Classical Electrodynamics

Lecture 03

Electrostatics

Energy

④ Interaction Energy

The energy density of the electric field in electrostatic is given by

$$\mathcal{E} = \frac{\varepsilon_0}{2} |\vec{E}|^2$$

If the density ρ is smoothly distributed over spacetime and $\phi \to 0$ at infinity, we can write the total energy as

$$U = \frac{\varepsilon_0}{2} \int |\vec{E}|^2 d^3x = \frac{\varepsilon_0}{2} \lim_{R\to\infty} \int_{r\leq R} |\vec{E}|^2 d^3x$$

$$= \frac{\varepsilon_0}{2} \lim_{R\to\infty} \int_{r\leq R} |\nabla\phi|^2 d^3x$$

$$= \frac{\varepsilon_0}{2} \lim_{R\to\infty} \int_{r\leq} [\nabla\cdot(\phi\nabla\phi) - \phi\nabla^2\phi] d^3x$$

0 (asymptotic conditions)

$$= \frac{\varepsilon_0}{2} \lim_{R\to\infty} \int_{r=R} (\phi\nabla\phi)\cdot\hat{n}\, ds - \frac{\varepsilon_0}{2} \int \phi \underbrace{\nabla^2\phi}_{= -\rho/\varepsilon_0} d^3x$$

$$\Rightarrow \boxed{U = \frac{1}{2} \int \phi\rho\, d^3x}$$

Now, let us suppose that we have

$$\rho = \rho_1 + \rho_2$$

$\rho_1 \rightarrow$ associated to the solution ϕ_1

$\rho_2 \rightarrow$ associated to the solution ϕ_2

$\rho \rightarrow$ associated to the solution $\phi = \phi_1 + \phi_2$

The total energy is

$$U = \frac{\varepsilon_0}{2}\int |\vec{E}_1 + \vec{E}_2|^2 d^3x$$

$$= \frac{\varepsilon_0}{2}\int |\vec{E}_1|^2 d^3x + \frac{\varepsilon_0}{2}\int (\vec{E}_2)^2 d^3x + \varepsilon_0 \int \vec{E}_1 \cdot \vec{E}_2 d^3x$$

The fields $\vec{E}_1$ and $\vec{E}_2$ are associated to the potentials ϕ_1 and ϕ_2. The last term is the interaction energy of the fields

$$U^{int} = \varepsilon_0 \int \vec{E}_1 \cdot \vec{E}_2 \, d^3x$$

$$= \int \rho_1 \phi_2 \, d^3x = \int \rho_2 \phi_1 \, d^3x$$

Let us now consider a charged body placed in an external field ϕ^{ext} that satisfies Laplace's equation in a neighborhood of the charged body

$$\nabla^2 \phi^{ext} = 0$$

If we assume that $\phi^{ext} \to 0$ as $|\vec{x}| \to \infty$ (ϕ^{ext} is created by distant charges), we can compute the interaction energy as

$$U^{int} = \int \rho \phi^{ext} d^3x$$

Using Taylor expansion we have

$$U^{int} = \int d^3x \rho \left[\phi^{ext}\Big|_{\vec{x}=0} + \sum_i \frac{\partial \phi^{ext}}{\partial x_i}\Big|_{\vec{x}=0} x_i + \frac{1}{6} \sum_{i,j} \frac{\partial^2 \phi^{ext}}{\partial x_i \partial x_j}\Big|_{\vec{x}=0} x_i x_j \right]$$

$$= q \phi^{ext}\Big|_{\vec{x}=0} + \vec{p} \cdot \nabla \phi^{ext}\Big|_{\vec{x}=0} + \frac{1}{6} \sum_{i,j} Q_{ij} \frac{\partial^2 \phi^{ext}}{\partial x_i \partial x_j} x_i x_j$$

Here we defined

total charge: $q = \int d^3x \, \rho$

Dipole moment $\vec{p} = \int d^3x \, \vec{x} \cdot \rho(\vec{x})$

Quadrupole moment $Q_{ij} = \int d^3x \, (3x_i x_j - r^2 \delta_{ij}) \rho(\vec{x})$

The force on the charged body in the external field is

$$\vec{F} = \int \vec{f} \, d^3x = \int d^3x \, \rho(\vec{x}) \vec{E}(\vec{x})$$

$$\vec{E}(\vec{x}) = \vec{E}^{self} + \vec{E}^{ext}$$

$\hookrightarrow$ self-field of the body

$$\vec{E}^{self}(\vec{x}) = -\nabla \phi^{self} = \frac{1}{4\pi\varepsilon_0} \nabla \int d^3x \, \frac{\rho(\vec{x})}{|\vec{x} - \vec{x}'|}$$

Now we write

$$\frac{1}{4\pi\varepsilon_0} \nabla \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x' = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|^3} (\vec{x} - \vec{x}') d^3x'$$

and the force due to the self-field as

$$\vec{F}^{self} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{x}')\rho(\vec{x}')}{|\vec{x}-\vec{x}'|^3}(\vec{x}-\vec{x}')\,d^3x\,d^3x' = 0$$

Since the integrand with $(\vec{x},\vec{x}')$ has the opposite sign of $(\vec{x}',\vec{x})$.

This is not zero, in general, in Electrodynamics.

Therefore, we have

$$\vec{F} = \int \rho(\vec{x})\,\vec{E}^{ext}(\vec{x})\,d^3x$$

In the same way we did before, by Taylor expanding $\vec{E}^{ext}(\vec{x}) = -\nabla\phi^{ext}$, we obtain the i-th component of the force as

$$F_i = -q\left.\frac{\partial\phi^{ext}}{\partial x_i}\right|_{\vec{x}=0} - \sum_j p_j \left.\frac{\partial^2\phi^{ext}}{\partial x_i\,\partial x_j}\right|_{\vec{x}=0} - \frac{1}{6}\sum_{j,k} Q_{jk}\left.\frac{\partial^3\phi^{ext}}{\partial x_i\,\partial x_j\,\partial x_k}\right|_{\vec{x}=0} + \dots$$

that can be written as

$$\vec{F} = q\vec{E}^{ext} + (\vec{p}\cdot\nabla)\vec{E}^{ext} + \frac{1}{6}\sum_{j,k}{}' Q_{jk}\frac{\partial^2 \vec{E}^{ext}}{\partial x_j \partial x_k} + \ldots$$

From this we can compute the torque $\vec{\tau}$ as

$$\vec{\tau} = \int \vec{x}\times\vec{f}\, d^3x = \int \vec{x}\times[\rho(\vec{x})\vec{E}^{ext}(\vec{x})]\,d^3x$$

$$= \vec{p}\times\vec{E}^{ext}(\vec{x}) + \ldots$$

Let us come back to the case of point charges.

The energy is

$$U = \frac{\varepsilon_0}{2}\int |E|^2 d^3x = \frac{1}{32\pi^2\varepsilon_0}\int \frac{q^2}{r^4}d^3x$$

$$= \frac{q^2}{32\pi^2\varepsilon_0}\left(-\frac{1}{3x^3}\right)\Bigg|_0^\infty \to \infty$$

As mentioned before, point charges do not yield a mathematically reasonable description of charged matter.

However, the interaction energy of the fields of two point charges is well defined

$$U^{int} = \int \rho_1 \phi_2 d^3x = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{|\vec{x}_1 - \vec{x}_2|}$$

The force on a point charge is simply $\vec{F} = q\vec{E}^{ext}$, leading to Coulomb's Law

$$\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{|\vec{x}_1 - \vec{x}_2|} \vec{x}_{12}$$

$\hat{x}_{12}$: unit vector pointing in the direction $\vec{x}_1 - \vec{x}_2$

Observation: Usually we start from this expression for the force and comput the work done to bring charge q_1 to $\vec{x}_1$ from infinity while holding q_2 fixed at $\vec{x}_2$. then we equate the work with the change in the electromagnetic energy.

But this holds only if there is no change in the internal energy of the charged body. While this is true in electrostatic (the motion must be quasi-static), it does not hold in general, not even in magnetostatic.

Therefore, the energy density

$$\mathcal{E} = \frac{\epsilon_0}{2} |\vec{E}|^2$$

should be viwed as a fundamental property of the field.