Classical Electrodynamics

Lecture 04

Multipole Expansion

Conducting Cavities

A Multipole expansion

If the charge density ρ is such that

$$\rho \to 0$$

sufficiently rapidly at infinity, the unique solution to Poisson's equation with $\phi \to 0$ as $\vec{x} \to \infty$ is given by

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'$$

Now, let us consider that

$$\rho \neq 0 \quad \text{for} \quad |\vec{x}'| < R$$

and we are interested in solutions at

$$|\vec{x}| > R$$

then $|\vec{x}| > |\vec{x}'|$. Let us consider the Taylor expansion of $|\vec{x} - \vec{x}'|$ about $\vec{x} = \vec{x}'$.

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{\left[(\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}')\right]^{1/2}}$$

$$= \left[|\vec{x}|^2 + |\vec{x}'|^2 - 2\,\vec{x}\cdot\vec{x}'\right]^{-1/2}$$

$$= |\vec{x}|^{-1}\left[1 - \frac{|\vec{x}'|^2}{|\vec{x}|^2} - 2\,\frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^2}\right]^{-1/2}$$

$$= |\vec{x}|^{-1}\left\{ 1 - \left(-\frac{1}{2}\right)2\frac{\vec{x}'}{|\vec{x}|^2}\Bigg|_{\vec{x}'=\vec{x}} \cdot \vec{x}' \right.$$

$$\left. - \frac{1}{2}\sum_{i,j}\left[3\,\frac{x_i x_j x_i' x_j'}{|\vec{x}|^5} - \frac{x_i' x_j'}{|\vec{x}|^3}\delta_{ij}\right] + \dots \right\}$$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{|\vec{x}|} + \frac{\vec{x}'\cdot\vec{x}}{|\vec{x}|^3} + \frac{1}{2}\left[\frac{3(\vec{x}'\cdot\vec{x})^2}{|\vec{x}|^5} - \frac{|\vec{x}'|^2}{|\vec{x}|^3}\right] + \dots$$

$$\boxed{\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{|\vec{x}|} + \frac{\vec{x}'\cdot\hat{x}}{|\vec{x}|^2} + \frac{1}{2}\,\frac{3(\vec{x}'\cdot\hat{x}) - |\vec{x}'|^2}{|\vec{x}|^3} + \dots}$$

Plugging this into the formal solution for Poisson's equation we obtain

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \rho(\vec{x}')\left[\frac{1}{|\vec{x}|} + \frac{\hat{x}\cdot\vec{x}'}{|\vec{x}|^2}\right.$$

$$\left. + \frac{3(\hat{x}\cdot\vec{x}')^2 - |\vec{x}'|^2}{|\vec{x}|^3} + \dots\right] d^3x'$$

$$\boxed{\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{|\vec{x}|} + \frac{\vec{P}\cdot\hat{x}}{|\vec{x}|^2} + \frac{1}{2}\sum_{i,j}{}' Q_{ij}\frac{\hat{x}_i\hat{x}_j}{|\vec{x}|^3} + \dots}$$

$$Q = \int d^3x\, \rho(\vec{x}) \qquad \vec{P} = \int d^3x\, \rho(\vec{x})\,\vec{x}$$

$$Q_{ij} = \int (3x_i x_j - |\vec{x}|^2 \delta_{ij})\,\rho(\vec{x})\, d^3x$$

These are the multipoles moments of $\phi(\vec{x})$

Let us see these expressions in spherical coordinates.

$$r^2 = x^2 + y^2 + z^2 \; ; \quad \cos\theta = \frac{z}{r} \; ; \quad tg\varphi = \frac{y}{x}$$

In order to do this, we compute Green function in spherical coordinates.

The Laplacian takes the form

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}\right]$$

Let us define the Laplacian on the unit sphere

$$D^2 f \equiv \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$

$$\Rightarrow \nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + D^2 f$$

D^2 is a self-adjoint operator on the Hilbert space of complex-valued square integrable functions on the unit sphere. The inner product is defined as

$$\langle f_1, f_2 \rangle = \int_0^{2\pi} \int_0^{\pi} f_1^*(\theta,\varphi)\, f_2(\theta,\varphi) \sin\theta \, d\theta \, d\varphi$$

Therefore, there is a basis on this space formed by the eigenvectors of D^2. The eigenvalues of D^2 are

$$-\ell(\ell+1) \qquad \ell = 0, 1, 2, \ldots$$

The ℓ-th eigenvalue of D^2 has degeneracy $2\ell+1$. Given this, the choice of the basis is not unique. A convenient one is given by the spherical harmonics

$$Y_{\ell m}(\theta,\varphi)=\sqrt{\frac{2\ell+1}{4\pi}}\sqrt{\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\theta)\,e^{im\varphi}$$

with

$$P_\ell^m(x)=\frac{(-1)^m}{2^\ell\,\ell!}(1-x^2)^{m/2}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell$$

being the associated Legendre functions and $-\ell\leq m\leq \ell$.

The spherical harmonics are a basis of the Hilbert space, they are orthonormal with respect to the inner product and satisfy

$$D^2 Y_{\ell m}=-\ell(\ell+1)Y_{\ell m}$$

We can write any square integrable function $f(\theta,\varphi)$ as

$$f(\theta,\varphi)=\sum_{\ell,m}C_{\ell m}Y_{\ell m}(\theta,\varphi)$$

with

$$C_{lm} = \int_0^{2\pi}\int_0^{\pi} Y^*_{lm}(\theta,\varphi) f(\theta,\varphi) \sin\theta \, d\theta \, d\varphi$$

Let us now come back to the Green function

$$\nabla^2 G(\bar{x},\bar{x}') = -\frac{1}{\varepsilon_0}\delta(\bar{x}-\bar{x}') \qquad (*)$$

$$G(\bar{x},\bar{x}') \to 0 \quad \text{as} \quad |\bar{x}| \to \infty$$

In spherical coordinates, we have

$$\delta(\bar{x}-\bar{x}') = \frac{1}{r^2 \sin\theta}\delta(\bar{r}-\bar{r}')\delta(\theta-\theta')\delta(\varphi-\varphi')$$

$$\int \delta(\bar{r}-\bar{r}')\,dr = \int \delta(\theta-\theta')\,d\theta = \int \delta(\varphi-\varphi')\,d\varphi = 1$$

$$\int \delta(\bar{x}-\bar{x}')\,d^3x = \int \delta(\bar{x}-\bar{x}')\, r^2 \sin^2\theta \, dr \, d\theta \, d\varphi = 1$$

We now expand the Green function in terms of this basis

$$G(\bar{x},\bar{x}') = \sum_{l,m} G_{lm}(r,\bar{x}') Y_{lm}(\theta,\varphi)$$

with the coefficients

$$G_{lm}(r,\bar{x}') = \int_0^{2\pi}\int_0^{\pi} Y^*_{lm}(\theta,\varphi)\, G(r,\theta,\varphi,\bar{x}') \sin\theta\, d\theta\, d\varphi$$

By plugging this into (*) we obtain

$$\sum_{l,m}\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dG_{lm}}{dr}\right)Y_{lm} + \frac{G_{lm}}{r^2}D^2 Y_{lm}\right] = -\frac{1}{\varepsilon_0}\delta(\bar{x}-\bar{x}')$$

$$\sum_{l,m}\left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dG_{lm}}{dr}\right) - \frac{l(l+1)}{r^2}G_{lm}\right]Y_{lm}$$

$$= \frac{1}{\varepsilon_0 r^2\sin\theta}\delta(r-r')\delta(\theta-\theta')\delta(\varphi-\varphi')$$

Multiplying by $Y^*_{\ell' m'}$ and integrating over the sphere we obtain

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dG_{\ell m}}{dr}\right)-\frac{\ell(\ell+1)}{r^2}G_{\ell m}=-\frac{\delta(r-r')}{\varepsilon_0 r^2}Y^*_{\ell m}(\theta',\varphi')$$

Therefore, we can write

$$G_{\ell m}(r,\bar{x}')=g_{\ell m}(r,r')Y_{\ell m}(\theta',\varphi')$$

And

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dg_{\ell m}}{dr}\right)-\frac{\ell(\ell+1)}{r^2}g_{\ell m}=\frac{-\delta(r-r')}{\varepsilon_0 r^2}$$

Therefore

$$\frac{d}{dr}\left(r^2\frac{dg_{\ell m}}{dr}\right)-\ell(\ell+1)g_{\ell m}=-\frac{1}{\varepsilon_0}\delta(r-r')$$

for $r < r'$ and $r > r'$ we have the general solution as

$$g_{\ell m}(r, r') = \frac{1}{r^{\ell+1}} a_{\ell m}(r') + b_{\ell m}(r') r^{\ell}$$

when $r \to 0$, the term proportional to $1/r^{\ell+1}$ is unacceptable, while the term proportional to r^{ℓ} is unacceptable for $r \to \infty$.

Therefore, the general solution must be of the form

$$g_{\ell m}(r, r') = \begin{cases} b_{\ell m}(r') r^{\ell} & r < r' \\ \dfrac{a_{\ell m}(r')}{r^{\ell+1}} & r > r' \end{cases}$$

We now have to solve the equation for $g_{\ell m}$ when $r \to r'$. Let us integrate the equation from $r = r' - \varepsilon$ to $r = r' + \varepsilon$ and take the limit $\varepsilon \to 0$.

$$\int_{r'-\varepsilon}^{r'+\varepsilon} \frac{d}{dr}\left[r^2 \frac{dg_{\ell m}}{dr}\right] dr - (\ell+1)\int_{r'-\varepsilon}^{r'+\varepsilon} g_{\ell m}\, dr = -\frac{1}{\varepsilon_0}$$

$$r'^2\left[\left.\frac{dg_{\ell m}}{dr}\right|_{r\downarrow r'} - \left.\frac{dg_{\ell m}}{dr}\right|_{r\uparrow r'}\right] = -\frac{1}{\varepsilon_0}$$

Now, we know that $g_{\ell m}$ must be continuous at $r = r'$. First, we have from the previous solution that

$$\frac{a_{\ell m}}{r'^{\ell+1}} = b_{\ell m}\, r'^{\ell}$$

However, we must also have

$$-(\ell+1)\frac{a_{\ell m}}{r'^{\ell}} - \ell b_{\ell m} r'^{\ell+1} = -\frac{1}{\varepsilon_0}$$

These equations imply that

$$-(\ell+1) r'^{-\ell} r'^{2\ell+1} b_{\ell m} - \ell b_{\ell m} r'^{\ell+1} = -\frac{1}{\varepsilon_0}$$

$$\rightarrow b_{\ell m}(r') = -\frac{1}{\varepsilon_0}\frac{1}{-(\ell+1) r'^{\ell+1} - \ell r'^{\ell+1}}$$

$$b_{\ell m}(r') = \frac{1}{\varepsilon_0 (2\ell+1) r'^{(\ell+1)}}$$

and

$$a_{\ell m}(r') = \frac{r'^{\ell}}{\varepsilon_0 (2\ell+1)}$$

Now we go back to the Green function

$$G(\vec{x},\vec{x}') = \sum_{l,m} G_{lm}(r,\vec{x}')\, Y_{lm}(\theta,\varphi)$$

$$= \sum_{l,m} g_{lm}(r,r')\, Y^*_{lm}(\theta',\varphi')\, Y_{lm}(\theta,\varphi)$$

leading us to

$$\boxed{G(\vec{x},\vec{x}') = \begin{cases} \dfrac{1}{\varepsilon_0} \displaystyle\sum_{l,m} \dfrac{1}{(2l+1)} \dfrac{r^l}{r'^{l+1}}\, Y^*_{lm}(\theta',\varphi')\, Y_{lm}(\theta,\varphi) & r \leq r' \\[2ex] \dfrac{1}{\varepsilon_0} \displaystyle\sum_{l,m} \dfrac{1}{(2l+1)} \dfrac{r'^l}{r^{l+1}}\, Y^*_{lm}(\theta',\varphi')\, Y_{lm}(\theta,\varphi) & r \geq r' \end{cases}}$$

This provides us the general solution to Poisson's equation for an arbitrary charge density $\rho(\vec{x})$

$$\phi(\vec{x}) = \int G(\vec{x},\vec{x}')\rho(\vec{x}')d^3x'$$

$$= \frac{1}{\varepsilon_0}\int \sum_{l,m}' \frac{\rho\, r^l}{(2l+1)\, r'^{l+1}} Y^*_{lm}(\theta',\varphi')\, Y_{lm}(\theta,\varphi)\, r'^2 \sin\theta\, dr'd\theta' d\varphi'$$

$$+ \frac{1}{\varepsilon_0}\int \sum_{l,m} \frac{\rho\, r'^l}{(2l+1)\, r^{l+1}} Y^*_{lm}(\theta',\varphi')\, Y_{lm}(\theta,\varphi)\, r'^2 \sin\theta\, dr'd\theta' d\varphi$$

defining

$$\alpha_{lm}(r) = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \frac{\rho(r',\theta',\varphi')}{r'^{l+1}} Y^*_{lm}(\theta',\varphi')\, r'^2 dr'd\theta' d\varphi'$$

$$\beta_{lm}(r) = \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \rho(r',\theta',\varphi')\, r'^l\, Y^*_{lm}(\theta',\varphi')\, r'^2 dr'd\theta' d\varphi'$$

we obtain

$$\boxed{\phi(\vec{x}) = \frac{1}{\varepsilon_0}\sum_{l,m}' \frac{1}{2l+1}\left[\alpha_{lm}(r)\, r^l + \frac{\beta_{lm}}{r^{l+1}}\right] Y_{lm}(\theta,\varphi)}$$

In particular, if $\rho \neq 0$ only for $r < R$, we have for $r > R$

$$\phi(\bar{x}) = \frac{1}{\varepsilon_0} \sum_{l,m}' \frac{q_{lm}}{(2l+1)\, r^{l+1}} Y_{lm}(\theta, \varphi)$$

$$q_{lm} = \int \rho(\bar{x}')\, r'^{l}\, Y^{*}_{lm}(\theta', \varphi')\, d^3x'$$

This is the multipole expansion of the electrostatic potential.

③ Conducting cavities

Conductor: material with charges that are free to move within the material

This directly implies that, inside the conductor

$$\bar{E} = -\nabla \phi = 0$$

Inside the conductor, we then must have

$$\phi = \text{const.} \Rightarrow \nabla^2 \phi = 0$$

This implies that any charge inside the conductor must vanish. The charges are located at the surface.

The charge density is assumed to take the form

$$\rho(\bar{x}) = \sigma(\bar{x})\,\delta(s)$$

s is the normal distance of $\vec{x}$ to the surface.

Poisson's equation takes the form

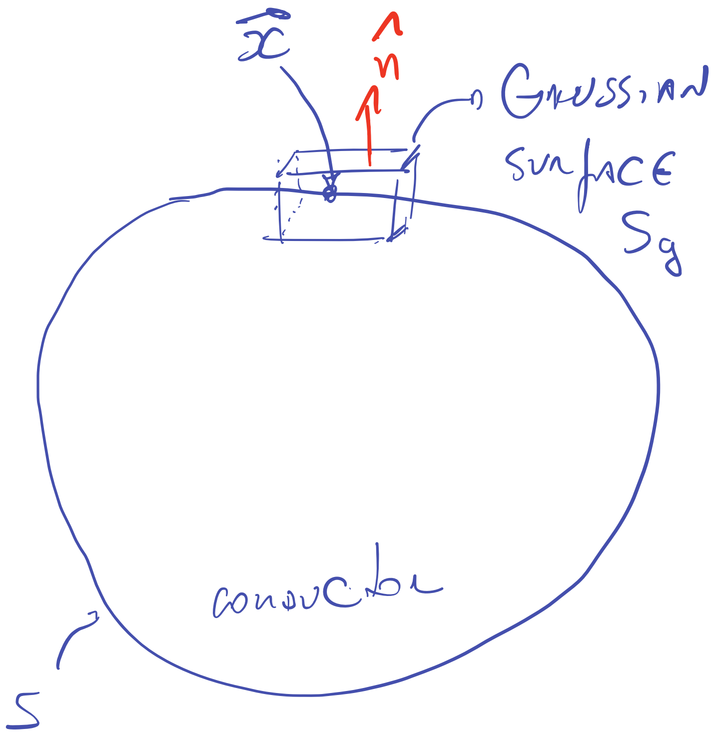
$$\nabla^2 \phi = -\frac{\sigma}{\varepsilon_0}\,\delta(s)$$

This implies that ϕ must be continuous (there is no derivative of the delta function) at the surface.

Similarly, the component of $\vec{E}$ tangencial to S must be continuos, implying that

$$\vec{E}_{\parallel} = 0 \quad \text{at } S$$

since $\vec{E} = 0$ inside the conductor. If this was not true, charges would still move on the surface and electrostatics would not apply.



The bottom and top surfaces of S_g are parallel to S. The top face lies outside the conductor.

From Gauss' Law it follows that

$$\nabla \cdot \vec{E}_S = \frac{\sigma}{\varepsilon_0}\,\delta(S)$$

$$\int_{S_g} \bar{n} \cdot \vec{E}\, ds = \frac{\sigma}{\varepsilon_0} A = \int \bar{E} \cdot \hat{n}\, ds = \bar{E}(\bar{x}) \cdot \hat{n} A$$

Therefore, just outside the conductor we have

$$\boxed{\bar{E}(\bar{x}) \cdot \hat{n} = \sigma / \varepsilon_0}$$

We are interested here in conducting cavities. So, we consider a conductor that encloses a volume V and study the electrostatics within V.

First, we need the following theorem

<u>Teorem</u> (Existence and Uniqueness): Let $V \subset \mathbb{R}^3$ be a bounded region whose boundary, $S = \partial V$, is a regular 2-dimensional surface.

Let $\rho(\vec{x})$ be an arbitrary continuous function on V, and consider the Poisson's equation

$$\nabla^2 \phi = -\rho/\varepsilon_0 \quad (*)$$

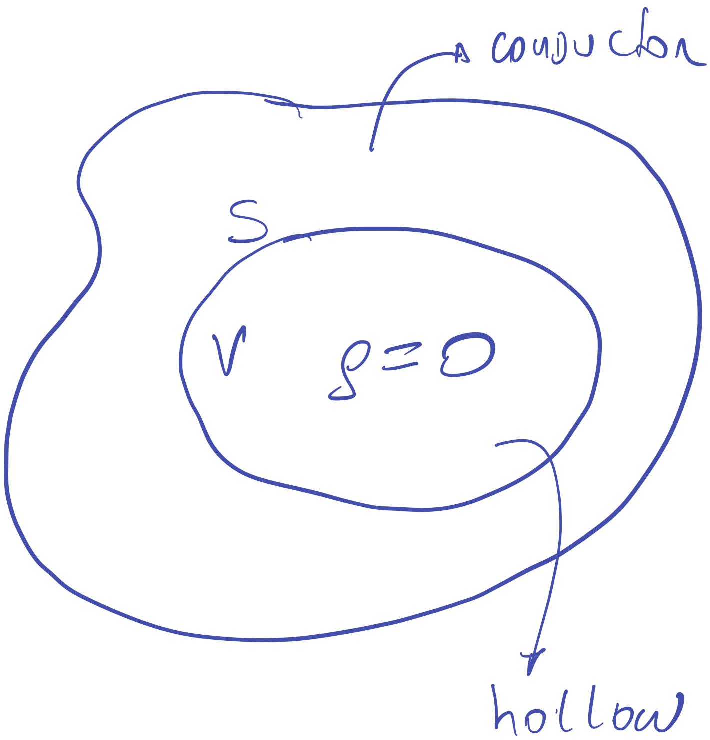
The following holds:

a) Dirichlet conditions: let ψ be an arbitrary continuous function on S. Then there exists a unique solution ϕ to $(*)$ in V such that $\phi|_S = \psi$.

b) Neumann condition: let χ be an arbitrary continuous function on S such that

$$\int_S \chi = -\frac{1}{\varepsilon_0}\int_V \rho$$

Then there exists a solution to $(*)$ on V such that $[\hat{n}\cdot\nabla\phi]|_S = \chi$. This solution is unique up to an additive constant.

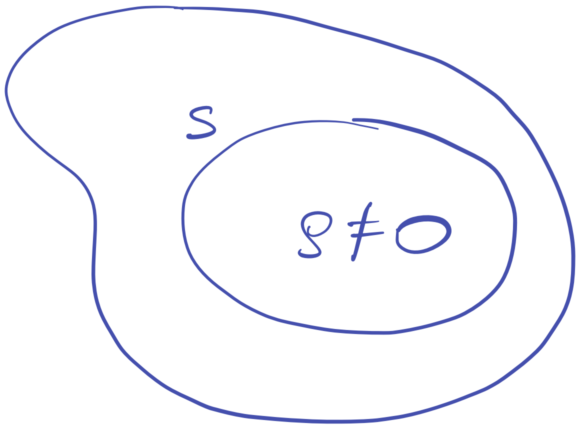


$$\phi|_S = \text{const.}$$

If $\rho = 0$ in V, then there is a unique solution to $\nabla^2 \phi = 0$ in V. This, of course, leads to $\phi = C$ throughout V. Therefore, we conclude that $\vec{E} = 0$ INSIDE ANY conducting cavity with no charges.

This is usually referred as the <u>Faraday cage</u>.

Let us now turn to the case where $\rho \neq 0$.



we still have

$$\phi|_S = \text{const.}$$

Since we have a gauge freedom, we can set this constant to zero. Therefore, on S we must have

$$\phi|_S = 0$$

↳ Dirichlet boundary condition.

We need to solve

$$\nabla^2 \phi = -\rho/\varepsilon_0$$

under this condition. The general solution follows from

$$\nabla^2 G_D(\bar{x}, \bar{x}') = -\frac{1}{\varepsilon_0}\delta(\bar{x} - \bar{x}')$$

subject to

$$G_D(\bar{x},\bar{x}') = 0 \quad \forall\, \bar{x} \in S$$

The general solution for ϕ is then

$$\phi(\bar{x}) = \int_V G_D(\bar{x},\bar{x}')\,\rho(\bar{x}')\,d^3x'$$

For a spherical cavity of radius R, it turns out that G_D is given by

$$G_D(\bar{x},\bar{x}') = \frac{1}{4\pi\varepsilon_0}\frac{1}{|\bar{x}-\bar{x}'|} + \frac{1}{4\pi\varepsilon_0}\frac{\alpha}{|\bar{x}-\bar{x}''|}$$

with

$$\bar{x}'' = \bar{x}'\frac{R}{|\bar{x}'|^2} \, ; \quad \alpha = -\frac{R}{|\bar{x}'|}$$

For a general cavity, finding G_D usually demands numerical methods.

Now, we know that

$$G(\bar{x},\bar{x}') = \frac{1}{4\pi\varepsilon_0|\bar{x}-\bar{x}'|}$$

satisfies the equation

$$\nabla^2 G_D(\bar{x},\bar{x}') = -\frac{1}{\varepsilon_0}\delta(\bar{x}-\bar{x}')$$

But it does not satisfy the boundary condition in S. However, we can correct this by writing

$$G_D(\bar{x},\bar{x}') = \frac{1}{4\pi\varepsilon_0|\bar{x}-\bar{x}'|} + F_D(\bar{x},\bar{x}')$$

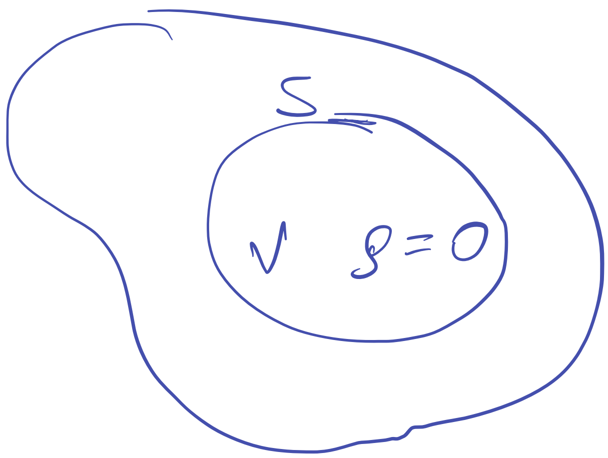
$$\nabla_x^2 F_D(\bar{x},\bar{x}') = 0$$ Derivatives with respect to $\bar{x}$.

subject to

$$F_D(\bar{x},\bar{x}') = -\frac{1}{4\pi\varepsilon_0|\bar{x}-\bar{x}'|} \quad \forall\ \bar{x}\in S$$

F_D must exist and also be smooth. Therefore, G_D must differ from $1/4\pi\varepsilon_0|\bar{x}-\bar{x}'|$ by at most a smooth function.

The Dirichlet problem



$\phi|_S = \psi$

ψ is arbitrary

(not a conductor)

In V it holds that $\nabla^2 \phi = 0$. What is the general solution to this problem?

We need Green's theorem

<u>Theorem</u> (Green's theorem): let $V \subset \mathbb{R}^3$ be a bounded region whose boundary is a two-dimensional surface $S = \partial V$. Let ϕ_1 and ϕ_2 be arbitrary continuous functions in V, and assume that ϕ_1 and ϕ_2 satisfy

$$\nabla^2\phi_1 = -\rho_1/\varepsilon_0 \qquad \nabla^2\phi_2 = -\rho_2/\varepsilon_0$$

We have

$$\int_S \hat{n}\cdot(\phi_1\nabla\phi_2 - \phi_2\nabla\phi_1)\,ds = \frac{1}{\varepsilon_0}\int_V(\phi_1\rho_2 - \phi_2\rho_1)\,d^3x$$

The proof follows directly from Gauss theorem.

Let now $\phi_1(\bar{x}) = \phi(\bar{x})$ and $\phi_2(\bar{x}) = G_D(\bar{x},\bar{x}')$, with ϕ being a solution to $\nabla^2\phi = 0$ with $\phi|_S = \psi$. From Green's theorem we obtain

$$\int_S \hat{n}\cdot(\phi(\bar{x})\nabla_x G_D - G_D\nabla\phi)\,ds = -\frac{1}{\varepsilon_0}\int_V \phi\,\rho_2\,d^3x$$

where we have used $\rho_1 = 0$. We know that $\rho_2 = \delta(\bar{x}-\bar{x}')$, since $\phi_2 = G_D(\bar{x},\bar{x}')$

Now we rewrite the left-hand-side of the last equation as

$$\int \phi(\hat{n}\cdot\nabla G_D)ds - \int G_D(\hat{n}\cdot\nabla\phi)ds$$

the last term vanishes because $G_D = 0$ on S. Therefore, we obtain

$$\int_S \phi(\vec{x})\,\hat{n}\cdot\nabla_x G_D(\vec{x},\vec{x}')\,ds = -\frac{1}{\varepsilon_0}\phi(\vec{x}')$$

Reversing the roles of $\vec{x}$ and $\vec{x}'$ we have

$$\phi(\vec{x}) = -\varepsilon_0\int_S \phi(\vec{x}')\,\hat{n}\cdot\nabla_{x'} G_D(\vec{x}',\vec{x})\,dS'$$

which is the general solution to the Dirichlet problem!