Classical Electrodynamics

Lecture 01

Introduction

(A) The theory

Maxwell's theory was fully developed in the end of the nineteenth century. It was constructed over many centuries of knowledge on electric, magnetic and optical phenomena.

This is a classical theory that cannot correctly describe quantum phenomena, but the quantum theory is based on the classical one.

Maxwell's equations relate the electric and magnetic fields ($\vec{E}$ and $\vec{B}$) to each other and to charge density ρ and the current density $\vec{J}$.

$$\nabla\cdot\vec{E} = \rho/\varepsilon_0$$
$$\nabla\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \mu_0\vec{J}$$
$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$
$$\nabla\cdot\vec{B} = 0$$

$\rho(\vec{x})$ is the electric charge per unity volume at $\vec{x}$.

For any unity vector $\hat{n}$, $\hat{n}\cdot\vec{J}$ is the flux of charges per unity area through an area element perpendicular to $\hat{n}$.

One important consequence of these equations follows from taking the time derivative of Gauss law and the divergence of the Ampère's law ($c^2 = \varepsilon_0\mu_0$)

$$\nabla\cdot\frac{\partial\vec{E}}{\partial t} = \frac{1}{\varepsilon_0}\frac{\partial\rho}{\partial t} \qquad \cancelto{0}{\nabla\cdot\nabla\times\vec{B}} - \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\cdot\vec{E} = \mu_0\nabla\cdot\vec{J}$$

$\Rightarrow$ continuity equation $\boxed{\dfrac{\partial\rho}{\partial t} = -\nabla\cdot\vec{J}}$

Usually, we introduce the electric charge by means of Coulomb's law, which motivates the introduction of $\vec{E}$. Energy is assigned to the electrostatic interaction via mechanical work. Similarly, in magnetostatics, one normally starts with Biot-Savart law for the force between current elements, thus motivating the introduction of $\vec{B}$. The dynamical terms of $\vec{E}$ and $\vec{B}$ are introduced to get the full set of equations.

The scalar ϕ and vector $\vec{A}$ potentials

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \nabla \times \vec{A}$$

are introduced as a convenient way of solving Maxwell's equations.

From this we are taken to conclude that

i) $\vec{E}$ and $\vec{B}$ are fundamental

ii) The energy, momentum and stress properties of the fields are considered to be properties derived from the interaction of the field with charged matter.

iii) Electromagnetic fields are considered to be produced by charged matter.

iv) Point charges are taken to be a fundamental description of matter.

But these conclusions are not true! Let us see why. Remember that most of what follows will be elucidated during the course. This is just to give the reader a modern perspective on electrodynamics.

Before going into these ideas, it is important to say some words regarding electrodynamics and special relativity. Maxwell's equations are not compatible with Newtonian space and time, unless we have a preferred rest frame. It was largely believed that such frame was provided by the luminiferous aether, the mechanical medium through which the electromagnetic fields propagated. However, this idea resulted in severe difficulties that were resolved by special relativity. This theory replaced the time function and the metric of the space in Newtonian theory by the space-time metric. Classical electrodynamics is fully compatible with special relativity. No aether is needed!

In special relativity, the potential is a four (dual)-vector

$$A_\mu = \left(-\frac{\phi}{c}, \vec{A}\right)$$

And the fields arise from the strength tensor

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \qquad x^\mu = (x^0 = ct, x^1, x^2, x^3)$$

The six independent components of $F_{\mu\nu}$ are

$$E_i = c F_{i0} \qquad B_i = F_{jk} \qquad i = 1, 2, 3$$

Note that, since distinct observers define different notions of time, the fields are not invariant, just the tensor $F_{\mu\nu}$ is.

Therefore, Maxwell's equations must be written in terms of $F_{\mu\nu}$ and

$$J^\mu = (c\rho, \vec{J})$$

i) The fundamental variables are the potential

The fundamental description of the electromagnetic field is given in terms of the potentials ϕ and $\bar{A}$, not the field strengths $\bar{E}$ and $\bar{B}$

However, the potentials do not uniquely describe the field, since ϕ' and $\bar{A}'$ are equivalent if they differ by a gauge transformation

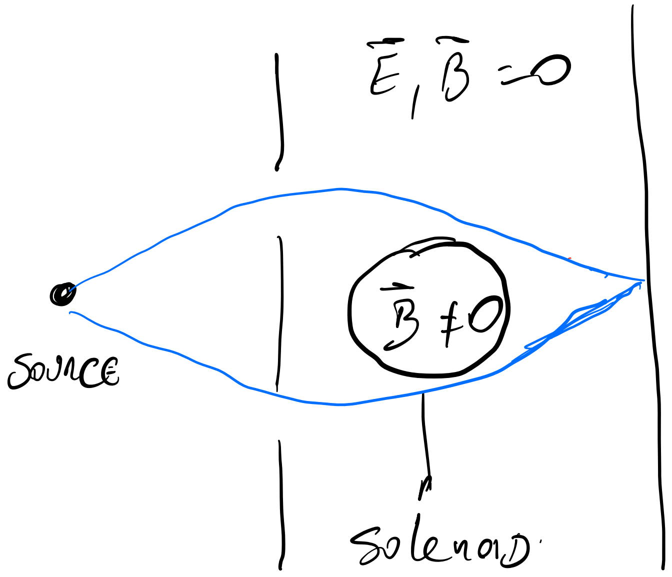
$$\phi' = \phi - \frac{\partial \chi}{\partial t} \qquad \bar{A}' = \bar{A} + \nabla \chi$$

for some function $\chi(\bar{x}, t)$.

The coupling of the electromagnetic field to charged matter can be described only in terms of the potentials.

There are situations in which the fields does not contain all the information

Aharonov-Bohm effect



$$\oint \vec{A} \cdot d\vec{\ell} \neq 0$$

for any loop (This is gauge invariant)

A quantum mechanical particle will suffer a shift in the interference pattern.

ii) Energy, momentum and stress

Classical electrodynamics arise from the Lagrangean density

$$\mathcal{L} = \frac{1}{2}\left[\epsilon_0 |\vec{E}|^2 - \frac{1}{\mu_0}|\vec{B}|^2\right] - \phi \rho + \vec{A} \cdot \vec{J}$$

The Euler-Lagrange equations are obtained by varying $\mathcal{L}$ with respect to the potentials. ρ and $\vec{J}$ are externally given quantities.

The energy, momentum and stress of the electromagnetic field is obtained by its coupling to gravity. The stress-energy-momentum tensor is given by

$$T^{\mu\nu} = -2 \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}}$$

This procedure gives rise to

$$\mathcal{E} = \frac{1}{2}\left(\varepsilon_0 |\vec{E}|^2 + \frac{|\vec{B}|^2}{\mu_0}\right) \qquad \vec{P} = \varepsilon_0 \vec{E} \times \vec{B}$$

$$\Theta_{ij} = \varepsilon_0 E_i E_j + \frac{B_i B_j}{\mu_0} - \delta_{ij} \mathcal{E}$$

Using Maxwell's equations, we obtain the following conservation laws

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{1}{c^2} \nabla \cdot \vec{P} = -\vec{J} \cdot \vec{E}$$

$$\frac{\partial P_i}{\partial t} - \sum_{j=1}^{3} \frac{\partial \Theta_{ij}}{\partial x^j} = -\rho E_i - (\vec{J} \times \vec{B})_i$$

If $\rho = 0$ and $\vec{J} = 0$, these equations express the local conservation of the energy and momentum of the electromagnetic field.

(iii) We should not view the electromagnetic field as being produced by changes and currents

The electromagnetic field has its own independent dynamical degrees of freedom.

These degrees of freedom are characterized by the initial value formulation of the Maxwell's equations. The initial conditions for $\vec{E}$ and $\vec{B}$ can be freely specified, showing that the field has its own dynamics. The solutions are not determined by ρ and $\vec{J}$.

This fact cannot be viewed in the static situation.

$$\frac{\partial \rho}{\partial t} = \frac{\partial \vec{J}}{\partial t} = 0 \Rightarrow$$ unique solutions $\vec{E}$ and $\vec{B}$ that are determined by ρ and $\vec{J}$.

iv) Matter is continuous

Maxwell's equations must be simultaneously solved with the equations of motion of matter, with the fields influencing the matter and vice-versa.

The notion of point charge is introduced in order to solve the idealized problems:

a) Given ρ and $\vec{J}$, find $\vec{E}$ and $\vec{B}$.

b) Given $\vec{E}$ and $\vec{B}$, find the motion of a charged body.

$$\rho(\vec{x},t) = q\,\delta(\vec{x} - \vec{X}(t))$$

$$\vec{J}(\vec{x},t) = q\,\frac{d\vec{X}}{dt}\,\delta(\vec{x} - \vec{X}(t))$$

$\vec{X}(t)$: world line of the point charge.

This leads to problems like infinite energy of a point charge.

Also, $\vec{E}$ is singular at the location of the point charge.