Information Theory — Foundations and Applications

Quantum critical systems

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Equilibrium quantum phase transition

A fundamental quantity in the statistical description of open quantum systems is the equilibrium partition function

$$Z = \mathrm{Tr}\left[e^{-\beta H}\right]$$

From which we can define all the thermodynamic potentials, as the free energy

$$F = -\beta^{-1} \ln Z$$

Phase transition

Thermodynamic potentials become nonanalytic as a function of the control parameter.

Dynamical quantum phase transition

Let us consider a quantum quench:

- The system is prepared in the ground state of the initial Hamiltonian

$$H_0(\lambda = \lambda_0)\left|\psi_0\right\rangle = \epsilon_0 \left|\psi_0\right\rangle$$

- At $t = 0$ the control parameter λ suffers a sudden change $\lambda_0 \to \lambda_f$, associated with the change in the Hamiltonian $H_0 \to H_f$.
- The evolved state of the system, at any time $t > 0$ is given by

$$\left|\psi_t\right\rangle = e^{-iH_f t}\left|\psi_0\right\rangle$$

Dynamical quantum phase transition

The central quantity here is the Loschmidt echo $\mathcal{L}_t$

$$G_t = \langle \psi_0 | \psi_t \rangle \qquad \mathcal{L}_t = |G_t|^2$$

Since the ground state can be degenerate, we can consider the return probability to the ground state manifold

$$P_t = \sum_{\alpha=1}^{g} P_\alpha \qquad P_\alpha = \langle \psi_\alpha | \psi_t \rangle$$

$\{|\psi_\alpha\rangle\}_{\alpha=1}^{g}$ is the set of degenerate ground states of the initial Hamiltonian.

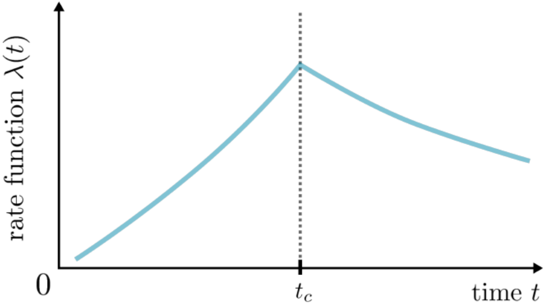
> **But why is this quantity relevant?**
> Because of the formal similarity of Loschmidt amplitudes with partition functions!

The dynamical quantum phase transition[1]

The DQPT will be identified as the nonanalytical behaviour of the rate function

$$r(t) = -\lim_{N\to\infty} \frac{1}{N} \ln \mathcal{L}_t$$

as a function of time, based on the similarities between such a quantity and the partition function.



[1] M. Heyl, A. Polkovnikov and S. Kehrein, Phys. Rev. Lett. **110**, 135704 (2013)

The Fisher zeros[2]

Equilibrium system with boundary conditions imposed on two ends a distance R apart from each other. The boundary partition function can be written as

$$Z = \langle\psi|e^{-HR}|\phi\rangle\,,$$

$|\psi\rangle$ and $|\phi\rangle$ encodes the boundary conditions while H is the bulk Hamiltonian.

Formally, the Loschmidt amplitude can be identified as $R = it$. Accordingly, the initial state $|\psi_0\rangle$ plays the role of the boundary conditions in time!

[2] M. E. Fisher, The nature of critical points (Lectures in Theoretical Physics vol VIIC) (1968)
C. N. Yang and T. D. Lee, Phys. Rev. **87**, 404 (1952)
T. D. Lee and C. N. Yang, Phys. Rev. **87**, 410 (1952)

Fisher zeros

In the eigenbasis of the Hamiltonian we can write the complex Loschmidt amplitude with $t \to z = t + i\tau$ as

$$\mathcal{G}_z = \sum_\alpha \langle E_\alpha|\psi_0\rangle \, e^{-iE_\alpha z}$$

This is an analytical function since it is a sum of analytical ones. The Weierstrass theorem says that

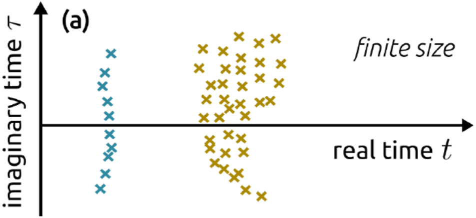
$$\mathcal{G}_z = e^{\mu(z)}\Pi_i \left(z_i - z\right)$$

with z_i being the i-th zero of $\mathcal{G}_z$ in the complex plane while $\mu(z)$ is analytic. Therefore, the nonanalytic behaviour of $\mathcal{G}_z$ is contained in the structures of the zeros z_i.

Fisher zeros

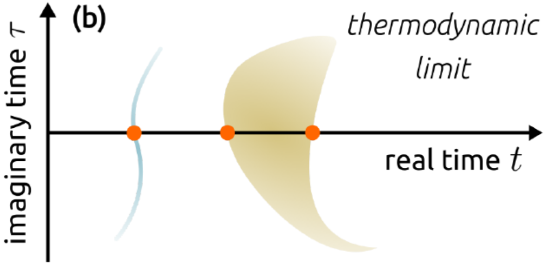
Disregarding $\mu(z)$, the singular contribution to the rate function comes from

$$-\frac{1}{N}\sum_i \ln\left[z_i - z\right]$$



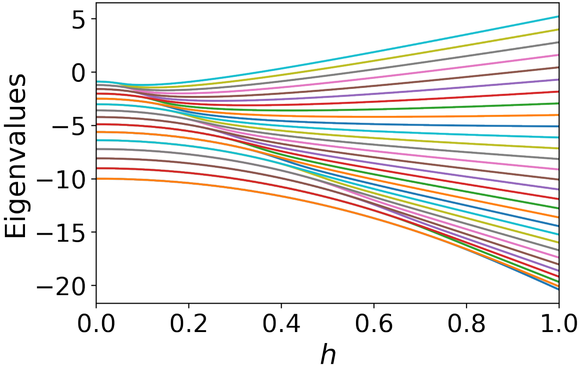
Fisher zeros — Thermodynamic limit

Nonanalyticities and thus phase transitions occur whenever a line or a boundary of an area of the Fisher zeros crosses the real line.



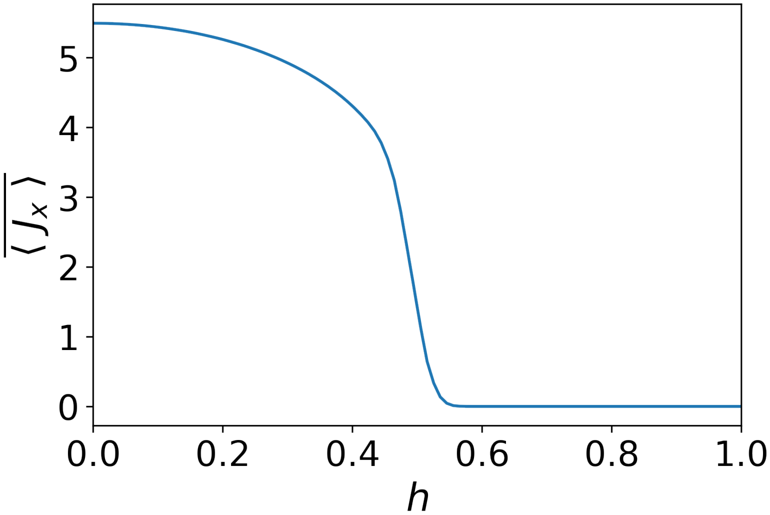
Example — LMG model[3]

$$H = -hJ_z - \frac{\gamma_x}{2j} J_x^2$$

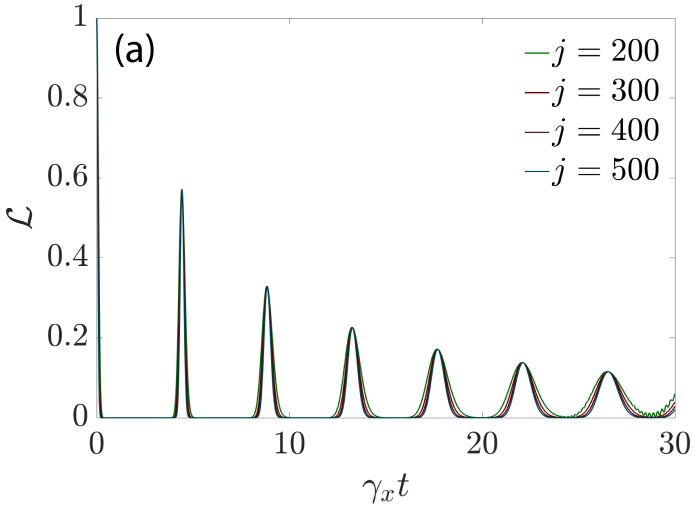


[3] H. Lipkin, N. Meshkov and A. Glick, Nucl. Phys. **62**, 188 (1965); N. Meshkov, A. Glick and H. Lipkin, Nucl. Phys. **62**, 199 (1965); A. Glick, H. Lipkin and N. Meshkov, Nucl. Phys. **62**, 211 (1965)

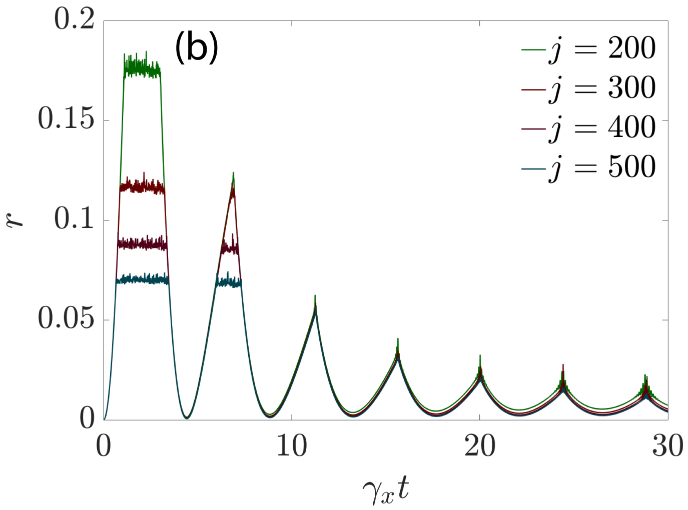
Example — LMG model



Example — LMG model



Example — LMG model



Q-Husimi function

For any state ρ, the Q-Husimi function is defined as

$$Q = \langle \Omega | \rho | \Omega \rangle$$

The spin coherent state $|\Omega\rangle$ can be constructed from the maximum angular momentum state $J_z |m = j\rangle = j |m = j\rangle$ as

$$|\Omega\rangle = e^{-i\phi J_z} e^{-i\theta J_y} |m = j\rangle$$

$\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. The expectation values of the angular momentum operators in this state obeys

$$(\langle J_x \rangle, \langle J_x \rangle, \langle J_x \rangle) = j (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$$

Wehrl entropy production[4]

Based on this quasi-probability distribution, we can define the associated entropy

$$S_Q = \frac{2j+1}{4\pi} \int \mathrm{d}\Omega \, Q(\Omega) \ln Q(\Omega)$$

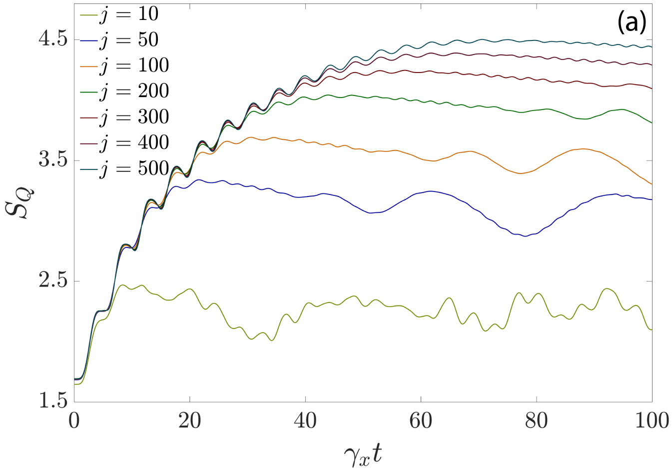
The entropy production rate can them be defined as

$$\frac{\mathrm{d}S_Q}{\mathrm{d}t} = \Pi_Q \geq 0$$

The last inequality follows from the second law of thermodynamics, which demands S_Q to be a monotonically increasing function of time for a closed system.

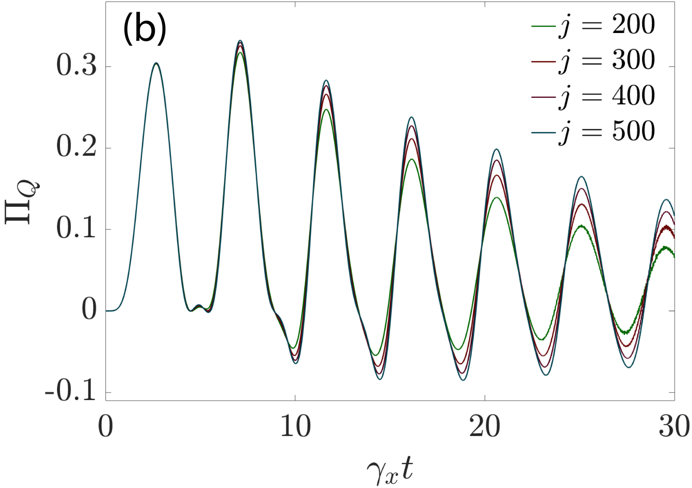
[4] A. Wehrl, Rev. Mod. Phys. **50**, 221 (1978)

Entropy for the LMG model[5]

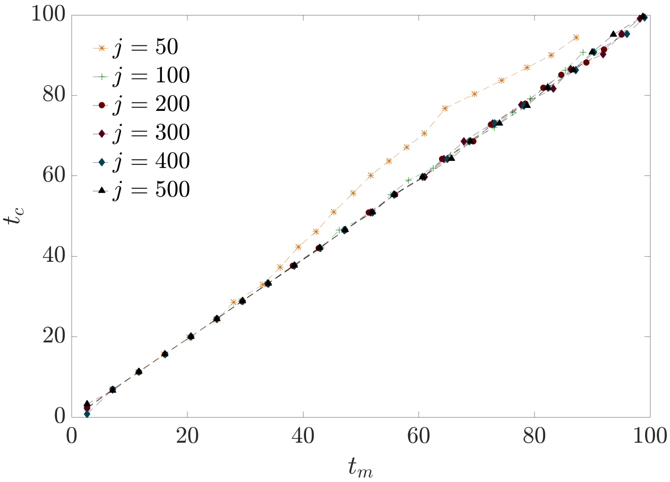


[5] B. O. Goes, G. T. Landi, E. Solano, M. Sanz and LCC, Phys. Rev. Research **2**, 033419 (2020)

Entropy production for the LMG model



The critical times



Take home messages

- A quantum phase transition can occur in complex systems in the time domain. They are characterised by the structure of the Fisher zeros of the Loschmidt amplitude.
- Although entropy production rate is not always positive, on average it monotonically increases in time, indicating that information is being spread over the system due to a DPT.
- Just like in the case of equilibrium quantum phase transition, a thermodynamic theory for the dynamical case seems possible[6].

[6] A. B. Nascimento and LCC, Phys. Rev. A **110**, 052223 (2024)
P. H. S. Bento, A. del Campo and LCC, Phys. Rev. B **109**, 224304 (2024)

Thank you for your attention

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