, J_j] &= i\epsilon_{ijk} J_k, \\
[J_i, P_j] &= i\epsilon_{ijk} P_k, \\
[J_i, K_j] &= i\epsilon_{ijk} K_k, \\
[K_i, H] &= iP_i,
\end{aligned}$$

with all other commutators vanishing. These relations fully characterize the Lie algebra of the Galilean group and reflect the underlying symmetries of non-relativistic spacetime.

3.5 The Poincarè group

The Poincaré group is the fundamental symmetry group of Minkowski spacetime, and thus of special relativity. It describes all transformations that leave the spacetime interval invariant. These transformations include Lorentz transformations (rotations and boosts) and spacetime translations. In mathematical terms, the Poincaré group is the semidirect product of the Lorentz group $O(1,3)$ with the four-dimensional vector space of spacetime translations $\mathbb{R}^{1,3}$:

$$\mathbb{P} = O(1,3) \ltimes \mathbb{R}^{1,3}.$$

Understanding the Poincaré group is essential for the formulation of relativistic field theories, the classification of elementary particles, and the study of general relativity in the local tangent space.

Let $x^\mu = (ct, \vec{x})$ denote coordinates in Minkowski spacetime with metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. A Poincaré transformation maps $x^\mu \mapsto x^{\mu'}$ according to

$$x^{\mu'} = \Lambda^\mu{}_\nu x^\nu + a^\mu, \tag{3.5}$$

where $\Lambda \in O(1,3)$ is a Lorentz transformation satisfying

$$\eta_{\rho\sigma}\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu = \eta_{\mu\nu},$$

and $a^\mu \in \mathbb{R}^{1,3}$ is a constant translation four-vector. The group operation is the composition of transformations, and the inverse of (Λ, a) is given by $(\Lambda^{-1}, -\Lambda^{-1}a)$.

Physically, Lorentz transformations rotate and boost the spacetime axes while preserving the spacetime interval, whereas translations shift the origin of spacetime. The full Poincaré group therefore encodes all isometries of Minkowski spacetime.

The Poincaré group is a ten-parameter Lie group in four dimensions: three parameters for rotations, three for boosts, and four for translations. The infinitesimal generators can be defined as follows: $i)$ P_μ, the generators of translations, corresponding to energy and momentum; $ii)$ J_i, the generators of spatial rotations; $iii)$ K_i, the generators of Lorentz boosts.

These generators satisfy the Poincaré algebra, which can be expressed in covariant notation using the antisymmetric tensor $M_{\mu\nu}$ for rotations and boosts:

$$\begin{aligned}
[M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho} + \eta_{\mu\sigma}M_{\nu\rho}), \\
[P_\mu, P_\nu] &= 0, \\
[M_{\mu\nu}, P_\rho] &= i(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu).
\end{aligned}$$

These relations encode the structure of spacetime symmetries in special relativity: the boosts and rotations form the Lorentz subgroup, while translations commute with each other but mix with boosts under commutation.

In quantum field theory, a particle is associated with a Hilbert space carrying an irreducible unitary representation of the Poincaré group. The eigenvalues of the Casimir operators correspond to physical invariants: the particle's mass m and intrinsic spin s. For massive particles, one can go to the rest frame where $P^\mu = (m, \vec{0})$, and the remaining symmetry subgroup is the rotation group $SO(3)$, which determines the spin. For massless particles, there is no rest frame, and the little group is $ISO(2)$, leading to helicity classification.

While the Poincaré group describes global symmetries in flat Minkowski spacetime, general relativity generalizes this concept to curved spacetime. Locally, any small region of a curved manifold is approximately flat, and the tangent space at each point has a local Poincaré symmetry. This underlies the notion of local inertial frames and motivates the use of tetrads and spin connections when coupling fields with spin to gravity.