Classical Electrodynamics

Lecture 02

Electrostatic

Ⓐ Uniqueness of solutions

Let us consider the case where the charge density ρ and the current density $\vec{J}$ are time independent

$$\frac{\partial \rho}{\partial t} = 0 \qquad \frac{\partial \vec{J}}{\partial t} = 0$$

We also have

$$\frac{\partial \phi}{\partial t} = 0 \qquad \frac{\partial \vec{A}}{\partial t} = 0$$

In this case, the equations for $\vec{E}$ and $\vec{B}$ in terms of the potentials completely decouple

$$\vec{E} = -\nabla \phi \qquad \vec{B} = \nabla \times \vec{A}$$

In addition to stationarity, we have

$\vec{J} = 0$; $\vec{A} = 0$ $\qquad\qquad$ $\rho = 0$ $\phi = 0$

Electrostatic $\qquad\qquad$ Magnetostatic

We here consider the electrostatic case.
Maxwell's equation reads

$$\nabla \cdot \vec{E} = \rho/\varepsilon_0$$

$$\nabla \times \vec{E} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \quad \xrightarrow{\text{Electrostatic}} \quad \nabla \cdot \vec{E} = \rho/\varepsilon_0$$

$$\nabla \times \vec{B} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0$$

The other equations are trivial.

Since $\vec{E} = -\nabla\phi$, we obtain

$$\nabla \cdot (\nabla \phi) = \rho/\varepsilon_0 \Rightarrow \boxed{\nabla^2 \phi = -\rho/\varepsilon_0}$$

this is the Poisson equation

The other equations are trivial since

$$\vec{A} = 0 \Rightarrow \vec{B} = 0$$

so, the last two equations are identically vanish.

Ampère's law takes the form

$$\nabla \times \vec{E} = \nabla \times (\nabla \phi) \equiv 0.$$

Therefore, Poisson equation is the only relevant equation for electrostatic.

Now, how about the gauge freedom?

Since ϕ is time independent, we must have

$$\phi' = \phi - \frac{\partial \chi}{\partial t} \Rightarrow \chi = \alpha t \qquad \alpha = \text{const.}$$

We also have $\vec{A}=0$. This implies that the only allowed gauge transformation is

$$\phi \to \phi + \alpha$$

To prove the uniqueness theorem for solutions to Poisson equation we need Gauss theorem.

Gauss Theorem: Let $\vec{v}$ be an arbitrary differentiable vector field on $\mathbb{R}^3$. Let $V \subset \mathbb{R}^3$ be a bounded region whose boundary, $S \equiv \partial V$, is a regular two-dimensional surface. Then

$$\int_V \vec{\nabla}\cdot\vec{v}\, d^3x = \int_S \vec{v}\cdot\hat{n}\, dS$$

when $\hat{n}$ is the outward pointing unity normal to S.

Theorem: Uniqueness For a given charge density $\rho(\vec{x})$, there is at most one solution to Poisson's equation such that $\phi \to 0$ as $r \to \infty$ in such a way that $r\phi$ remains bounded and $r|\nabla \phi| \to 0$.

$\boxed{\text{Proof:}}$ For the given ρ, let ϕ and ϕ' be solutions to Poisson's equation that goes to zero at infinity. Let

$$\psi = \phi - \phi'$$

Then, we have

$$\nabla^2 \psi = \nabla^2 \phi - \nabla^2 \phi' = \frac{\rho}{\varepsilon_0} - \frac{\rho}{\varepsilon_0} = 0$$

Also, ψ and $|\nabla \psi|$ goes to zero at infinity. Now we integrate $\psi \nabla^2 \psi$ over a ball of radius R

$$\int_{r\leq R} \psi \nabla^2 \psi \, d^3x = \int_{r\leq R} \left[\nabla \cdot (\psi \nabla \psi) - |\nabla \psi|^2 \right] d^3x = 0$$

Using Gauss theorem

$$\int_{r\leq R} \nabla \cdot (\psi \nabla \psi) \, d^3x = \int_{r=R} \psi \, \hat{n} \cdot \nabla \psi \, ds$$

The area of a sphere of radius R varies as R^2, the asymptotic conditions on ψ and $\nabla \psi$ imply that

$$\int_{r=R} \psi \, \hat{n} \cdot \nabla \psi \, ds \to 0 \quad \text{as } R \to \infty$$

Therefore

$$\lim_{R\to\infty} \int_{r\leq R} \psi \nabla^2 \psi \, d^3x = - \int |\nabla \psi| \, d^3x = 0$$

where the integral now extends over all space. This implies that

$$\nabla \psi = 0 \Rightarrow \psi = \text{const.}$$

Since $\psi \to 0$ at infinity, const. $= 0$.

Thus

$$\boxed{\phi = \phi'}$$

$\square$

Observe that this does not establish the existence of the solution, just its uniqueness.

(B) Point charges and Green's functions

Let us consider point charges

$$\rho(\vec{x}) = q\,\delta(\vec{x} - \vec{x}_0)$$

We want to solve Poisson's equation

Let us choose $\vec{x}_0 = \vec{0}$. There can be at most one solution to Poisson's equation such that $\phi \to 0$ at infinity, which must be spherically simmetric, otherwise we could generate new solutions by rotations.

Let us integrate Gauss Law over a sphere of radius R

$$\int_{r \leq R} \nabla \cdot \vec{E}\, d^3x = \frac{1}{\varepsilon_0} \int_{r \leq R} q\, \delta(\bar{x})\, d^3r = q/\varepsilon_0$$

$\vec{E} \equiv E(r)\hat{r}$ (rotationally invariant)

Therefore, since $\hat{r}$ is normal to the surface of the sphere, we have

$$\int_{r \leq R} \nabla \cdot \bar{E}\, d^3x = \int_{r=R} \bar{E} \cdot \hat{r}\, ds = E(R) \int_{r=R} ds$$

Therefore, we obtain

$$4\pi R^2 E(R) = q/\varepsilon_0 \Rightarrow \boxed{\vec{E}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}}$$

Then, from $\vec{E} = -\nabla\phi$, we obtain

$$\boxed{\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}}$$

by imposing $\phi \to 0$ when $r \to \infty$.

This is the unique solution for a point charge located at the origin.

If we have n point charges q_i located at $\vec{x}_0^i$, we have

$$\boxed{\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{|\vec{x} - \vec{x}_0^i|}}$$

Now, if we have a continuous distribution $\rho(\bar{x})$, we can break up space into small volumes ΔV_i centered about $\vec{x}_0^i$. Then

$$\Delta q_i = \rho(x_0^i)\,\Delta V_i$$

should act as point charges for very small volume. This suggests that the solution for a continuous distribution can be written as

$$\phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\bar{x}')}{|\vec{x}-\vec{x}'|}\,d^3x'$$

The point charge solution, when used to generate solutions with continuous $\rho(\bar{x})$ is called Green's function. In electrostatic, the Green function is

$$\boxed{G(\bar{x},\bar{x}') = \frac{1}{4\pi\varepsilon_0}\,\frac{1}{|\bar{x}-\bar{x}'|}}$$

In general, if $\mathcal{L}$ is any linear partial differential operator, a Green's function for the equation

$$\mathcal{L}\psi = f$$

is a solution to the equation

$$\mathcal{L}\,G(\vec{x},\vec{x}') = \delta(\vec{x}-\vec{x}')$$

that satisfies the required boundary conditions.

If $G(\vec{x},\vec{x}')$ is known, the solution to $\mathcal{L}\psi = f$ can be written as

$$\psi = \int G(\vec{x},\vec{x}')\,f(\vec{x}')\,d^3x'$$

provided the integral converge.