UNIVERSIDADE FEDERAL DE GOIÁS

INSTITUTO DE FÍSICA

PROGRAMA DE PÓS GRADUAÇÃO EM FÍSICA

Matheus Capela

Markov monogamy inequalities

Goiânia

2022

Matheus Capela

Markov monogamy inequalities

Tese apresentada ao Programa de Pós-Graduação em Física, do Instituto de Física (IF) da Universidade Federal de Goiás (UFG), como requisito para obtenção do título de Doutor em Física.

Área de concentração: Física

Linha de pesquisa: Óptica Quântica e Informação Quântica

Orientador: Prof. Dr. Lucas Chibebe Céleri

Co-orientador: Prof. Dr. Rafael Chaves Souto Araujo

Goiânia

2022

Resumo

O estudo dos limites na comunicação com canais ruidosos é um problema central na teoria de informação. *Data processing inequalities* são de fundamental importância sob esta perspectiva. Entretanto, estas não são as as únicas condições existentes sob o processamento de informação. Neste trabalho, apresentamos uma nova categoria de condições chamadas *Markov monogamy inequalities*. Estes novos resultados são aplicados ao estudo e caracterização de processos não-Markovianos clássicos e quânticos. De grande importância, as condições *Markov monogamy inequalities* revelam-se serem mais eficientes que as *data processing inequalities* na certificação de processos não-Markovianos em determinados exemplos.

Palavras-chave: desigualdades informacionais

Abstract

The study of the limits on the communication with noisy channels is of fundamental concern in information theory. Data processing inequalities are ubiquitous in this sense. Nevertheless, those are not a complete set of conditions on the processing of information. Here, we develop a novel class of information conditions called Markov monogamy inequalities. We apply the novel information inequalities to the problem of witnessing classical and quantum non-Markov processes. Importantly, we show that Markov monogamy may certify non-Markovianity beyond what is possible with data processing inequalities.

Keywords: information inequalities

Publications during candidature

The original results of this thesis have been previously reported in the following papers in collaboration with Lucas C. Céleri, Kavan Modi and Rafael Chaves [1, 2].

1. Quantum Markov monogamy inequalities
   Matheus Capela, Lucas C. Céleri, Rafael Chaves and Kavan Modi,
   arXiv preprint arXiv:2108.11533, 2021.

2. Monogamy of temporal correlations: Witnessing non-markovianity beyond data processing
   Matheus Capela, Lucas C. Céleri, Kavan Modi and Rafael Chaves
   Physical Review Research, vol. 2, no. 1, p. 013350, 2020.

The following publication in collaboration with Fabio Costa during candidature is unrelated to this thesis.

3. Matheus Capela and Fabio Costa
   Preprocessing operations and the reverse compression
   arXiv preprint arXiv:2101.02277, 2021.

Acknowledgments

Funding sources

Contents

List of Figures

Chapter 1

Introduction

This thesis is concerned with classical and quantum information inequalities, defined as constraints on information processing under well-defined class of operations. Here, we focus on the case of Markov processes. In this chapter we aim to motivate the reader on the relevance of the conditions considered here.

1.1 Classical information inequalities

Information theory primarily deals with the problem of communication between distant parties [3]. Therefore, it is of ubiquitous relevance the description and understanding of the physical medium allowing for communication (channel). Let us start with the original formulation for which the communication terminals and channels are represented as classical probabilistic events.

The channel in a communication scheme can be imperfect. That is, the input terminal might eventually send a message x, while the output terminal receives a different message y. This effect is called noise, and usually it is a variable out of control of the parties willing for communication. The overall action of a classical channel is described as a conditional probability distribution $p(y|x)$. So we might as well wonder whether would the reliable communication between distant terminals through a noisy channel be possible. Information theory answers this question affirmatively, providing the appropriate strategies to deal with inherent noise, and moreover, setting the limits under which reliable communication is pos-

sible. The following short description is based on standard text-books on information theory [4, 5, 6].

The strategy to deal with noise consists in considering several independent uses of the given channel in order to simulate (binary) noiseless channels. The communication protocol is defined by local operations performed by the input and output terminal as follows. The process starts with the input terminal encoding k input binary variables $U^k = U_1 \times \cdots \times U_k$ into n channel's input variables $X^n = X_1 \times \cdots \times X_n$. Then, the block X^n is sent through n independent uses of the channel $p(Y|X)$, that is, $p(Y^n|X^n) = p(Y_1|X_1) \cdots p(Y_n|X_n)$. Finally, the output joint variable Y^n is decoded into the final binary variable V^k representing the message received. Importantly, the complete communication process is represented as a Markov process $U^k \rightarrow X^n \rightarrow Y^n \rightarrow V^k$.

The goal of the encoding-decoding scheme is thus to maximize the communication ratio $R = k/n$, while keeping the probability of error $\Pr\{U^k \neq V^k\}$ small as possible. The channel coding theorem states that communication can be made reliable in the asymptotic regime, that is, taking the limit $n \rightarrow \infty$.

The direct part of the channel-coding theorem assures there is a reliable encoding-decoding scheme with communication ratio R no larger than the channel capacity defined as

$$C = \max_{p(X)} I(X : Y), \tag{1.1}$$

where $I(X : Y)$ is the mutual information of the input and output variables, and the maximization is over every possible probability distribution of the input terminal. Therefore, mutual information shows up to be a fundamental correlation function in communication problems.

The converse part of the channel-coding theorem states that there is no reliable encoding-decoding scheme with communication ratio R larger than the channel capacity C. The key ingredient in the proof of this result is the data processing inequality

$$I(X^n : Y^n) \geq I(U^k : V^k). \tag{1.2}$$

Information inequalities such as the condition above are of remarkable relevance towards

2

deriving necessary conditions on the possible operations on communication systems. This is enforced by the channel-coding theorem. Therefore, data processing inequalities are fundamental tools in information theory [7, 8, 9, 10, 11, 12, 13, 14]. Nevertheless, would data processing inequalities be the only constraints on encoding-decoding protocols? This is the main question addressed in this thesis. We show the existence of a novel class of information inequalities called Markov monogamy inequalities.

Interestingly, the channel capacity also imply on information inequalities for the transformation of channels under encoding and decoding with shared randomness. This is referred as Shannon inclusion [15] and has been extensively explored in literature [16, 17, 18]. The particular case of transforming channels under sole postprocessing operations is called the theory of comparison [19, 20]. Nevertheless, we do not consider this type of information inequalities and operations in this thesis.

1.2   Quantum information inequalities

The development of quantum mechanics has changed dramatically the paradigms in science, and it has not been different with information theory. The possibility of more efficient and secure communication via quantum processes became a central perspective leading to the so-called quantum information theory.

Now, the coherent information $I_c(\rho; \Lambda)$ – defined in chapter 6 – of a quantum system in the state ρ with respect to the quantum channel Λ becomes the relevant quantity for the definition of quantum information inequalities [21]. The quantum data processing inequality reads

$$I_c(\rho; \Lambda_1) \geq I_c(\rho; \Lambda_2 \circ \Lambda_1), \tag{1.3}$$

for every quantum state ρ, and quantum channels Λ_1, Λ_2.

The quantum data processing inequality above is also ubiquitous for deriving quantum operational results. For instance, there is a recovery operation R for which [21]

$$\mathrm{R} \circ \Lambda(\rho) = \rho, \tag{1.4}$$

3

whenever the ρ and Λ satisfy the condition $I_c(\rho; \mathrm{id}) = I_c(\rho; \Lambda)$, where id denotes the noiseless identity quantum channel. Clearly, this condition must hold for every quantum state ρ if Λ is a unitary channel.

The development of quantum information inequalities and constraints on quantum information processing has been extensively explored as well since then [22, 23, 24, 25, 26, 27, 28, 29, 30]. Once again, the quantum data processing inequality in Eq. (1.3) is not the only possible information inequality for quantum processes. Here, we show that the Markov monogamy inequalities can be extended to the quantum realm. This is the main original result reported in this thesis.

1.3 Outline

This thesis is divided into two parts. The first part deals with classical Markov monogamy inequalities, and is presented in chapter 2, chapter 3 and chapter 4. The second part treats quantum Markov monogamy inequalities, and is related to chapters 5, 6, 7 and 8. The remaining chapters of this thesis are organized as follows.

Chapter 2 briefly reviews probabilistic processes. The concepts are presented in such way that a broad audience can understand the main results in the thesis. We consider the definition of classical systems, operations and processes. Then, we introduce Markov processes accordingly. We consider a detailed example of classical processes violating the Markov regime.

Chapter 3 deals with data processing inequalities for classical Markov processes. Here, we show how information measures such as mutual information imply constraints on classical Markovianity. We derive this particular instance of classical information inequalities.

Chapter 4 presents the Markov monogamy inequalities for classical Markov processes. This is original result first announced in [2, 1]. We discuss in detail the derivation of several classical information inequalities. Then, we show how the classical information inequalities can be used in witnessing non-Markovianity. Fi-

nally, we make a conjecture on the general form of classical Markov monogamy inequalities.

Chapter 5 reviews quantum systems, operations and processes. We consider the notation, definitions and results used throughout the next chapters. This is done in order to make the text accessible to a broad audience. The exposition do not contain the derivation of the main results, nevertheless, enough references are provided to the interested reader. The expert reader can skip this chapter without further consequences.

Chapter 6 treats the quantum counterpart of the data processing inequalities introduced in chapter 3. There, we consider the information inequalities for quantum processes previously reported in literature. Importantly, this sets an appropriate framework for the establishment of novel information inequalities.

Chapter 7 deals with the quantum version of the Markov monogamy inequalities. This is the main original result in this thesis, previously reported in [1]. We present the derivation of all the novel information inequalities considered. Then, we explore how the quantum information inequalities can be used in witnessing non-Markov processes. We conclude the discussion arguing on the conjecture on quantum Markov monogamy inequalities.

Chapter 8 shows how the quantum information inequalities can be extended in order to allow for interventions along the process. This is done using the process tensor formalism. Particularly, we derive three versions of the quantum Markov monogamy inequality for a four-time-step quantum Markov process allowing for interventions on the quantum system of interest. We also show how this approach can be useful in witnessing quantum non-Markovianity in the examples previously considered.

Chapter 9 concludes the thesis with a summary of the results and a discussion on the possible directions which could be undertaken in future studies.

Part I

Classical information inequalities

Chapter 2

Classical processes

2.1 Classical systems

To each classical system it is associated an outcome space. Here, we denote both a classical system and its associated alphabet with the same symbol. Let X be a classical system. The state of a classical system is a probability distribution. That is, a function $p : X \to \mathbb{R}$ assigning a value

$$p(x) \geq 0, \qquad (2.1)$$

to each outcome x in X. Here, $\mathbb{R}$ denotes the set of real numbers. In addition, p also satisfy the normalization condition

$$\sum_{x \in X} p(x) = 1. \qquad (2.2)$$

The state of a system X is denoted $p(X)$ in order to explicitly define its domain when necessary. We only consider classical systems with finite cardinality, that is, systems with finite number of outcomes. Here, we only present a brief description of classical probabilistic processes in similar way as introduced in standard text-books in information theory such as [4, 5, 6]. The interested reader is also referred to Ref. [31, 32, 33] for a discussion on the mathematical theory of probability.

In turn, n-time-step classical processes are multi-partite classical systems $X_1 \times \cdots \times X_n$. The symbol $\times$ represents the Cartesian product, defining the ordered lists of outcomes. Intuitively, a classical process represents a classical system observed in different instances of

time. Furthermore, the subsystems of a classical process do not necessarily have the same outcome space. That is because we allow for discarding and adding classical systems through the process, for instance.

An example of a process with different subsystems would be defined by throwing a dice first, then flipping a coin. This would be defined by the Cartesian product of a set with cardinality six and a set with cardinality 2, thus accounting for every possible result of the total experiment. We can also define a process for which the first round is defined by flipping a coin, but in the second round we add a dice to the total system. This would be described by the Cartesian product of a set with cardinality 2 and a set with cardinality 12. We could also have started the first round with a total classical system defined as a dice and a coin, and in the second round we could discard the coin.

Hence, a classical process $X_1 \times \cdots \times X_n$ is described by the joint probability distribution

$$p : X_1 \times \cdots \times X_n \to \mathbb{R}. \tag{2.3}$$

Important is the definition of a marginal system. Let $X_1 \times X_2$ be a bipartite classical system drawn according to the joint probability distribution p. We define the classical state of a marginal system X_1 by the probability distribution with values

$$p(x_1) = \sum_{x_2} p(x_1, x_2), \tag{2.4}$$

for any outcome x_1 of X_1. Note we use the same symbol p to denote the classical state of the joint system $X_1 \times X_2$ and of the marginal system X_1.

Similarly, the probability masses of the marginal system X_2 are defined with

$$p(x_2) = \sum_{x_1} p(x_1, x_2), \tag{2.5}$$

for every outcome x_2 of X_2. Once again the states of $X_1 \times X_2$ and X_2 are denoted with the same symbol. When necessary, we use $p(X_1)$, $p(X_2)$ and $p(X_1, X_2)$ to distinguish the states of the systems X_1, X_2 and $X_1 \times X_2$, respectively. Similar convention is also adopted in the following definition for multi-partite classical systems.

The concept of marginalization can be generalized to arbitrary n-partite classical systems $X_1 \times \cdots \times X_n$. We now define the distribution of a marginal system with $1 \leq m \leq n$ arbitrary variables. Define a bijective function $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $f(1) \leq \cdots \leq f(m)$ and $f(m+1) \leq \cdots \leq f(n)$. The labels $f(1), \cdots, f(m)$ identify the subsystem of interest, while the remaining indexes $f(m+1), \cdots, f(n)$ refers to the systems to be removed. The probability distribution of the m-partite subsystem $X_{f(1)} \times \cdots \times X_{f(m)}$ is defined accordingly with probability masses

$$p(x_{f(1)}, \ldots, x_{f(m)}) = \sum_{x_{f(m+1)}, \ldots, x_{f(n)}} p(x_1, \ldots, x_n), \tag{2.6}$$

for all outcomes $(x_{f(1)}, \ldots, x_{f(m)})$ of the joint system $X_{f(1)} \times \cdots \times X_{f(m)}$.

The definition of marginalization for bipartite classical systems can be recovered from the general definition in Eq. (2.6). For instance, let $n = 2$ and $m = 1$. Thus, we have Eq. (2.4) defining $f : \{1, 2\} \to \{1, 2\}$ with the values $f(1) = 1$ and $f(2) = 2$. On the other hand, defining the bijective function $f : \{1, 2\} \to \{1, 2\}$ with values $f(1) = 2$ and $f(2) = 1$, we have Eq. (2.5).

2.2 Classical operations and Markov processes

Marginalization allow us to define classical channels, representing noisy operations on classical systems. This is done through the definition of conditional probability distributions. Consider the bipartite system $X_1 \times X_2$ in the state p. The probability distribution of the system X_2 conditional on X_1 is defined with conditional probability masses

$$p(x_2|x_1) = \frac{p(x_1, x_2)}{p(x_1)}, \tag{2.7}$$

for every outcome x_1 in X_1 with positive probability $p(x_1)$, and for each x_2 in X_2. The set of outcomes with positive probability is denoted as the support of a random variable X, and denoted as $\mathrm{supp}(X)$. The function $p(X_2|X_1) \colon (x_1, x_2) \in \mathrm{supp}(X_1) \times X_2 \mapsto p(x_2|x_1) \in \mathbb{R}$ is called conditional probability distribution. The mapping $p(X_2|X_1)$ represents a classical noisy operation with input system X_1 and output system X_2.

Conditional probability distributions can also be generalized to arbitrary n-partite systems $X_1 \times \cdots \times X_n$. Let $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a bijective function. Now, consider integers $1 \leq r \leq n$ and $1 \leq s \leq n$ such that $r + s \leq n$. The probability distribution of $X_{f(1)} \times \cdots \times X_{f(r)}$ conditional on $X_{f(r+1)} \times \cdots \times X_{f(s)}$ is defined by

$$p(X_{f(1)}, \ldots, X_{f(r)} | X_{f(r+1)}, \ldots, X_{f(s)}) = \frac{p(X_{g(f(1))}, \ldots, X_{g(f(r))}, X_{g(f(r+1))}, \ldots, X_{g(f(s))})}{p(X_{f(r+1)}, \ldots, X_{f(s)})},$$

(2.8)

where g is a permutation of $(f(j))_{j=1}^s$ with ordering $g(f(1)) \leq \cdots \leq g(f(s))$.

In general, classical processes $X_1 \times \cdots \times X_n$ can be described by conditional probability distributions with the assistance of the formula

$$p(X_1, \ldots, X_n) = p(X_n | X_1, \ldots, X_{n-1}) \cdots p(X_3 | X_1, X_2) p(X_2 | X_1) p(X_1).$$

(2.9)

Here, the indexes in the process' variables represent time instances. The above equation easily follows from the definition of conditional probability distribution. In particular,

$$p(X_1, \ldots, X_t) = p(X_t | X_1, \ldots, X_{t-1}) p(X_1, \ldots, X_{t-1}),$$

(2.10)

with $1 < t \leq n$.
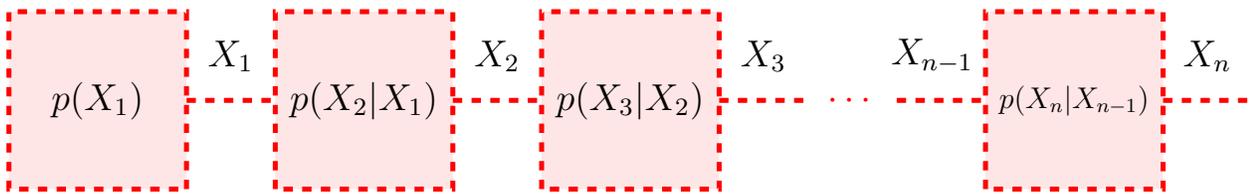
Figure 2.1: **Diagrammatic reasoning for classical processes and Markov conditions.** A classical channel $p(X_2|X_1)$ acting on a classical state $p(X_1)$ results in the classical state $p(X_1, X_2) = p(X_2|X_1)p(X_1)$. This situation can be pictorially represented as follows. Classical channels are boxes with input and output systems represented by left and right wires, respectively. Classical states are boxes with sole right wires. This emphasizes that states are channels with trivial input systems, that is, input variables with single outcome. Throughout this thesis we use dashed lines to denote classical states and channels. This makes clear distinction with the diagrams for quantum processes, which are drawn with solid lines in the following chapters. A Markov process $X_1 \to \cdots \to X_n$ is given by the application of $n - 1$ classical channels on a classical state. That is supported by the joint probability distribution of a Markov process in Eq. (2.14).

This thesis deals with the properties of Markov processes. A classical process is called Markov when the future is independent of the past conditionally on the present. For instance, a tripartite classical system $X_1 \times X_2 \times X_3$ is a Markov process if

$$p(X_3|X_1, X_2) = p(X_3|X_2). \tag{2.11}$$

On the same way, a four-partite system $X_1 \times X_2 \times X_3 \times X_4$ is a Markov process if the condition in Eq. (2.11) is fulfilled altogether with

$$p(X_4|X_1, X_2, X_3) = p(X_4|X_3). \tag{2.12}$$

In general, discrete-time Markov processes are defined as classical systems $X_1 \times \cdots \times X_n$ satisfying

$$p(X_t|X_1, \cdots, X_{t-1}) = p(X_t|X_{t-1}), \tag{2.13}$$

with $3 \leq t \leq n$. The $n-2$ equalities in Eq. (2.13) are referred to as Markov conditions. A Markov process is denoted as $X_1 \to X_2 \to \cdots \to X_n$.

The Markov conditions imply the process considered is identified with the consecutive action of local classical operations on an initial classical state. That is,

$$p(X_1, \cdots, X_n) = p(X_n|X_{n-1}) \cdots p(X_3|X_2)p(X_2|X_1)p(X_1) \tag{2.14}$$

holds whenever $X_1 \to \cdots \to X_n$. This follows directly from Eq. (2.9), and is diagrammatically represented in Figure 2.1.

On the other hand, a process is non-Markov whenever it violates any condition in Eq. (2.13). Equivalently, for a non-Markov process $X_1 \times \cdots \times X_n$ we have

$$p(X_1, \cdots, X_n) \neq p(X_n|X_{n-1}) \cdots p(X_3|X_2)p(X_2|X_1)p(X_1), \tag{2.15}$$

since at least one Markov condition is not respected.

The goal of this study is to further characterize information constraints on Markov processes, and thus, defining novel witnesses of non-Markov behaviour. We now move to consider

a concrete example of classical non-Markov process.
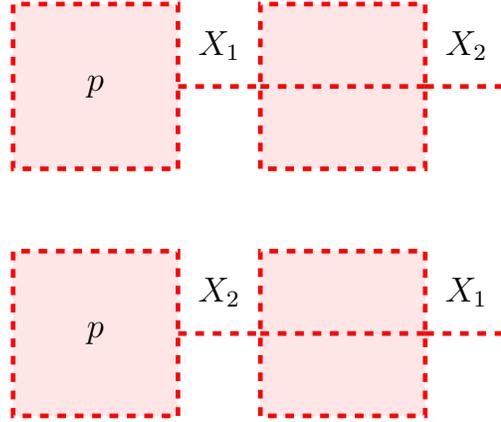
Figure 2.2: **Identical copy of classical systems.** Two systems are identical copies whenever one is the output of the identity operation acting on the other.

Consider the initial bipartite classical system $X_1 \times Y_1$, with binary subsystems $X_1, Y_1 = \{0, 1\}$. Let the joint system be in the classical state

$$p(x_1, x_2) = \frac{\delta_{x_1, x_2}}{2}. \tag{2.16}$$

The variables X_1 and X_2 are identical copies of each other, with uniform marginal distributions. That is, one is obtained by the action of a noiseless channel on the other. See Figure 2.2.

Now consider the classical bipartite channel

$$\Omega_\eta(x, y | a, b) = \begin{cases} 1 - \eta, & \text{if } x = b \text{ and } y = a \\ \frac{\eta}{3}, & \text{otherwise} \end{cases}, \tag{2.17}$$

with classical bits $a, b, x, y = 0, 1$. In Fig. 2.3 the classical probabilistic operation Ω_η is represented diagrammatically.

We define a bipartite classical process as follows. Let $X_2 \times Y_2$ be the output system for the channel Ω_η defined in Eq. (2.17), with input system $X_1 \times Y_1$ in the state defined in Eq. (2.16). Then, feed the channel Ω_η with the input system $X_2 \times Y_2$ to get the output system $X_3 \times Y_3$. Finally, define $X_4 \times Y_4$ as the output system of Ω_η with input $X_3 \times Y_3$. Therefore, the state
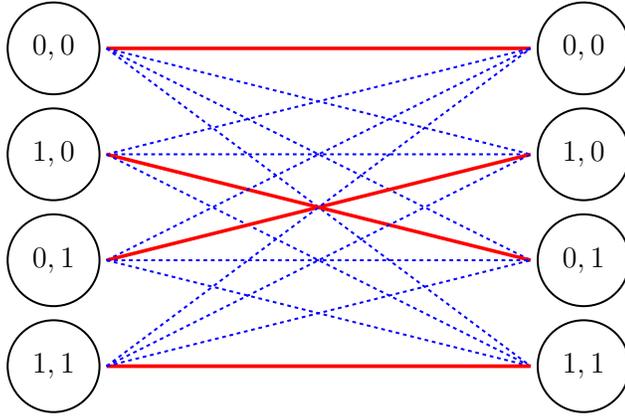
Figure 2.3: **Channel diagrams.** The diagram represents the classical channel Ω_η in Eq. (2.17). In the left side we represent the outcomes the bipartite input variable, while in the right side we represent the possible results of the bipartite output system. The lines connecting the nodes represent the probability of the output conditional on the input. Here, the red lines represent conditional probabilities with a value of $1 - \eta$, and blue lines represent conditional probabilities with a value of $\eta/3$. Note that for each input outcome it follows four lines summing up to unit, thus, defining a conditional probability distribution. Here, the same holds for the output outcomes. Therefore, Ω_η is said to be a symmetric classical channel.

of the system $X_1 \times Y_1 \times \cdots \times X_4 \times Y_4$ is

$$p(X_1, Y_1, X_2, Y_2, X_3, Y_3, X_4, Y_4) =$$
$$\Omega_\eta(X_4, Y_4 | X_3, Y_3)\Omega_\eta(X_3, Y_3 | X_2, Y_2)\Omega_\eta(X_2, Y_2 | X_1, Y_1)p(X_1, Y_1). \quad (2.18)$$

Figure 2.4 presents a diagram representing the process in Eq. (2.18). The protocol described results in the Markov process

$$(X_1, Y_1) \to (X_2, Y_2) \to (X_3, Y_3) \to (X_4, Y_4). \quad (2.19)$$

Although the bipartite classical process $\bigtimes_{t=1}^{4} X_t \times Y_t$ is Markov, the marginal process $\bigtimes_{t=1}^{4} X_t$ could potentially have non-trivial correlations between X_4 and $X_1 \times X_2$, and between X_3 and X_1. This would be provided by the non-trivial interaction with the variables $\bigtimes_{t=1}^{4} Y_t$. Would we have $X_1 \to X_2 \to X_3 \to X_4$? We show in the following that the process $\bigtimes_{t=1}^{4} X_t$ is non-Markov for most of the values η.
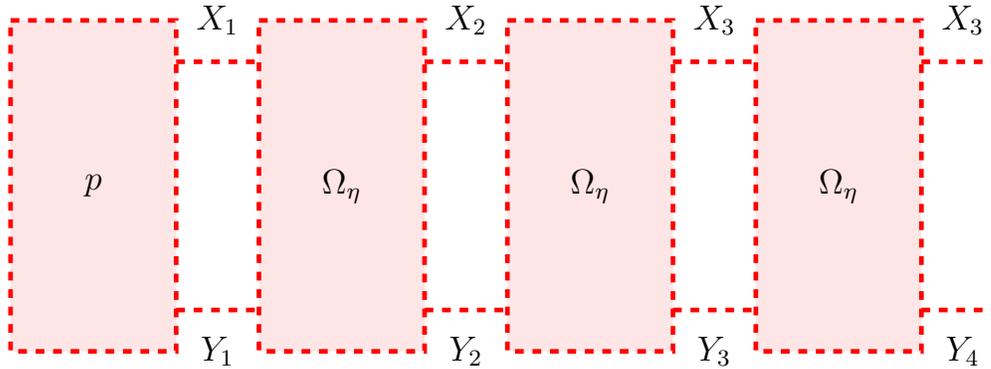
Figure 2.4: **Bipartite Markov Process.** A bipartite Markov process is defined with an initial bipartite classical state $p(X_1, Y_1)$ going through evolution given by Ω.

In order to check if the process is Markov we need to check all the Markov conditions. Thus, we need to have access to the joint probability distribution $p(X_1, X_2, X_3, X_4)$. For a four-time-step process we have to check Eqs. (2.11) and (2.12). Figure 2.5 compares the conditional probability distributions $p(X_3|X_1, X_2)$ and $p(X_3|X_2)$. Figure 2.6 compares the conditional probability distributions $p(X_4|X_1, X_2, X_3)$ and $p(X_4|X_3)$.

The conditions (2.11) and (2.12) hold for $X_1 \times X_2 \times X_3 \times X_4$ only when $\eta = 0.75$. This can be checked visualizing the plots in Figs. 2.5 and 2.6. Therefore, for any other value of η the process is regarded as non-Markov.

In this example the Markov condition (2.11) solely was sufficient for characterizing the process as non-Markovian. Thus, is it possible to be sure a process is non-Markov without the knowledge of the full joint probability distribution? In the following chapters we show the existence of relaxed conditions witnessing non-Markovianity with the knowledge of pair-wise probability distributions $p(X_i, X_j)$, with $1 \leq i < j \leq 4$.
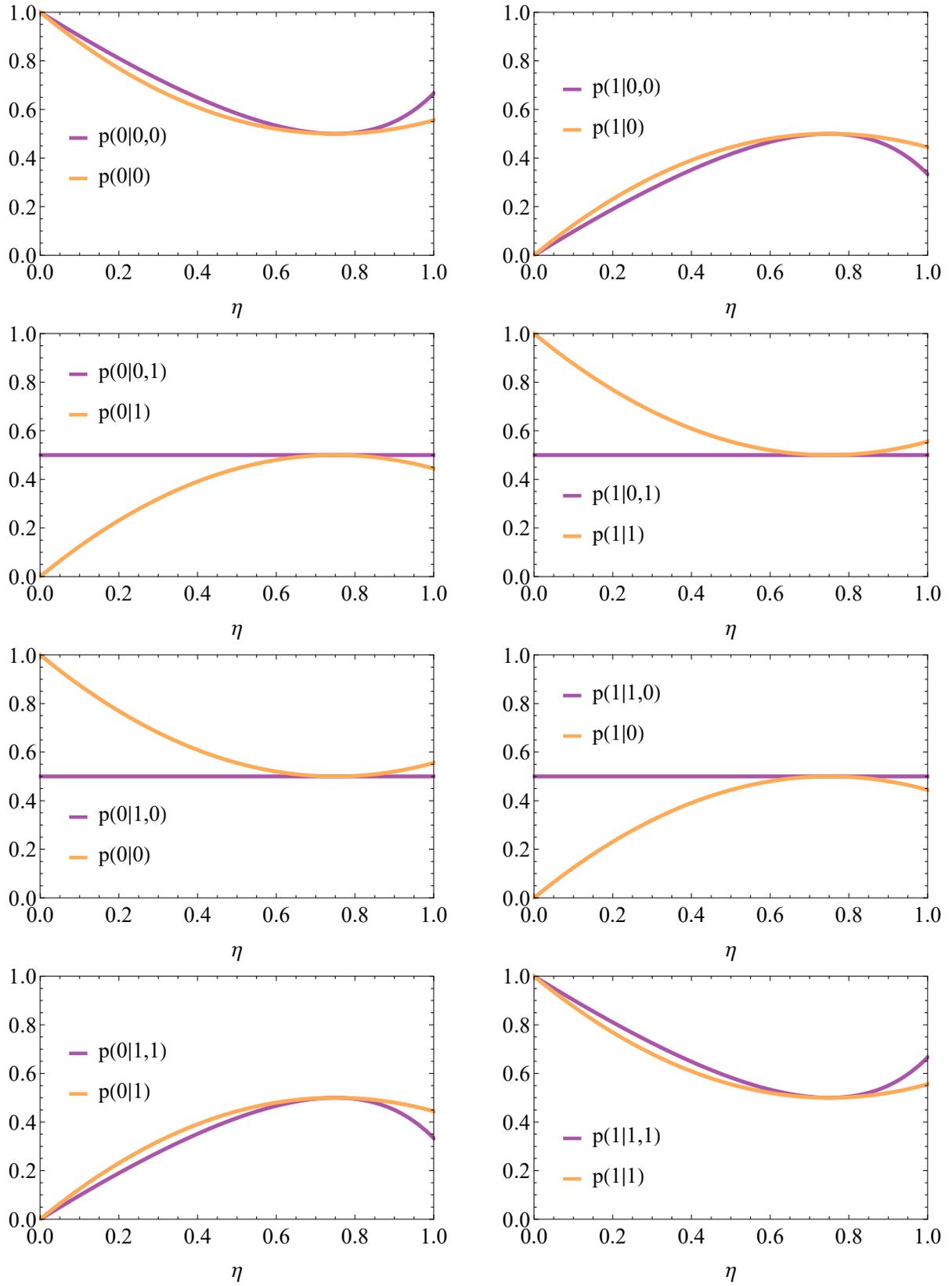
Figure 2.5: **Violation of Markov condition in Eq.** (2.11). The equality for conditional probability masses $p(x_3|x_1, x_2) = p(x_3|x_2)$ do not hold for most of the values of η.
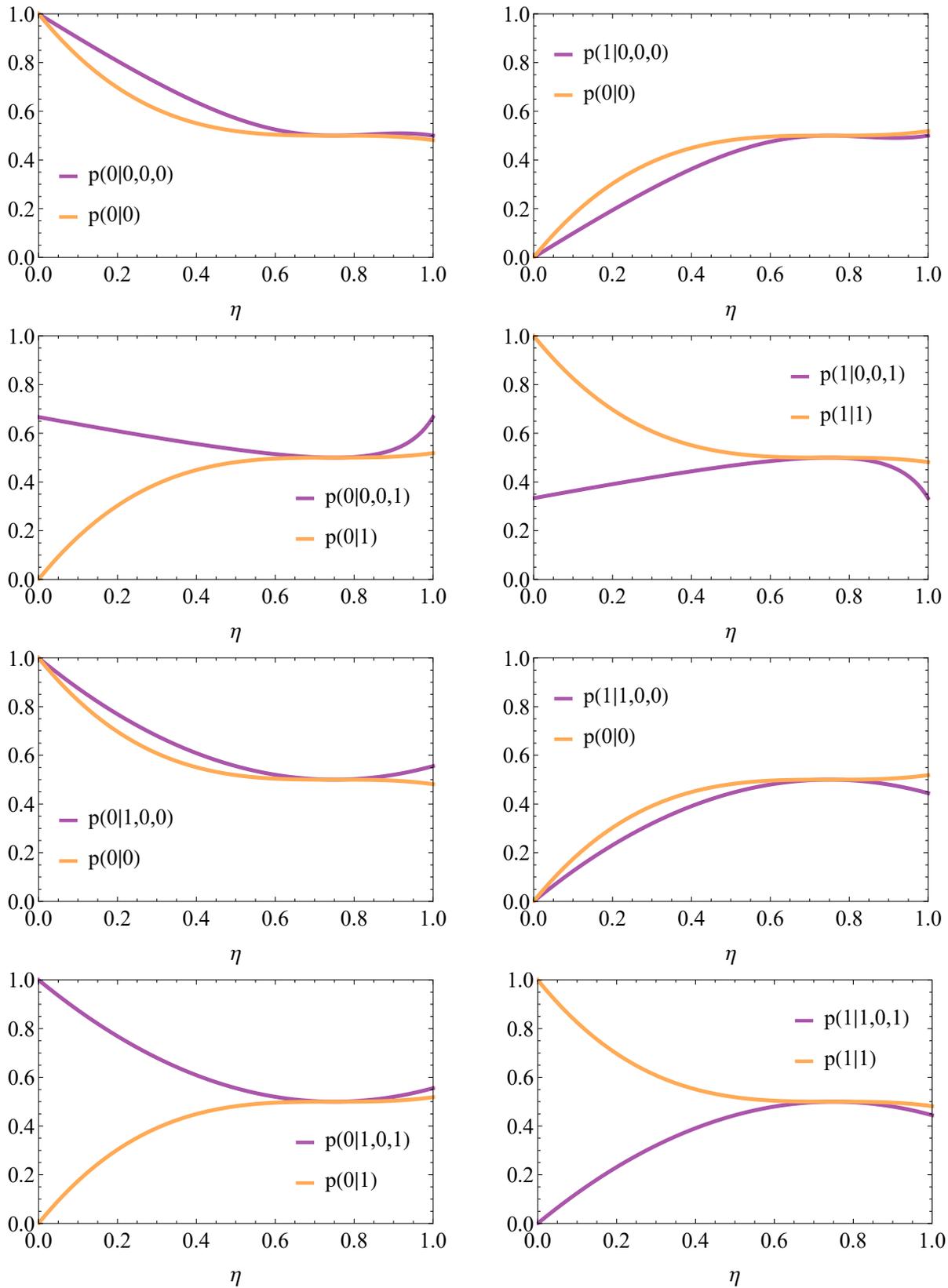
Figure 2.6: **Violation of Markov condition in Eq.** (2.12) **(Figure continues on next page).** The equality for conditional probability masses $p(x_4|x_1, x_2, x_3) = p(x_4|x_3)$ do not hold for most of the values of η.
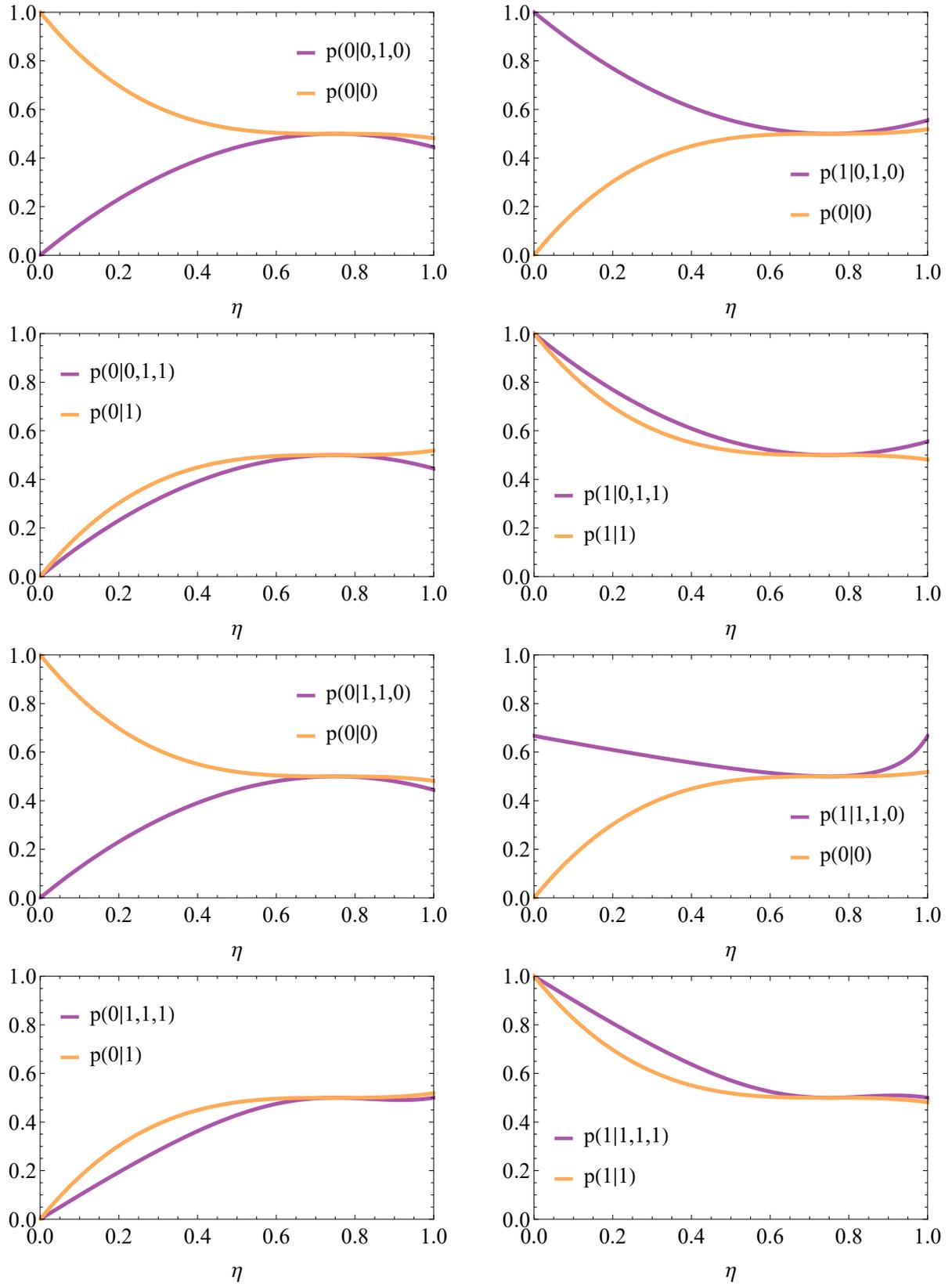
Figure 2.6: **Violation of Markov condition in Eq.** (2.12). The equality for conditional probability masses $p(x_4|x_1, x_2, x_3) = p(x_4|x_3)$ do not hold for most of the values of η.

Chapter 3

Data processing inequalities

Markovianity implies constraints on the statistical dependence of variables in a process. In chapter 2 we have considered a concrete example of four-time-step non-Markov process. We have also certified its non-Markovianity by directly checking the violation of the Markov conditions. In doing so we made use of the state of the full probability distribution of the process. That was necessary to compute the conditional probability distributions $p(X_3|X_1, X_2)$ and $p(X_4|X_1, X_2, X_3)$ in order to compare them with $p(X_3|X_2)$ and $p(X_4|X_3)$, respectively.

In that particular example, the condition $p(X_3|X_1, X_2) = p(X_3|X_2)$ was enough to witness non-Markovianity. So the marginal probability distribution of the variables $X_1 \times X_2 \times X_3$ was sufficient for invalidating the condition $X_1 \to X_2 \to X_3 \to X_4$. It suggests that we could systematically witness non-Markovianity even without the knowledge of the full joint distribution of process, that is, $p(X_1, X_2, X_3, X_4)$.

In fact, we may witness non-Markovianity with the knowledge of the distributions of bipartite subsystems of the process only. This method is provided by information measures, which we define now.

Here, the central quantity is the mutual information of classical systems X and Y defined as [4, 5]

$$I(X : Y) = H(X) + H(Y) - H(X, Y), \tag{3.1}$$

with classical entropy [4, 5]

$$H(Z) = - \sum_{p(z)>0} p(z) \log_2 p(z). \tag{3.2}$$

The Markov conditions clearly impose constraints on the information measures of the random variables of a stochastic process. Particularly, it does impose constraints on the mutual information of pairs of random variables: the data processing inequalities.

3.1 Three-time-step classical Markov processes

The elementary data processing inequalities emerges from three-time-step Markov processes. Every process $X_1 \to X_2 \to X_3$ satisfies [4, 5]

$$I(X_1 : X_2) \geq I(X_1 : X_3) \tag{3.3}$$

and

$$I(X_2 : X_3) \geq I(X_1 : X_3). \tag{3.4}$$

The conditions in Eqs. (3.3) and (3.4) are called data processing inequalities for the following reason. Consider the the data generated by a random source represented by the classical system X_1. Suppose the input variable X_1 is then transformed into the variable X_2 by some communication channel $p(X_2|X_1)$. The input and output variables are not perfectly correlated whenever p is a noisy operation. Thus, the message is corrupted through the process. One could try to recover the original message by acting locally on the output terminal. Equation (3.3) asserts that it is impossible. A similar interpretation holds for Eq. (3.4), but the transformation $p(X_2|X_1)$ is regarded now as a noisy pre-processing of data, and thus possibly decreasing correlations between the input and output terminals of a communication system. Figure 3.1 introduces diagrams for the information inequalities considered here.

In order to prove that Eqs. (3.3) and (3.4) hold for any three-time-step process $X_1 \to$
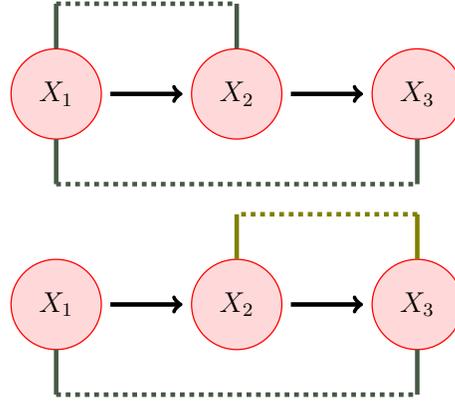
Figure 3.1: **Information inequalities.** The wires connecting two systems through dashed lines refers to their mutual information. The diagrams follow the convention that the mutual information terms defined by the bottom links cannot be greater than the top ones. Therefore, the top panel represents Eq. (3.3), while the bottom panel represents Eq. (3.4).

$X_2 \to X_3$, consider the strong subadditivity of classical entropy [4, 5]

$$H(X_1, X_2, X_3) + H(X_3) \leq H(X_1, X_3) + H(X_2, X_3) \tag{3.5}$$

and

$$H(X_1, X_2, X_3) + H(X_1) \leq H(X_1, X_2) + H(X_1, X_3). \tag{3.6}$$

The conditions above follow from the non-negativity of the conditional mutual information defined as $I(X : Y|Z) = H(X, Z) + H(Y, Z) - H(X, Y, Z) - H(Z)$ [4, 5]. Then, by using the Markov condition we have

$$H(X_1, X_2, X_3) = H(X_1, X_2) + H(X_2, X_3) - H(X_2). \tag{3.7}$$

Finally, use Eq. (3.7) in Eqs. (3.5) and (3.6) in order to get the desired data processing inequalities.

3.2 Four-time-step classical Markov processes

Similarly, four-partite Markov processes satisfy data processing inequalities. But now the number of constraints are considerably larger. For any four-time-step Markov process $X_1 \to X_2 \to X_3 \to X_4$ it holds that [4, 5]

$$I(X_1 : X_2) \geq I(X_1 : X_3), \tag{3.8}$$

$$I(X_2 : X_3) \geq I(X_2 : X_4), \tag{3.9}$$

$$I(X_1 : X_3) \geq I(X_1 : X_4), \tag{3.10}$$

$$I(X_1 : X_2) \geq I(X_1 : X_4), \tag{3.11}$$

$$I(X_2 : X_3) \geq I(X_1 : X_3), \tag{3.12}$$

$$I(X_2 : X_4) \geq I(X_1 : X_4), \tag{3.13}$$

$$I(X_3 : X_4) \geq I(X_2 : X_4), \tag{3.14}$$

$$I(X_3 : X_4) \geq I(X_1 : X_4), \tag{3.15}$$

$$I(X_2 : X_3) \geq I(X_1 : X_4). \tag{3.16}$$

The data processing inequalities for four-time-step Markov processes follow from the elementary data processing conditions for three-time-step Markov processes.

For instance, Eqs. (3.8)-(3.11) have the same form of Eq. (3.3), thus, representing the impossibility of fixing the communication of a given channel by post-processing with a noisy operation. In particular, Eq. (3.11) goes further, stating this constraint for post-processing the output terminal twice. On the other hand, Eqs. (3.12)-(3.15) are of the same type of Eq. (3.4), and furthermore, have the same interpretation. Eq. (3.15) considers pre-processing the input of a communication channel twice.

The validity of Eqs. (3.8)-(3.15) follows since marginal processes preserve Markovianity [5]. In the case considered, we have that $X_1 \to X_2 \to X_3 \to X_4$ implies $X_1 \to X_2 \to X_3$, $X_2 \to X_3 \to X_4$, $X_1 \to X_3 \to X_4$ and $X_1 \to X_2 \to X_4$.

The remaining condition in Eq. (3.16) is not of the elementary type. Nevertheless, it is a valid data processing condition obtained adding Eqs. (3.9) and (3.13). Moreover, it represents a more involved condition directly related to the proof of the converse part of the channel

coding theorem: it is impossible to fix the noise introduced by a communication channel with joint pre-and-post-processing noisy operations.

3.3 General classical Markov processes

The pattern in Eqs. (3.8)-(3.16) can be generalized for Markov processes with arbitrary number of variables. That is, for any n-time-step Markov process $X_1 \to X_2 \to \cdots \to X_n$ it holds that

$$I(X_i : X_j) \leq I(X_r : X_s), \tag{3.17}$$

with $1 \leq i \leq r < s \leq j \leq n$.

Data processing inequalities hold for any Markov process. Hence, one may assure a process is non-Markov by witnessing their violation. As expected a non-Markov process may violate those inequalities, as it necessarily violates the conditions under which they have been derived from. It is interesting to note, though, that a non-Markov process may satisfy all the data processing inequalities. We provide an example of such behaviour in the next chapter. That is because the data processing conditions are only necessary conditions for Markovianity. The data processing inequalities are not the only witnesses of non-Markovianity. In the next chapter we address different necessary conditions for Markov processes.

Chapter 4

Markov monogamy inequalities

In chapter 3 we have gone through the formal derivation of data processing inequalities. A key feature of them is their dependence on two-time-step correlation functions only. It has been claimed there that the data processing conditions provide a suitable way to certify the non-Markovianity of processes. Here, we show how this is done by considering concrete examples. But we go further, considering the existence of non-trivial information inequalities witnessing non-Markov behaviour beyond what is possible with data processing inequalities. We discuss here the definition of the Markov monogamy inequalities holding for classical processes. Moreover, Markov monogamy inequalities are the central result present in this thesis.

This chapter is organized as follows. We first consider the most basic Markov monogamy inequality arising for four-time-step processes. We also show how all the classical information inequalities derived so far can be used to witness non-Markovianity in particular examples. Then, we move to the definition of Markov monogamy inequalities of six- and eight-time-step classical Markov processes. Finally, we use this results to support our conjecture on the general form of Markov monogamy conditions for arbitrarily long classical Markov chains.

4.1   Four-time-step classical Markov processes

Markov monogamy are information inequalities that every Markov process satisfy. To this matter, data processing and Markov monogamy inequalities can be considered asser-

tions of the same type, that is, necessary conditions on Markovianity. The simplest Markov monogamy condition emerges from four-time-step Markov processes, and is defined as follows.

For every four-time-step Markov process $X_1 \to X_2 \to X_3 \to X_4$, it holds that [4, 2]

$$I(X_1 : X_4) + I(X_2 : X_3) \geq I(X_1 : X_3) + I(X_2 : X_4). \tag{4.1}$$

The Markov monogamy inequality in Eq. (4.1) appeared first in Ref. [4]. The authors in Ref. [2] derived it independently and also considered several applications. For instance, we considered how Eq. (4.1) connects to causal modelling, and how it can be used to witness non-Markov behaviour. The general form of Markov monogamy inequalities is novel result reported in [1].

In order to prove that Eq. (4.1) holds for every four-time-step Markov process, we need to add the strong subadditivity inequalities

$$I(X_2 : X_3 | X_1, X_4) \geq 0, \tag{4.2}$$

$$I(X_1 : X_4 | X_2) \geq 0, \tag{4.3}$$

$$I(X_1 : X_4 | X_3) \geq 0, \tag{4.4}$$

and then, to use the Markov conditions

$$H(X_4 | X_1, X_2, X_3) = H(X_4 | X_3), \tag{4.5}$$

$$H(X_3 | X_1, X_2) = H(X_3 | X_2). \tag{4.6}$$

Markov monogamy is not equivalent to any data processing inequality, nor any combination of them. That is, we have that $I(X_1 : X_3) \geq I(X_1 : X_4)$ and $I(X_2 : X_3) \geq I(X_2 : X_4)$, or $I(X_1 : X_3) \geq I(X_2 : X_3)$ and $I(X_2 : X_4) \geq I(X_1 : X_4)$, are the only combinations with same mutual information terms as in Eq. (4.1). Nevertheless, none of those pairs result in a Markov monogamy inequality when added together.

For the case of four-time-step Markov processes $X_1 \to X_2 \to X_3 \to X_4$ we have a total of nine data processing conditions in Eqs. (3.8)-(3.16). Nevertheless, not all of them are independent. That is, it is possible to combine two data processing inequalities to define a

new one. For instance, the data processing inequality $I(X_1 : X_2) \geq I(X_1 : X_4)$ is obtained by adding

$$I(X_1 : X_2) \geq I(X_1 : X_3), \text{ and} \tag{4.7}$$

$$I(X_1 : X_3) \geq I(X_1 : X_4). \tag{4.8}$$

Similarly, the information inequality $I(X_2 : X_3) \geq I(X_1 : X_4)$ is obtained with

$$I(X_2 : X_3) \geq I(X_2 : X_4), \text{ and} \tag{4.9}$$

$$I(X_2 : X_4) \geq I(X_1 : X_4). \tag{4.10}$$

Finally, we have that $I(X_2 : X_3) \geq I(X_1 : X_4)$ follows from the inequalities

$$I(X_3 : X_4) \geq I(X_2 : X_4), \text{ and} \tag{4.11}$$

$$I(X_2 : X_4) \geq I(X_1 : X_4). \tag{4.12}$$

Therefore, the data processing conditions for four-time-step Markov processes can be reduced to the following six inequalities

$$I(X_1 : X_2) \geq I(X_1 : X_3), \tag{4.13}$$

$$I(X_1 : X_3) \geq I(X_1 : X_4), \tag{4.14}$$

$$I(X_2 : X_3) \geq I(X_1 : X_3), \tag{4.15}$$

$$I(X_2 : X_3) \geq I(X_2 : X_4), \tag{4.16}$$

$$I(X_3 : X_4) \geq I(X_2 : X_4), \tag{4.17}$$

$$I(X_2 : X_4) \geq I(X_1 : X_4). \tag{4.18}$$

The Markov monogamy inequality in Eq. (4.1) can be used to further reduce the number of independent information inequalities for four-time-step Markov processes. The six information inequalities above can be achieved from the Markov monogamy and the four data

processing inequalities

$$I(X_1 : X_4) + I(X_2 : X_3) \geq I(X_1 : X_3) + I(X_2 : X_4), \qquad (4.19)$$

$$I(X_1 : X_2) \geq I(X_1 : X_3), \qquad (4.20)$$

$$I(X_1 : X_3) \geq I(X_1 : X_4), \qquad (4.21)$$

$$I(X_3 : X_4) \geq I(X_2 : X_4), \qquad (4.22)$$

$$I(X_2 : X_4) \geq I(X_1 : X_4). \qquad (4.23)$$

The information inequalities in Eqs. (4.20)-(4.23) are of the type of the elementary data processing inequalities for three-time-step Markov processes. Therefore, this reasoning provides a different route to prove the nine data processing conditions for four-time-step Markov processes.

4.2 Witnessing non-Markovianity with information inequalities

The information inequalities developed so far are necessary conditions for classical Markov processes. That is, classical processes for which the Markov conditions hold also satisfy data processing and Markov monogamy inequalities. We address here the method of witnessing non-Markovianity with information inequalities.

For every four-time-step process $X_1 \times X_2 \times X_3 \times X_4$ we define real-valued functions

$$\mathrm{DP}_1 := I(X_1 : X_2) - I(X_1 : X_3), \qquad (4.24)$$

$$\mathrm{DP}_2 := I(X_1 : X_3) - I(X_1 : X_4), \qquad (4.25)$$

$$\mathrm{DP}_3 := I(X_2 : X_3) - I(X_1 : X_3), \qquad (4.26)$$

$$\mathrm{DP}_4 := I(X_2 : X_3) - I(X_2 : X_4), \qquad (4.27)$$

$$\mathrm{DP}_5 := I(X_3 : X_4) - I(X_2 : X_4), \qquad (4.28)$$

$$\mathrm{DP}_6 := I(X_2 : X_4) - I(X_1 : X_4), \qquad (4.29)$$

$$\mathrm{M4} := I(X_1 : X_4) + I(X_2 : X_3) - I(X_1 : X_3) - I(X_2 : X_4). \qquad (4.30)$$

The quantities above are non-negative for every Markov process $X_1 \to X_2 \to X_3 \to X_4$. Thus, in case any of the quantities above assume a negative value for a process $X_1 \times X_2 \times X_3 \times X_4$ we are sure it is not in agreement with at least one Markov condition. Furthermore, the quantities DP_i ($i \in \{1, \ldots, 6\}$) and M4 are called witnesses of non-Markovianity. Note a non-Markov process does not necessarily imply a negative value for any of the quantities in Eqs. (4.24)-(4.30). Therefore, a non-negative value for $DP_1, \ldots, DP_6$ and M4 is not conclusive evidence a process is Markovian.[1]

We start considering the non-Markov process defined in chapter 2. There, we have studied a four-time-step bipartite Markov process $X_1 \times Y_1 \to X_2 \times Y_2 \to X_3 \times Y_3 \to X_4 \times Y_4$. Its full probability distribution was defined by a bipartite classical channel Ω_η acting successively on the initial state of $X_1 \times Y_1$, and depending upon the parameter $0 \leq \eta \leq 1$. The marginal process $X_1 \times X_2 \times X_3 \times X_4$ was also shown to be non-Markov for most of the values of the variable η. This was done explicitly by checking the Markov conditions for four-time-step processes.

Now, we check how information inequalities can be useful witnessing non-Markov processes in this example. Figure 4.1 shows the plot of $DP_1, \ldots, DP_6$ and M4 as a function of the parameter η. The quantities DP_5 and M4 are identical for every value of η in this example, and moreover, are the only quantities witnessing non-Markovianity. Note that for $\eta = 0.75$ all the witnesses are non-negative. This is already expected to happen as the process is Markovian for this situation.

The next sections consider the Markov monogamy inequalities arising from $X_1 \to \cdots \to X_n$, with $n = 6, 8$. In what follows it is convenient to define the Markov entropy

$$H_{\text{Markov}}(X_1, \cdots, X_n) := \sum_{i=1}^{n-1} H(X_i, X_{i+1}) - \sum_{i=2}^{n-2} H(X_i). \tag{4.31}$$

The quantity H_{Markov} is defined to have the same form as the joint entropy of a Markov process [2]. That is, we have $H(X_1, \cdots, X_n) = H_{\text{Markov}}(X_1, \cdots, X_n)$ for every Markov process $X_1 \to \cdots \to X_n$. Note that non-Markov chains can have joint entropy with a different form.

---

[1] In fact, the data processing inequality in Eq. (3.3) can be modified to be a necessary and sufficient condition on three-time-step Markov processes. This is done by considering the min-entropy instead of the classical entropy [34].
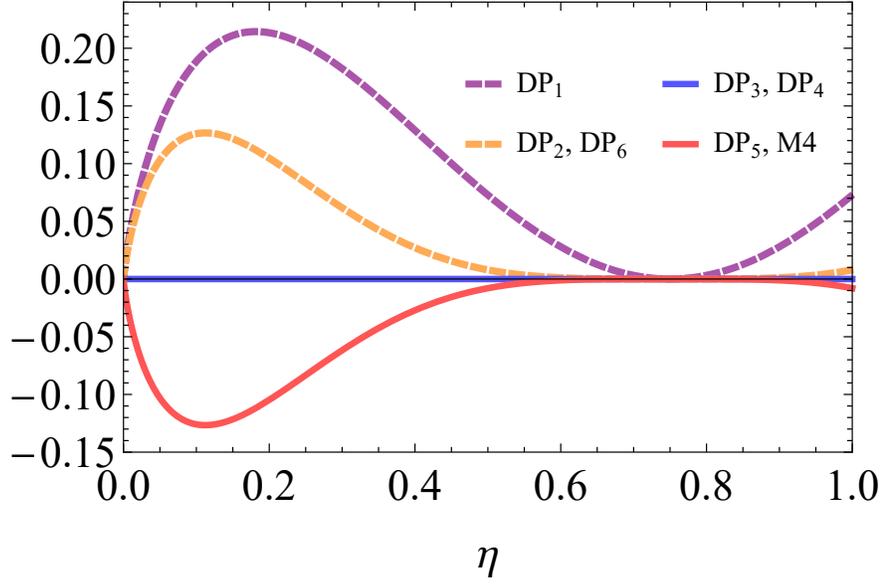
Figure 4.1: **Witnessing non-Markovianity with information inequalities.** The non-Markov process defined in chapter 2 implies in the violation of M4 and DP_5. All the other information inequalities are obsolete for every value of η.

4.3 Six-time-step classical Markov processes

Now, we consider the Markov monogamy inequalities for six-time-step Markov processes. For this sake, let $X_1 \rightarrow \cdots \rightarrow X_6$ be an arbitrary Markov process. Then, it follows that [2, 1]

$$I(X_1 : X_6) + I(X_2 : X_5) + I(X_3 : X_4) \geq I(X_1 : X_4) + I(X_2 : X_6) + I(X_3 : X_5), \quad (4.32)$$

$$I(X_1 : X_6) + I(X_2 : X_5) + I(X_3 : X_4) \geq I(X_1 : X_5) + I(X_2 : X_4) + I(X_3 : X_6). \quad (4.33)$$

The validity of the information inequalities above can be checked with the help of a program such as ITIP [35]. Nevertheless, we go through the analytical derivation of the first of them. We follow the proof presented in Ref. [2]. This is done so the reader can have an idea of how this information inequalities are proved. First, we add the following strong subadditivity inequalities

$$I(X_3 : X_4 | X_1, X_2, X_5, X_6) \geq 0, \quad (4.34)$$

$$I(X_1 : X_3 | X_2, X_5, X_6) \geq 0, \quad (4.35)$$

$$I(X_4 : X_6 | X_1, X_2, X_5) \geq 0, \quad (4.36)$$

$$I(X_3 : X_6 | X_2, X_5) \geq 0, \tag{4.37}$$

$$I(X_2 : X_4 | X_1, X_5) \geq 0, \tag{4.38}$$

$$I(X_2 : X_5 | X_1, X_6) \geq 0, \tag{4.39}$$

$$I(X_1 : X_6 | X_5) \geq 0, \tag{4.40}$$

$$I(X_1 : X_5 | X_4) \geq 0, \tag{4.41}$$

$$I(X_2 : X_5 | X_3) \geq 0, \tag{4.42}$$

$$I(X_1 : X_6 | X_2) \geq 0, \tag{4.43}$$

in order to get

$$H_{\mathrm{Markov}}(X_1, \ldots, X_6) - H(X_1, \ldots, X_6) \geq$$
$$I(X_1 : X_4) + I(X_2 : X_6) + I(X_3 : X_5)$$
$$- I(X_1 : X_6) - I(X_2 : X_5) - I(X_3 : X_4). \tag{4.44}$$

Then, the use of the Markov conditions concludes the proof. In general, the proof of Markov monogamy inequalities follows similar lines. We simply add strong subadditivity inequalities, and then we use the Markov conditions. The challenge here is to find the appropriate elementary conditions summing up to the desired information inequality. As we are going to see in the next sections, the number of strong subadditivity conditions necessary to prove Markov monogamy inequalities increases with the number of variables in the process.

4.4 Eight-time-step classical Markov processes

We move to define the Markov monogamy inequalities for eight-time-step Markov process. Thus, let $X_1 \to \cdots \to X_8$ be any Markov process. Then, it holds that [2, 1]

$$I(X_1 : X_8) + I(X_2 : X_7) + I(X_3 : X_6) + I(X_4 : X_5) \geq$$
$$I(X_1 : X_5) + I(X_2 : X_8) + I(X_3 : X_7) + I(X_4 : X_6), \tag{4.45}$$

$$I(X_1 : X_8) + I(X_2 : X_7) + I(X_3 : X_6) + I(X_4 : X_5) \geq$$
$$I(X_1 : X_7) + I(X_2 : X_5) + I(X_3 : X_8) + I(X_4 : X_6), \quad (4.46)$$

$$I(X_1 : X_8) + I(X_2 : X_7) + I(X_3 : X_6) + I(X_4 : X_5) \geq$$
$$I(X_1 : X_6) + I(X_2 : X_8) + I(X_3 : X_5) + I(X_4 : X_7), \quad (4.47)$$

$$I(X_1 : X_8) + I(X_2 : X_7) + I(X_3 : X_6) + I(X_4 : X_5) \geq$$
$$I(X_1 : X_5) + I(X_2 : X_6) + I(X_3 : X_8) + I(X_4 : X_7), \quad (4.48)$$

$$I(X_1 : X_8) + I(X_2 : X_7) + I(X_3 : X_6) + I(X_4 : X_5) \geq$$
$$I(X_1 : X_7) + I(X_2 : X_6) + I(X_3 : X_5) + I(X_4 : X_8), \quad (4.49)$$

$$I(X_1 : X_8) + I(X_2 : X_7) + I(X_3 : X_6) + I(X_4 : X_5) \geq$$
$$I(X_1 : X_6) + I(X_2 : X_5) + I(X_3 : X_7) + I(X_4 : X_8), \quad (4.50)$$

$$I(X_1 : X_8) + I(X_2 : X_7) + I(X_3 : X_6) + I(X_4 : X_5) \geq$$
$$I(X_1 : X_5) + I(X_2 : X_6) + I(X_3 : X_7) + I(X_4 : X_8). \quad (4.51)$$

Here, we provide a proof of the Markov monogamy condition in shown in Eq. (4.45). First, we add the strong subadditivity inequalities

$$I(X_4 : X_5 | X_1, X_2, X_3, X_6, X_7, X_8) \geq 0, \quad (4.52)$$
$$I(X_1 : X_4 | X_2, X_3, X_6, X_7, X_8) \geq 0, \quad (4.53)$$
$$I(X_5 : X_8 | X_1, X_2, X_3, X_6, X_7) \geq 0, \quad (4.54)$$
$$I(X_4 : X_8 | X_2, X_3, X_6, X_7) \geq 0, \quad (4.55)$$

$$I(X_3 : X_8 | X_1, X_2, X_6, X_7) \geq 0, \tag{4.56}$$

$$I(X_1 : X_3 | X_2, X_5, X_6, X_7) \geq 0, \tag{4.57}$$

$$I(X_2 : X_5 | X_1, X_6, X_7) \geq 0, \tag{4.58}$$

$$I(X_3 : X_5 | X_2, X_6, X_7) \geq 0, \tag{4.59}$$

$$I(X_2 : X_4 | X_3, X_6, X_7) \geq 0, \tag{4.60}$$

$$I(X_2 : X_7 | X_1, X_6, X_8) \geq 0, \tag{4.61}$$

$$I(X_1 : X_7 | X_5, X_6) \geq 0, \tag{4.62}$$

$$I(X_3 : X_6 | X_2, X_7) \geq 0, \tag{4.63}$$

$$I(X_4 : X_7 | X_3, X_6) \geq 0, \tag{4.64}$$

$$I(X_2 : X_6 | X_1, X_8) \geq 0, \tag{4.65}$$

$$I(X_1 : X_8 | X_6, X_7) \geq 0, \tag{4.66}$$

$$I(X_1 : X_8 | X_2) \geq 0, \tag{4.67}$$

$$I(X_2 : X_7 | X_3) \geq 0, \tag{4.68}$$

$$I(X_3 : X_6 | X_4) \geq 0, \tag{4.69}$$

$$I(X_1 : X_6 | X_5) \geq 0, \tag{4.70}$$

$$I(X_5 : X_7 | X_6) \geq 0, \tag{4.71}$$

$$I(X_6 : X_8 | X_7) \geq 0, \tag{4.72}$$

to obtain

$$H_{\mathrm{Markov}}(X_1, \ldots, X_8) - H(X_1 \ldots, X_8) \geq$$
$$I(X_1 : X_5) + I(X_2 : X_8) + I(X_3 : X_7) + I(X_4 : X_6)$$
$$- I(X_1 : X_8) - I(X_2 : X_7) - I(X_3 : X_6) - I(X_4 : X_5). \tag{4.73}$$

Again, the use of the Markov conditions concludes the proof.

4.5 Conjecture on classical Markov monogamy inequalities

Now, we make a conjecture on the general form of Markov monogamy inequalities. This is supported by the pattern appearing in the information inequalities considered so far. For that, define an arbitrary classical Markov process

$$X_m \to \cdots \to X_1 \to Y_1 \to \cdots \to Y_m, \tag{4.74}$$

with integer $m \geq 2$.

The systems X_1 and Y_1 are respectively interpreted as input and output variables of a classical channel. Therefore, the systems X_2 and Y_2 are pre- and post-processed variables of the given classical channel. This interpretation goes further identifying the systems X_t and Y_t, with $1 < t \leq m$, as representations of the pre-and-post-processing of input- and output-variables $t - 1$ times.

From these definitions, we make the following conjecture. For any process of the form in Eq. (4.74), it holds that[1]

$$\sum_{i=1}^{m} I(X_i : Y_i) \geq \sum_{i=1}^{m} I(X_i : Y_{f(i)}), \tag{4.75}$$

for any bijective function $f \colon \{1, \ldots, m\} \to \{1, \ldots, m\}$.

The Markov monogamy conditions considered so far are particular instances of the above inequality. For instance, consider the case of four-time-step classical Markov processes $X_2 \to X_1 \to Y_1 \to Y_2$. Then, define a bijective function $f \colon \{1, 2\} \to \{1, 2\}$ with $f(1) = 2$ and $f(2) = 1$. The conjecture above takes the form

$$I(X_1 : Y_1) + I(X_2 : Y_2) \geq I(X_1 : Y_2) + I(X_2 : Y_1). \tag{4.76}$$

The information inequality above is equivalent to the Markov monogamy in Eq. (4.19), which we know to be valid for any four-time-step Markov process. The conjecture also specify the validity of Eq. (4.75) for four-time-step Markov processes with respect to the bijective

function $g \colon \{1,2\} \to \{1,2\}$ for which $g(1) = 1$ and $g(2) = 2$. Nevertheless, this defines a trivially valid information inequality

$$I(X_1 : Y_1) + I(X_2 : Y_2) \geq I(X_1 : Y_1) + I(X_2 : Y_2) \tag{4.77}$$

Furthermore, we have that (4.76) is the only non-trivial Markov monogamy inequality for four-time-step-processes. We invite the interested reader to check that the only non-trivial Markov monogamy inequalities for six-time step processes are represented in Eqs. (4.32) and (4.32), and for eight-time-step processes are described in Eqs. (4.45)-(4.51).

Here, we do not provide a proof for the general form of Markov monogamy in Eq. (4.75). This remains as an open problem. Nevertheless, we have proved that several particular cases do hold. Namely, the information inequalities in Eqs. (4.1), (4.32) and (4.45) have been introduced here with detailed derivation. In order to close the discussion about the conjecture on classical Markov monogamy inequalities, we consider a last example.

Define a bijective function $f \colon \{1,2,3,4,5\} \to \{1,2,3,4,5\}$ assigning

$$f(1) = 2, \tag{4.78}$$
$$f(2) = 3, \tag{4.79}$$
$$f(3) = 4, \tag{4.80}$$
$$f(4) = 5, \tag{4.81}$$
$$f(5) = 1. \tag{4.82}$$

Thus, our conjecture states that the Markov monogamy inequality

$$I(X_1 : Y_1) + I(X_2 : Y_2) + I(X_3 : Y_3) + I(X_4 : Y_4) + I(X_5 : Y_5) \geq$$
$$I(X_1 : Y_2) + I(X_2 : Y_3) + I(X_3 : Y_4) + I(X_4 : Y_5) + I(X_5 : Y_1) \tag{4.83}$$

holds for any ten-time-step Markov process $X_5 \to \cdots \to X_1 \to Y_1 \to \cdots \to Y_5$.

Writing the process $X_5 \to \cdots \to X_1 \to Y_1 \to \cdots \to Y_5$ as $X_1 \to \cdots \to X_{10}$, the condition

in Eq. (4.83) becomes

$$I(X_1 : X_{10}) + I(X_2 : X_9) + I(X_3 : X_8) + I(X_4 : X_7) + I(X_5 : X_6) \geq$$
$$I(X_1, X_6) + I(X_2, X_{10}) + I(X_3, X_9) + I(X_4, X_8) + I(X_5, X_7). \quad (4.84)$$

We now prove the information inequality above by adding the following strong subadditivity inequalities [2]

$$I(X_5 : X_6 | X_1, X_2, X_3, X_4, X_7, X_8, X_9, X_{10}) \geq 0, \quad\quad (4.85)$$
$$I(X_1 : X_5 | X_2, X_3, X_4, X_7, X_8, X_9, X_{10}) \geq 0, \quad\quad (4.86)$$
$$I(X_6 : X_{10} | X_1, X_2, X_3, X_4, X_7, X_8, X_9) \geq 0, \quad\quad (4.87)$$
$$I(X_5 : X_{10} | X_2, X_3, X_4, X_7, X_8, X_9) \geq 0, \quad\quad (4.88)$$
$$I(X_1 : X_4 | X_2, X_3, X_6, X_7, X_8, X_9) \geq 0, \quad\quad (4.89)$$
$$I(X_4 : X_{10} | X_1, X_2, X_3, X_7, X_8, X_9) \geq 0, \quad\quad (4.90)$$
$$I(X_2 : X_5 | X_3, X_4, X_7, X_8, X_9) \geq 0, \quad\quad (4.91)$$
$$I(X_2 : X_6 | X_1, X_3, X_7, X_8, X_9) \geq 0, \quad\quad (4.92)$$
$$I(X_4 : X_6 | X_2, X_3, X_7, X_8, X_9) \geq 0, \quad\quad (4.93)$$
$$I(X_2 : X_7 | X_1, X_3, X_8, X_9, X_{10}) \geq 0, \quad\quad (4.94)$$
$$I(X_5 : X_9 | X_3, X_4, X_7, X_8) \geq 0, \quad\quad (4.95)$$
$$I(X_3 : X_6 | X_1, X_7, X_8, X_9) \geq 0, \quad\quad (4.96)$$
$$I(X_2 : X_4 | X_3, X_7, X_8, X_9) \geq 0, \quad\quad (4.97)$$
$$I(X_2 : X_8 | X_1, X_3, X_9, X_{10}) \geq 0, \quad\quad (4.98)$$
$$I(X_3 : X_8 | X_1, X_7, X_9, X_{10}) \geq 0, \quad\quad (4.99)$$
$$I(X_3 : X_5 | X_4, X_7, X_8) \geq 0, \quad\quad (4.100)$$
$$I(X_6 : X_8 | X_1, X_7, X_9) \geq 0, \quad\quad (4.101)$$
$$I(X_4 : X_7 | X_3, X_8, X_9) \geq 0, \quad\quad (4.102)$$
$$I(X_1 : X_3 | X_2, X_9, X_{10}) \geq 0, \quad\quad (4.103)$$
$$I(X_3 : X_7 | X_1, X_9, X_{10}) \geq 0, \quad\quad (4.104)$$

38

$$I(X_1 : X_8 | X_7, X_9, X_{10}) \geq 0, \qquad (4.105)$$

$$I(X_5 : X_8 | X_4, X_7) \geq 0, \qquad (4.106)$$

$$I(X_6 : X_9 | X_1, X_7) \geq 0, \qquad (4.107)$$

$$I(X_4 : X_9 | X_3, X_8) \geq 0, \qquad (4.108)$$

$$I(X_2 : X_9 | X_1, X_{10}) \geq 0, \qquad (4.109)$$

$$I(X_3 : X_{10} | X_2, X_9) \geq 0, \qquad (4.110)$$

$$I(X_1 : X_9 | X_7, X_{10}) \geq 0, \qquad (4.111)$$

$$I(X_7 : X_9 | X_8, X_{10}) \geq 0, \qquad (4.112)$$

$$I(X_4 : X_7 | X_5) \geq 0, \qquad (4.113)$$

$$I(X_1 : X_7 | X_6) \geq 0, \qquad (4.114)$$

$$I(X_3 : X_8 | X_4) \geq 0, \qquad (4.115)$$

$$I(X_1 : X_{10} | X_2) \geq 0, \qquad (4.116)$$

$$I(X_2 : X_9 | X_3) \geq 0, \qquad (4.117)$$

$$I(X_1 : X_{10} | X_7) \geq 0, \qquad (4.118)$$

$$I(X_7 : X_{10} | X_8) \geq 0, \qquad (4.119)$$

$$I(X_8 : X_{10} | X_9) \geq 0, \qquad (4.120)$$

we have the inequality

$$H_{\text{Markov}}(X_1, \ldots, X_{10}) - H(X_1, \ldots, X_{10}) \geq$$
$$I(X_1 : X_6) + I(X_2 : X_{10}) + I(X_3 : X_9) + I(X_4 : X_8) + I(X_5 : X_7)$$
$$- I(X_1 : X_{10}) - I(X_2 : X_9) - I(X_3 : X_8) - I(X_4 : X_7) - I(X_5 : X_6). \quad (4.121)$$

Then, using the Markov conditions completes the proof.

Note that we have proved that the Markov monogamy inequalities of the particular type

$$\sum_{i=1}^{m} I(X_i : X_{2m+1-i}) \geq I(X_1 : X_{m+1}) + \sum_{i=2}^{m} I(X_i : X_{2m+2-i}), \qquad (4.122)$$

hold for every Markov process $X_1 \to \cdots \to X_{2m}$, with $m = 2, 3, 4, 5$. That is, we have proved that the conjecture in Eq. (4.75) holds for the bijective functions $f : \{1, \ldots, m\} \to \{1, \ldots, m\}$ for which

$$f(i) = (i+1) \bmod m, \tag{4.123}$$

with $m = 2, 3, 4, 5$.

The remaining Markov monogamy inequalities for different bijective functions f can be quickly tested to be valid with numerical calculations provided by packages such as the one described in Ref. [35].

Part II

Quantum information inequalities

Chapter 5

Quantum processes

In chapter 2 we have considered classical processes. The state of a classical system was defined as a probability distribution, and a classical channel as a conditional probability distribution. In turn, a classical process was represented by a joint probability distribution of several classical systems, and thus interpreted as physical quantities observed at different instances of time.

Importantly, classical processes are obtained by the sequential action of classical channels on the joint past systems (cf. Eq. (2.9)). Here, we focus our study on the particular class of Markov processes, represented as the action of local classical operations on single systems only (cf. Eq. (2.14)).

Moreover, Markov processes are the very description of communication systems. Thus it is crucial to develop and to understand the operational meaning of information inequalities. That enforces the limits on what is possible under this model. For instance, one possibility are the data processing inequalities. We have addressed the first part of the problem and defined novel information inequalities, the Markov monogamy inequalities. In addition, we have considered how information inequalities can be used to witness non-Markov behaviour from a practical perspective.

It is of current understanding that classical probabilistic processes are not enough to predict all the phenomena present in modern physical sciences. For this reason we need the framework provided by quantum mechanics. Now, we go through a brief description of quantum processes necessary for the subsequent chapters. The goal here is twofold. Firstly,

we set up a precise terminology and notation used in the text to follow. Secondly, in doing so we introduce a more general audience to the basic elements and results necessary to understand the next chapters. So this is done without much physical motivation, although the interested reader is referred to the standard textbook in Ref. [36]. Much of the content presented in this chapter can be found in greater detail and with complete proofs in standard references in quantum information theory [37, 38, 39, 40, 41].

5.1   Quantum systems

To each quantum system S it is associated a Hilbert space. We use the same symbol to denote interchangeably a quantum system and its associated Hilbert space. Here, we only consider quantum systems with finite-dimensional spaces. So the concept of a Hilbert space is reduced to finite-dimensional complex vector spaces equipped with an inner product. Furthermore, all the mathematical tools necessary to understand the results presented here coincide with the ones from basic linear algebra, which can be found in Refs. [42, 43, 44, 45].

The state of a quantum system S is a linear operator $\rho : S \to S$ for which holds the conditions

$$\rho \geq 0, \tag{5.1}$$

$$\mathrm{tr}[\rho] = 1. \tag{5.2}$$

The set of all linear operators on S is denoted as $L(S)$. The collection of quantum states of S is then a subset of $L(S)$. We say the state of a quantum system S is pure whenever there is a unit vector $|\psi\rangle$ such that $\rho = \psi := |\psi\rangle \langle\psi|$. Otherwise, the quantum state is called mixed.

Let R and S be quantum systems. The Hilbert space of the compound quantum system of R and S is given by the tensor product $R \otimes S$. Every quantum system S in the state ρ can be extended to a bipartite system $R \otimes S$ in a pure state ψ with respect to some reference system R. The state ψ is called a purification of ρ, and R the purification system of S. That is, for each quantum state $\rho \in L(S)$ there is a pure bipartite quantum state $\psi \in L(R \otimes S)$ for

which $\text{tr}_R[\psi] = \rho$ [38]. The purification of a quantum system is not unique. Note this result becomes trivial when ρ is already pure. In such case we can take the trivial one-dimensional system as purification system.

5.2 Quantum operations

Now, quantum channels representing the transformation of quantum systems are described by a linear completely positive and trace preserving map. That is, the physical transformation of a quantum system S_1 into a quantum system S_2 is defined as a linear map $\Lambda : L(S_1) \to L(S_2)$ for which

$$\text{id}_R \otimes \Lambda \geq 0 \tag{5.3}$$

holds for every reference system R, and

$$\text{tr}_{S_2} \circ \Lambda = \text{tr}_{S_1} . \tag{5.4}$$

Every quantum channel has a Kraus decomposition [38]. That is, to each quantum channel $\Lambda : L(S_1) \to L(S_2)$ there is a collection of linear operators $L_k : S_1 \to S_2$ for which

$$\Lambda(\rho) = \sum_k L_k \rho L_k^\dagger \tag{5.5}$$

holds for every input operator ρ in $L(S_1)$. The Kraus decomposition of a quantum channel is not unique. Nevertheless, the number of Kraus operators may be made to be no larger than $\dim(S_1) \dim(S_2)$.

Kraus decomposition is not the only representation of quantum channels. Important to the developments in this study is the Stinespring dilation representation. To each quantum channel $\Lambda : L(S_1) \to L(S_2)$ it is associated a unitary linear operator $U : S_1 \otimes E_1 \to S_2 \otimes E_2$ and a pure state $\varphi \in L(E_1)$ such that

$$\Lambda(\rho) = \text{tr}_{E_2}[U(\rho \otimes \varphi)U^\dagger], \tag{5.6}$$

for each input operator ρ in $L(S_1)$. The quantum systems E_1 and E_2 are called input and

output environmental systems of the channel Λ, respectively.

In this way, we can identify the action of a noisy channel Λ in a state ρ as a global system-environment unitary evolution of a separable bipartite input state $\rho \otimes \varphi$ given by $\mathrm{U}(\bullet) = U \bullet U^\dagger$, and then followed by discarding the environment system E_2. Therefore, this is also called a unitary representation or a dilation of a quantum channel. Once again, the unitary representation of a given channel is not unique.

Now we have all the basic elements we need in order to define quantum processes, and in particular quantum Markov processes. This will be a crucial step towards the setting of quantum information inequalities. Nevertheless, we consider a last example before we move to their proper definition. Here, we study the case of a non-Markov classical process arising from a quantum process.

5.3   Quantum measurements

The way to extract a classical system X out of a quantum system S is by performing a measurement. A quantum measurement is defined by assigning to each possible outcome $x \in X$ a measurement operator $M_x : S \to S$. Measurement operators are expected to respect the normalization condition

$$\sum_x M_x^\dagger M_x = \mathbb{1}_S. \tag{5.7}$$

Here, the set of outcomes $X = \{x\}$ define a classical system in the state with probability masses

$$p(x) = \mathrm{tr}[\mathrm{M}_x(\rho)]. \tag{5.8}$$

where the measurement maps $\mathrm{M}_x \colon L(S) \to L(S)$ are defined as $\mathrm{M}_x(\rho) = M_x \rho M_x^\dagger$, for every $\rho \in L(S)$. Therefore, the measurements are defined with linear completely positive maps M_x. The normalization condition (5.7) implies that the sum of the measurement maps $\sum_{x \in X} \mathrm{M}_x$ is also a trace-preserving quantum operation.

The post-measurement quantum state of S given the outcome x was observed is given by

$$\rho_x = \frac{\mathrm{M}_x(\rho)}{p(x)}. \tag{5.9}$$

The quantum measurement process defines a single-time-step classical process X with state $p(X)$. Now, we show how consecutive quantum measurements gives rise to a classical process. We consider the presentation in Ref. [40]. Suppose in the following we are given with a quantum system S in the quantum state ρ.

A classical system X_1 is defined in the state $p(x_1) = \text{tr}[A_{x_1}(\rho)]$ by a collection of quantum measurement maps $\{A_{x_1} : x_1 \in X_1\}$. Provided the classical system X_1 occurred in the particular outcome x_1, we know for sure that the quantum state of the system S after the measurement process is given by

$$\rho_{x_1} = \frac{A_{x_1}(\rho)}{p(x_1)}. \tag{5.10}$$

Suppose also that after a given outcome x_1 occurred, a subsequent quantum measurement is performed on the system S. This process defines a second classical system X_2 conditional on the value x_1 of X_1. Let the second measurement process be described by measurement maps B_{x_2} with $x_2 \in X_2$. Note that the maps B_{x_2} are not necessarily the same as A_{x_1}. The probability of X_2 conditional on X_1 is given by

$$p(x_2|x_1) = \text{tr}[B_{x_2}(\rho_{x_1})]. \tag{5.11}$$

Thus, the joint probability distribution of X_1 and X_2 is $p(x_1, x_2) = \text{tr}[B_{x_2} \circ A_{x_1}(\rho)]$. Furthermore, the post-measurement quantum state of the system S given the joint outcome (x_1, x_2) is

$$\rho_{x_1, x_2} = \frac{B_{x_2} \circ A_{x_1}(\rho)}{p(x_1, x_2)}. \tag{5.12}$$

Equation (5.12) shows that two sequential measurements are described with the composition of the measurement maps in the correct order, that is, the measurement maps resulting the joint classical system $X_1 \times X_2$ is $M_{x_1, x_2} = B_{x_2} \circ A_{x_1}$. We can keep this reasoning iteratively in order to show that an arbitrary number of measurements on a quantum system is thus described by the composition of the measurements maps.

For instance, we can define a four-time-step classical process $X_1 \times X_2 \times X_3 \times X_3$ by defining quantum measurement maps $\{A_{x_1} : x_1 \in X_1\}$, $\{B_{x_2} : x_2 \in X_2\}$, $\{C_{x_3} : x_3 \in X_3\}$ and
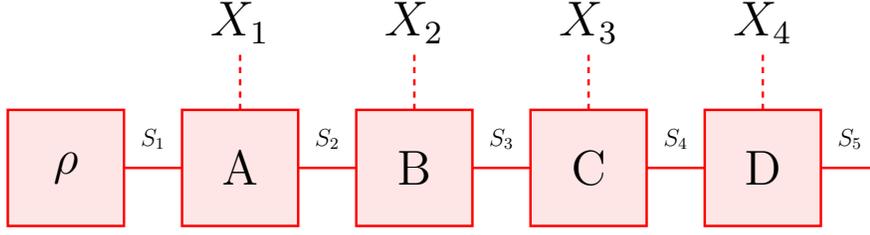
Figure 5.1: **Sequential quantum measurement process**. A quantum measurement is defined with measurement operators. This process generates a classical system associated with the possible outcomes. The measurement operators of a sequential measurement is defined by the concatenation of the measurement operators corresponding to each step of the process.

$\{D_{x_4} \colon x_4 \in X_4\}$. In turn, this classical process is represented with probability masses

$$p(x_1, x_2, x_3, x_4) = \mathrm{tr}[D_{x_4} \circ C_{x_3} \circ B_{x_2} \circ A_{x_1}(\rho)]. \tag{5.13}$$

Figure 5.1 represents the four-time-step quantum measurement process. Here, we consider a particular example. We take $A_x = C_x$ and $B_y = D_y$, with $x \in \{0, 1, 2, 3\}$ and $y \in \{0, 1\}$. Therefore, we set a measurement process with corresponding maps

$$M_{x_1, x_2, x_3, x_4} = B_{x_4} \circ A_{x_3} \circ B_{x_2} \circ A_{x_1}. \tag{5.14}$$

Let the maps A_x, with $x \in X_1 = X_3$, be defined with measurement operators represented in the canonical basis $\{|0\rangle, |1\rangle\}$ as

$$A_0 \;=\; \begin{bmatrix} +0.4953 + \mathrm{i}\,0.0687 & +0.0874 - \mathrm{i}\,0.2751 \\ +0.2751 + \mathrm{i}\,0.0874 & +0.1327 + \mathrm{i}\,0.2564 \end{bmatrix}, \tag{5.15}$$

$$A_1 \;=\; \begin{bmatrix} +0.1327 + \mathrm{i}\,0.2564 & 0.2751 + \mathrm{i}\,0.0874 \\ +0.0874 - \mathrm{i}\,0.2751 & +0.4953 + \mathrm{i}\,0.0687 \end{bmatrix}, \tag{5.16}$$

$$A_2 \;=\; \begin{bmatrix} +0.1327 + \mathrm{i}\,0.2564 & -0.2751 - \mathrm{i}\,0.0874 \\ -0.0874 + \mathrm{i}\,0.2751 & +0.4953 + \mathrm{i}\,0.0687 \end{bmatrix}, \tag{5.17}$$

$$A_3 \;=\; \begin{bmatrix} +0.4953 + \mathrm{i}\,0.0687 & -0.0874 + \mathrm{i}\,0.2751 \\ -0.2751 - \mathrm{i}\,0.0874 & +0.1327 + \mathrm{i}\,0.2564 \end{bmatrix}. \tag{5.18}$$
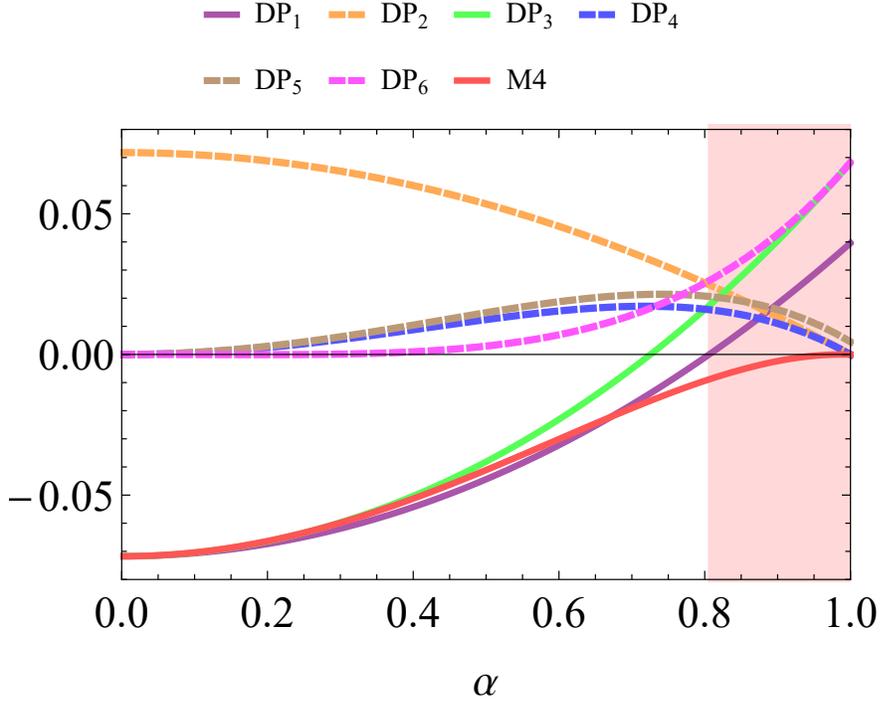
Figure 5.2: **Markov monogamy violation in non-projective quantum measurements.**
The plot shows the information inequalities as a function of the parameter α. Markov
monogamy is the only inequality witnessing non-Markovianity in the region shaded in red
color.

Define the maps B_y, with $y \in X_2 = X_4$, with measurement operators

$$B_0 \;=\; \sqrt{\frac{1+\alpha}{2}}\,|+\rangle\langle+| + \sqrt{\frac{1-\alpha}{2}}\,|-\rangle\langle-|, \tag{5.19}$$

$$B_1 \;=\; \sqrt{\frac{1-\alpha}{2}}\,|+\rangle\langle+| + \sqrt{\frac{1+\alpha}{2}}\,|-\rangle\langle-|, \tag{5.20}$$

where $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ and $0 \leq \alpha \leq 1$. The parameter α determines to each extent
$\{B_0, B_1\}$ is a projective measurement. For the limiting case $\alpha = 1$ we have a completely
projective measurement in the basis $\{|+\rangle, |-\rangle\}$. On the other hand, for $\alpha = 0$ we have the
opposite situation of a completely non-informative measurement.

If the initial state is $\rho = |+\rangle\langle+|$ we find that the Markov monogamy inequality is violated
in the region greater than $\alpha \simeq 0.8$. Nevertheless, none of the data processing inequalities
are violated, as shown in Fig. 5.2. In turn, this shows how relevant the Markov monogamy
inequalities can be.

We have constructed a class of classical processes by the action of non-projective quantum measurements on a quantum system for which Markov monogamy witness non-Markovianity beyond data processing inequalities. Nevertheless, apart from the fact that we have used a quantum system in generating classical correlations, the process of interest is completely classical. This motivates the question whether it is possible to properly define information inequalities in purely quantum terms. This is the main question addressed in the following chapters.

Chapter 6

Quantum data processing inequalities

Classical and quantum processes have a key distinguishing feature. A quantum operation $\Lambda: L(S_1) \to L(S_2)$ transforms the state ρ of its input system S_1 into the state σ of its output system S_2. On the other hand, classical operations are represented by conditional probability distributions $p(X_2|X_1)$, which maps the state $p(X_1)$ of the input system X_1 into the state $p(X_1, X_2)$ of the joint input and output system $X_1 \times X_2$.

Therefore, we can naturally determine the temporal correlations spread through classical processes. Mutual information plays a central role in characterising classical Markov processes. Nevertheless, it does not seem possible to trivially extend the information inequalities to the quantum realm due to the impossibility of directly compute temporal correlations. We refer the reader to chapter 9 in Ref. [40] for a more detailed discussion on this regard.

In order to stablish the quantum data processing inequality, we need to replace classical mutual information by a more appropriate quantity for quantum processes. This is done with the coherent information. But let us first define the quantum entropy of a system S in the state ρ as [37]

$$H(S)_\rho = - \sum_{\lambda_i > 0} \lambda_i \log_2 \lambda_i, \tag{6.1}$$

where $\{\lambda_i \colon i = 1, \ldots, \dim(S)\}$ are the eigenvalues of ρ. The quantum entropy $H(S)_\rho$ is also denoted as $H(S)$ when the state is implicitly known. It is also denoted as $H(\rho)$ when the referred quantum system is clear from context.

Now, we define the coherent information of the state ρ of a quantum system S_1 with

respect to a quantum channel $\Lambda\colon L(S_1) \to L(S_2)$ as [21]

$$I_c(\rho; \Lambda) := H(\Lambda(\rho)) - H((\mathrm{id}_R \otimes \Lambda)(\psi)), \tag{6.2}$$

where $\psi \in L(R \otimes S_1)$ is a any purification of ρ.

Note that the coherent information can be expressed as the negative of a conditional quantum entropy. That is, $I_c(\rho; \Lambda) = -H(R|S_2) := H(S_2) - H(R, S_2)$. Therefore, coherent information is positive when there are non-classical correlations between the systems R and S_2.

In what follows, a sequence of quantum states $\rho_1 \in L(S_1)$, $\rho_2 \in L(S_2)$ and $\rho_3 \in L(S_3)\}$ is called a three-time-step quantum Markov process with respect to the quantum channels $\Lambda_1\colon L(S_1) \to L(S_2)$ and $\Lambda_2\colon L(S_2) \to L(S_3)$ whenever the following conditions are satisfied

$$\rho_2 = \Lambda_1(\rho_1), \tag{6.3}$$

$$\rho_3 = \Lambda_2(\rho_2). \tag{6.4}$$

The quantum data processing inequality is defined as follows for three-time-step quantum processes. For any quantum state ρ of S_1, and for any quantum channels $\Lambda_1\colon L(S_1) \to L(S_2)$ and $\Lambda_2\colon L(S_2) \to L(S_3)$, it holds that [21]

$$I_c(\rho_1; \Lambda_1) \geq I_c(\rho_1; \Lambda_2 \circ \Lambda_1). \tag{6.5}$$

Now we prove that the information inequality above holds for any three-time-step quantum Markov process, the so-called quantum data processing theorem.

Let ψ in $L(R \otimes S_1)$ be a purification of the initial state ρ in $L(S_1)$. Define also a unitary representation of the quantum channels Λ_1 and Λ_2 according to

$$\Lambda_1(\sigma_1) = \mathrm{tr}_{E_1}\left[U_1(\sigma_1 \otimes \varphi_1)U_1^\dagger \right] \tag{6.6}$$

and

$$\Lambda_2(\sigma_2) = \mathrm{tr}_{E_2}\left[U_2(\sigma_2 \otimes \varphi_2)U_2^\dagger \right], \tag{6.7}$$

for any operators σ_1 and σ_2 in $L(S_1)$ and $L(S_2)$, respectively. The linear transformations $U_1\colon S_1 \otimes F_1 \to S_2 \otimes E_1$ and $U_2\colon S_2 \otimes F_2 \to S_3 \otimes E_2$ are unitary operators, and the pure quantum states φ_1 and φ_2 are in $L(F_1)$ and $L(F_2)$, respectively.

Furthermore, we can define a purification for the quantum Markov process $\rho_1 \xrightarrow{\Lambda_1} \rho_2 \xrightarrow{\Lambda_2} \rho_3$. This is done by defining a sequence of three pure quantum states $\{\gamma_1, \gamma_2, \gamma_3\}$ with respect to the unitary quantum evolutions U_1 and U_2. The first stage of the purification of the Markov process is given by the reference-system-environment system in the pure state

$$|\gamma_1\rangle = |\psi\rangle \otimes |\varphi_1\rangle \otimes |\varphi_2\rangle . \tag{6.8}$$

Note that tracing out the systems R, F_1 and F_2 we get the initial state ρ_1, in such a way that $|\gamma_1\rangle$ is a purification of S_2 as much as $|\psi\rangle$.

Then, define the second stage of the purification of the Markov process acting the unitary operation U_1 on the local system $S_1 \otimes F_1$, and swapping the output system $S_2 \otimes E_1$. That is represented by the pure state

$$|\gamma_2\rangle = (\mathbb{1}_R \otimes \mathsf{SWAP}_{S_2,E_1} \otimes \mathbb{1}_{E_2})(\mathbb{1}_R \otimes U_1 \otimes \mathbb{1}_{F_2}) |\gamma_1\rangle , \tag{6.9}$$

where the linear operator $\mathsf{SWAP}_{A,B} : A \otimes B \to B \otimes A$ is defined requiring that

$$\mathsf{SWAP}_{A,B}(|i\rangle \otimes |j\rangle) = |j\rangle \otimes |i\rangle \tag{6.10}$$

for arbitrary orthonormal basis $\{|i\rangle\}$ and $\{|j\rangle\}$ of A and B, respectively. Note that tracing out the systems R, E_1 and F_2 we have the system S_2 in the state $\rho_2 = \Lambda_1(\rho_1)$.

The final stage of the three-time-step Markov process purification is obtained by acting U_2 on the $S_2 \otimes F_2$-part, and then swapping the output system $S_3 \otimes E_2$. That is,

$$|\gamma_3\rangle = (\mathbb{1}_R \otimes \mathbb{1}_{E_1} \otimes \mathsf{SWAP}_{S_3,E_2})(\mathbb{1}_R \otimes \mathbb{1}_{E_2} \otimes U_2) |\gamma_2\rangle . \tag{6.11}$$

Note that tracing out R, E_1, E_2 we have the system S_3 in the state $\rho_3 = \Lambda_2(\rho_2) = \Lambda_2(\Lambda_1(\rho_1))$.

Figure 6.1 shows a diagram representing the purification for the quantum Markov process
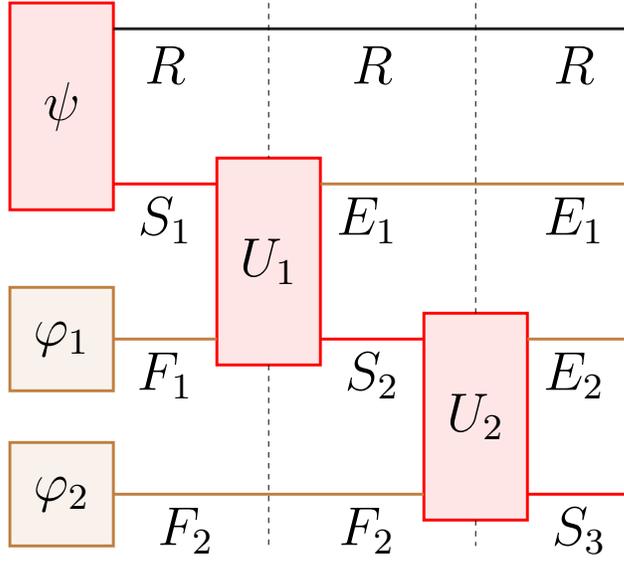
Figure 6.1: **Diagram representing the purification of three-time-step quantum Markov processes.** The diagram displays the pure final state obtained by acting successively the isometric representations $(\mathbb{1}_R \otimes U_i)(\bullet \otimes |\varphi_i\rangle)$ of Λ_i (with $i = 1, 2$) on the purification ψ of ρ. This picture was adapted from [1].

described above. Note that we can write the quantum data processing inequality depending upon the states $\{\gamma_1, \gamma_2, \gamma_3\}$. The terms in (6.5) are given as

$$I_c(\rho_1; \Lambda_1) = H(S_2)_{\gamma_2} - H(R, S_2)_{\gamma_2}, \tag{6.12}$$

$$I_c(\rho_1; \Lambda_2 \circ \Lambda_1) = H(S_3)_{\gamma_3} - H(R, S_3)_{\gamma_3}. \tag{6.13}$$

Now, consider the following assertions for the pure Markov process $\gamma_1 \xrightarrow{U_1} \gamma_2 \xrightarrow{U_2} \gamma_3$:

$$R \otimes E_1 \otimes E_2 \otimes S_3 \text{ is pure} \Rightarrow H(S_3)_{\gamma_3} = H(R, E_1, E_2)_{\gamma_3}, \tag{6.14}$$

$$R \otimes E_1 \otimes S_2 \text{ is pure} \Rightarrow H(S_2)_{\gamma_2} = H(R, E_1)_{\gamma_2}, \tag{6.15}$$

$$R \otimes E_1 \otimes S_2 \text{ is pure} \Rightarrow H(R, S_2)_{\gamma_2} = H(E_1)_{\gamma_2}, \tag{6.16}$$

$$R \otimes E_1 \otimes E_2 \otimes S_3 \text{ is pure} \Rightarrow H(R, S_3)_{\gamma_3} = H(E_1, E_2)_{\gamma_3}. \tag{6.17}$$

Note that a local quantum operation T: $L(B) \to L(C)$ acting on a bipartite system $A \otimes B$ in the state σ preserves the marginal system A. That is true because quantum operations

preserve the trace operation, thus, $\mathrm{tr}[\mathrm{T}(P)] = \mathrm{tr}[P]$ for every $P \in L(B)$. In order to check that $\mathrm{tr}_B[\sigma] = \mathrm{tr}_C[(\mathrm{id} \otimes \mathrm{T})(\sigma)]$ for every $\sigma \in L(A \otimes B)$, define an operator basis $\{P_j\}$ for $L(B)$. Then, every linear bipartite operator on $A \otimes B$ can be represented as $\sigma = \sum_j \sigma_j \otimes P_j$.

Furthermore, we have

$$\mathrm{tr}_C[(\mathrm{id} \otimes \mathrm{T})(\sigma)] = \sum_j \mathrm{tr}[\mathrm{T}(P_j)]\sigma_j \tag{6.18}$$

$$= \sum_j \mathrm{tr}[P_j]\sigma_j \tag{6.19}$$

$$= \mathrm{tr}_B[\sigma]. \tag{6.20}$$

Then, we can write the right-hand-side of Eqs. (6.15) and (6.16) as

$$H(R, E_1)_{\gamma_2} = H(R, E_1)_{\gamma_3}, \tag{6.21}$$

$$H(E_1)_{\gamma_2} = H(E_1)_{\gamma_3}. \tag{6.22}$$

Finally, the conditions $H(R, E_1)_{\gamma_3} = H(S_2)_{\gamma_2}$ and $H(R, S_2)_{\gamma_2} = H(E_1)_{\gamma_3}$ hold. Moreover, the strong subadditivity of quantum entropy [46, 47, 48]

$$H(R, E_1, E_2)_{\gamma_3} + H(E_1)_{\gamma_3} \leq H(R, E_1)_{\gamma_3} + H(E_1, E_2)_{\gamma_3}, \tag{6.23}$$

altogether with Eqs. (6.14)-(6.17), (6.12) and (6.13) imply the desired condition in Eq. (6.5).

Interestingly, the quantum data processing theorem can be reformulated to be stated in terms of the quantum mutual information $I(A : B)_\rho = H(A)_\rho + H(B)_\rho - H(A, B)_\rho$ of bipartite quantum systems $A \otimes B$ in the state ρ. For any bipartite quantum state $\rho \in L(A \otimes B)$, and for any quantum channel $\Lambda \colon L(B) \to L(C)$, it holds that [37]

$$I(A : B)_\rho \geq I(A : C)_{(\mathrm{id}_A \otimes \Lambda)(\rho)}. \tag{6.24}$$

The above result is also called quantum data processing theorem, and furthermore, Eq. (6.24) is also called quantum data processing inequality. In fact, the following sentences are equivalent:

(I) For any quantum state ρ of S_1, and for any quantum channels $\Lambda_1 \colon L(S_1) \to L(S_2)$ and $\Lambda_2 \colon L(S_2) \to L(S_3)$, it holds the data processing inequality

$$I_c(\rho; \Lambda_1) \geq I_c(\rho; \Lambda_2 \circ \Lambda_1).$$

(II) For any quantum state σ of a bipartite system $A \otimes B$, and for any quantum operation $\Lambda \colon L(B) \to L(C)$, it holds that the quantum mutual information is monotonically decreasing under the action of the local operation Λ. That is,

$$I(A : B)_\sigma \geq I(A : C)_{(\mathrm{id}_A \otimes \Lambda)(\sigma)}.$$

We start with the trivial assertion that (I) is a necessary condition for (II). That is, we prove that (II) $\Rightarrow$ (I). For this matter, suppose (II) is true. Let ρ be the state of an arbitrary quantum system S_1, and ψ be a purification with respect to a bipartite system $R \otimes S_1$. Consider also arbitrary quantum channels $\Lambda_1 \colon L(S_1) \to L(S_2)$ and $\Lambda_2 \colon L(S_2) \to L(S_3)$. Since (II) is true by hypothesis, and $\sigma := (\mathrm{id}_R \otimes \Lambda_1)(\psi)$ is a state of the bipartite system $R \otimes S_2$, we have

$$I(R : S_2)_\sigma \geq I(R : S_2)_{(\mathrm{id}_R \otimes \Lambda_2)(\sigma)}. \tag{6.25}$$

Subtracting $H(R)$ from both sides in (6.25) we have the desired inequality

$$I_c(\rho; \Lambda_1) \geq I_c(\rho; \Lambda_2 \circ \Lambda_1), \tag{6.26}$$

for arbitrary ρ, Λ_1 and Λ_2.

In order to prove the more involving assertion that (II) is a necessary condition for (I), we only need to use the following result.

Let ψ be a pure state of a bipartite quantum system $R \otimes S_1$, and let ρ be any state of a bipartite quantum system $R \otimes S_2$ for which

$$\mathrm{tr}_{S_1}[\psi] = \mathrm{tr}_{S_2}[\rho]. \tag{6.27}$$

Thus, the states ψ and ρ are different extensions of the same marginal system R. In

56

particular, ψ is a purification of R. In such a case, there is a quantum channel $\Lambda\colon L(S_1) \to L(S_2)$ such that [38]

$$\rho = (\mathrm{id}_R \otimes \Lambda)(\psi). \tag{6.28}$$

Now, let us prove (I) $\Rightarrow$ (II) is true. So suppose (I) is true. Following the derivation of (II) $\Rightarrow$ (I) we see that in order to prove its converse statement all we need to do is to prove that all bipartite quantum states can be written as $\sigma := (\mathrm{id}_R \otimes \Lambda_2)(\psi)$ for some pure state ψ of a bipartite system, and quantum channel Λ_1. Then, we can add $H(R)$ to both sides of Eq. (6.26) and we are done.

So let ρ be an arbitrary quantum state of any bipartite quantum system $R \otimes S_2$. Take its marginal with respect to the system R, that is, $\tau := \mathrm{tr}_R[\rho]$. Now let ψ be a purification of τ with respect to a purification system S_1 such that $\psi \in R \otimes S_1$. Thus we have proved the existence of a pure quantum state ψ for which $\mathrm{tr}_{S_1}[\psi] = \mathrm{tr}_{S_2}[\rho]$. Now, the above reasoning makes sure the existence of a quantum channel $\Lambda_1\colon L(S_1) \to L(S_2)$ fulfilling the desired property.

Equation (6.5) is the quantum version of the classical data processing inequality $I(X_1 : X_2) \geq I(X_1 : X_3)$, for three-time-step classical Markov processes. Nevertheless, that is not the only possibility for $X_1 \to X_2 \to X_3$. We would also expect the information inequality $I(X_2 : X_3) \geq I(X_1 : X_3)$ to have a quantum counterpart. Would the condition

$$I_c(\Lambda_1(\rho); \Lambda_2) \geq I_c(\rho; \Lambda_2 \circ \Lambda_1) \tag{6.29}$$

hold for all quantum states $\rho \in L(S_1)$, and for all quantum channels $\Lambda_1\colon L(S_1) \to L(S_2)$ and $\Lambda_2\colon L(S_2) \to L(S_3)$?

Unfortunately, we do not have a conclusive answer to this question. In fact, its validity is not trivial. Define a purification $\gamma_1 \xrightarrow{U_1} \gamma_2 \xrightarrow{U_2} \gamma_3$ to a quantum Markov process $\rho_1 \xrightarrow{\Lambda_1} \rho_2 \xrightarrow{\Lambda_2} \rho_3$ as represented in Fig. (6.1). Then, Eq. (6.29) can be expressed as

$$H(E_1|E_2)_{\gamma_3} \geq 0, \tag{6.30}$$

57

where the conditional quantum entropy is defined as $H(E_1|E_2) = H(E_1, E_2) - H(E_2)$.[1]

In despite of the non-negativity of classical conditional entropy, quantum conditional entropy can be negative. For instance, this is the case for maximally entangled systems. Moreover, the above inequality cannot be derived from the strong subadditivity of quantum entropy. Nevertheless, we argue on the validity of Eq. (6.30) to a particular example of quantum Markov processes.

Define the purification of a quantum Markov process as follows. Let all the subsystems in question be qubits with canonical basis $\{|0\rangle, |1\rangle\}$. Then, start the environmental systems F_1 and F_2 in the same state $\varphi_1 = \varphi_2 = |0\rangle$. Let the bipartite system $R \otimes S_1$ be in the maximally entangled state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle). \tag{6.31}$$

Then, define unitary linear operators $U_1 \colon S_1 \otimes F_1 \to S_2 \otimes E_1$ and $U_2 \colon S_2 \otimes F_2 \to S_3 \otimes E_2$ with identical representation in the canonical basis as

$$U^{(\lambda)} = \begin{bmatrix} 0 & -\sqrt{1-\lambda} & \sqrt{\lambda} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \sqrt{\lambda} & \sqrt{1-\lambda} & 0 \end{bmatrix}, \tag{6.32}$$

where $0 \leq \lambda \leq 1$.

The local evolution of the system represented with maps $\Lambda_i(\bullet) = \mathrm{tr}_{E_i}[U_i(\bullet \otimes \varphi_i)U_i^\dagger]$, for $i \in \{1, 2\}$, are then defined as the amplitude damping channel with parameter λ [37]. Figure (6.2) shows that the condition in (6.30) holds for any value of λ in this example. This is clearly not a proof that this quantum data processing condition holds for any quantum Markov process. Furthermore, it remains as an open problem.

---

[1]The condition in Eq. (6.29) holds with equality for isometric quantum channels Λ_1. See Equation (8.42) in Ref.[39]. That is equivalent of considering a trivial one-dimensional quantum system E_1 in Eq. (6.30), and thus, the conditional entropy vanishes $H(E_1|E_2)_{\gamma_3} = 0$. Nevertheless, we could not find a violation of the data processing condition in Eq. (6.29) for the examples of quantum Markov processes considered here, as argued in the Fig. 6.2.
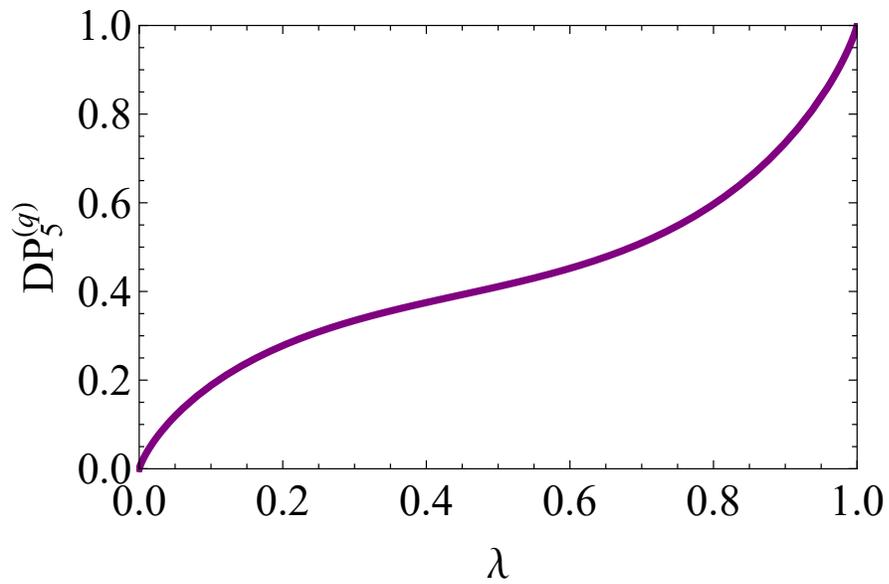
Figure 6.2: **Test of the condition in (6.30).** The conditional entropy $H(E_1|E_2)$ is non-negative for the pure Markov process in Figure 6.1 with system $R \otimes S_1$ in the maximally entangled state, enviromental systems F_1 and F_2 in the state $|0\rangle$, and the unitary operators U_1, U_2 as in Eq. (6.32).

Chapter 7

Quantum Markov monogamy inequalities

The quantum data processing inequality can be defined within the appropriate definition of quantum Markov processes. It has at least two equivalent forms. The first form states that coherent information is monotonically decreasing under the action of a quantum operation. The second form states that quantum mutual information is monotonically decreasing under the action of quantum local operations.

Furthermore, would it be possible to define Markov monogamy inequalities for quantum Markov processes? We answer this question affirmatively here, deriving the quantum Markov monogamy inequalities. We also show how they take place in two equivalent versions similar to the case of quantum data processing inequality.

Before moving to derive the quantum Markov monogamy inequalities, we formally define what is meant by n-time-step quantum Markov processes. A sequence of quantum states $\{\rho_1, \ldots, \rho_n\}$ is a quantum Markov process with respect to a sequence of quantum channels $\{\Lambda_1, \ldots, \Lambda_{n-1}\}$ if the condition

$$\rho_{i+1} = \Lambda_i(\rho_i) \tag{7.1}$$

holds for every $i \in \{1, \ldots, n-1\}$. Such situation is denoted by

$$\rho_1 \xrightarrow{\Lambda_1} \rho_2 \xrightarrow{\Lambda_2} \rho_3 \xrightarrow{\Lambda_3} \ldots \xrightarrow{\Lambda_{n-1}} \rho_n. \tag{7.2}$$

7.1 Four-time-step quantum Markov processes

Let us start considering four-time-step Markov processes. This is the case for which there is a single Markov monogamy inequality. Recall that for classical Markov processes $X_1 \to X_2 \to X_3 \to X_4$ the Markov monogamy inequality reads

$$I(X_1 : X_4) + I(X_2 : X_3) \geq I(X_1 : X_3) + I(X_2 : X_4). \tag{7.3}$$

Now, our goal here is to develop a condition of the same type for quantum Markov processes $\rho_1 \xrightarrow{\Lambda_1} \rho_2 \xrightarrow{\Lambda_2} \rho_3 \xrightarrow{\Lambda_3} \rho_4$. Thus, consider the reasonable substitution

$$I(X_1 : X_4) \to I_c(\rho_1; \Lambda_3 \circ \Lambda_2 \circ \Lambda_1), \tag{7.4}$$

$$I(X_1 : X_3) \to I_c(\rho_1; \Lambda_2 \circ \Lambda_1), \tag{7.5}$$

$$I(X_2 : X_3) \to I_c(\Lambda_1(\rho_1); \Lambda_2), \tag{7.6}$$

$$I(X_2 : X_4) \to I_c(\Lambda_1(\rho_1); \Lambda_3 \circ \Lambda_2). \tag{7.7}$$

In fact, this reasoning defines a valid condition on four-time-step quantum Markov processes. Let S_1 be any initial quantum system with state ρ_1. For every quantum channels $\Lambda_1 \colon L(S_1) \to L(S_2)$, $\Lambda_2 \colon L(S_2) \to L(S_3)$ and $\Lambda_3 \colon L(S_3) \to L(S_4)$, it holds that [1]

$$I_c(\rho_1, \Lambda_3 \circ \Lambda_2 \circ \Lambda_1) + I_c(\Lambda_1(\rho_1), \Lambda_2) \geq I_c(\rho_1, \Lambda_2 \circ \Lambda_1) + I_c(\Lambda_1(\rho_1), \Lambda_3 \circ \Lambda_2). \tag{7.8}$$

In order to prove the quantum Markov monogamy inequality in Eq. (7.8), let us define a purification $\gamma_1 \xrightarrow{U_1} \gamma_2 \xrightarrow{U_2} \gamma_3 \xrightarrow{U_3} \gamma_4$ for the quantum Markov process $\rho_1 \xrightarrow{\Lambda_1} \rho_2 \xrightarrow{\Lambda_2} \rho_3 \xrightarrow{\Lambda_3} \rho_4$. For this sake, let $\psi \in L(R \otimes S_1)$ be a purification for the initial state $\rho_1 \in L(S_1)$. Then, define a dilation for each of the quantum channels Λ_1, Λ_2 and Λ_3. That is, set unitary linear operators $U_i \colon S_i \otimes F_i \to S_{i+1} \otimes E_i$ and pure initial environmental quantum states $\varphi_i \in L(F_i)$, with $i = 1, 2, 3$, such that

$$\Lambda_i(\rho) = \mathrm{tr}_{E_i}\left[U_i(\rho \otimes \varphi_i)U_i^\dagger \right], \tag{7.9}$$

for any operator ρ in $L(S_i)$.

The pure quantum Markov process $\gamma_1 \xrightarrow{U_1} \gamma_2 \xrightarrow{U_2} \gamma_3 \xrightarrow{U_3} \gamma_4$ is then defined as follows.
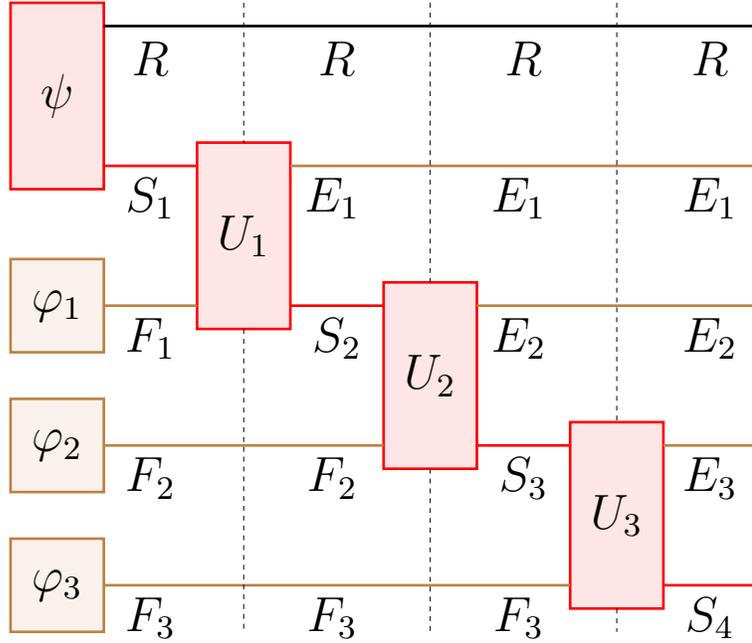
Figure 7.1: **Diagram representing the purification of the process** $\rho_1 \xrightarrow{\Lambda_1} \rho_2 \xrightarrow{\Lambda_2} \rho_3 \xrightarrow{\Lambda_3}$ ρ_4**.** The quantum state ψ is a purification of ρ_1. Thus, ρ_1 is obtained from ψ by tracing out the R system. The unitary operator U_i and the pure state φ_i provide a dilation of the quantum channel Λ_i, with $i \in \{1, 2, 3, 4\}$. The remaining quantum states ρ_2, ρ_3, ρ_4 are obtained by acting the unitary operations and tracing out the appropriate reference-environment systems. Picture adapted from [1].

Let the initial state γ_1 be the tensor product of ψ and the initial environmental states φ_1, φ_2 and φ_3. Now, we can define the process recursively. For any $i \in \{1, 2, 3\}$, the quantum state γ_{i+1} is obtained acting the local unitary operation U_i on the quantum state γ_i and then swapping its output systems. Figure 7.1 is a pictorial representation of the this pure process. Explicitly, we have the sequence of pure states

$$|\gamma_1\rangle = |\psi\rangle \otimes |\varphi_1\rangle \otimes |\varphi_2\rangle \otimes |\varphi_3\rangle \,, \tag{7.10}$$

$$|\gamma_2\rangle = (\mathbb{1}_R \otimes \mathsf{SWAP}_{S_2, E_1} \otimes \mathbb{1}_{F_2 \otimes F_3})(\mathbb{1}_R \otimes U_1 \otimes \mathbb{1}_{F_2 \otimes F_3}) |\gamma_1\rangle \,, \tag{7.11}$$

$$|\gamma_3\rangle = (\mathbb{1}_{R \otimes E_1} \otimes \mathsf{SWAP}_{S_3, E_2} \otimes \mathbb{1}_{F_2})(\mathbb{1}_{R \otimes E_1} \otimes U_2 \otimes \mathbb{1}_{F_3}) |\gamma_2\rangle \,, \tag{7.12}$$

$$|\gamma_4\rangle = (\mathbb{1}_{R \otimes E_1 \otimes E_2} \otimes \mathsf{SWAP}_{S_4, E_3})(\mathbb{1}_{R \otimes E_1 \otimes E_2} \otimes U_3) |\gamma_3\rangle \,. \tag{7.13}$$

Now, the coherent information terms involved in Eq. (7.8) can be computed with respect

to the purification of the quantum Markov process as

$$I_c(\rho_1; \Lambda_3 \circ \Lambda_2 \circ \Lambda_1) = H(S_4)_{\gamma_4} - H(R, S_4)_{\gamma_4}, \tag{7.14}$$

$$I_c(\Lambda_1(\rho_1); \Lambda_2) = H(S_3)_{\gamma_3} - H(R, E_1, S_3)_{\gamma_3}, \tag{7.15}$$

$$I_c(\rho_1; \Lambda_2 \circ \Lambda_1) = H(S_3)_{\gamma_3} - H(R, S_3)_{\gamma_3}, \tag{7.16}$$

$$I_c(\Lambda_1(\rho_1); \Lambda_3 \circ \Lambda_2) = H(S_4)_{\gamma_4} - H(R, E_1, S_4)_{\gamma_4}. \tag{7.17}$$

Once again, the purification of the quantum Markov process has been introduced in order to easy the computation and facilitate the derivation of the quantum Markov monogamy inequality. Furthermore, it holds the equality conditions

$$H(R, S_4)_{\gamma_4} = H(E_1, E_2, E_3)_{\gamma_4}, \tag{7.18}$$

$$H(R, E_1, S_3)_{\gamma_3} = H(E_2)_{\gamma_3} = H(E_2)_{\gamma_4}, \tag{7.19}$$

$$H(R, S_3)_{\gamma_3} = H(E_1, E_2)_{\gamma_3} = H(E_1, E_2)_{\gamma_4}, \tag{7.20}$$

$$H(R, E_1, S_4)_{\gamma_4} = H(E_2, E_3)_{\gamma_4}. \tag{7.21}$$

Now we are ready to check that Eq. (7.8) holds for every four-time-step quantum Markov process, that is, for every quantum state ρ_1 and quantum channels Λ_1, Λ_2 and Λ_3. Consider the marginal system $E_1 \otimes E_2 \otimes E_3$ with quantum state $\mathrm{tr}_{R,S_4}[\gamma_4]$. Then, strong subadditivity of quantum entropy implies that

$$H(E_1, E_2, E_3) + H(E_2) \leq H(E_1, E_2) + H(E_2, E_3). \tag{7.22}$$

Solving the above inequality for $I_c(\rho_1; \Lambda_3 \circ \Lambda_2 \circ \Lambda_1)$, $I_c(\Lambda_1(\rho_1); \Lambda_2)$, $I_c(\rho_1; \Lambda_2 \circ \Lambda_1)$ and $I_c(\Lambda_1(\rho_1); \Lambda_3 \circ \Lambda_2)$ with the assistance of Eqs. (7.18)-(7.21), we conclude the proof that the desired information inequality in Eq. (7.8) holds for every four-time-step quantum Markov process.

The information inequality in Eq. (7.8) can be formulated in terms of the conditional quantum mutual information defined in the following. Let ρ be a tripartite state in $L(A \otimes$

$B \otimes C$). Then, the mutual information of A and B conditional on C is given by [37]

$$I(A:B|C)_\rho = H(A,C) + H(B,C) - H(A,B,C) - H(C). \tag{7.23}$$

With this definition, the following statements are equivalent [1]:

(I) For each quantum state $\rho_1 \in L(S_1)$, and for each quantum channels $\Lambda_1\colon L(S_1) \to L(S_2)$, $\Lambda_2\colon L(S_2) \to L(S_3)$ and $\Lambda_3\colon L(S_3) \to L(S_4)$, it holds that

$$I_c(\rho_1, \Lambda_3 \circ \Lambda_2 \circ \Lambda_1) + I_c(\Lambda_1(\rho_1), \Lambda_2) \geq I_c(\rho_1, \Lambda_2 \circ \Lambda_1) + I_c(\Lambda_1(\rho_1), \Lambda_3 \circ \Lambda_2). \tag{7.24}$$

(II) For any tripartite quantum state $\sigma \in L(A \otimes B \otimes C)$, and for any quantum operation $\Lambda\colon L(B) \to L(D)$, it holds that the conditional quantum mutual information is monotonically decreasing under the action of the local operation Λ. That is,

$$I(A:B|C)_\sigma \geq I(A:D|C)_{(\mathrm{id}_A \otimes \Lambda \otimes \mathrm{id}_A)(\sigma)}. \tag{7.25}$$

The proof of (I) $\Leftrightarrow$ (II) is similar to the one for the two versions of quantum data processing theorem in chapter 6. For that, let $\psi \in L(R \otimes S_1)$ be a pure quantum state and $\rho \in L(R \otimes S_2)$ be an arbitrary quantum state. Recall that there is a quantum channel $\Lambda\colon L(S_1) \to L(S_2)$ such that $\rho = \mathrm{id}_R \otimes \Lambda(\psi)$ if $\mathrm{tr}_{S_1}[\psi] = \mathrm{tr}_{S_2}[\rho]$. We now follow the proof in Ref. [1].

The sentence (II) $\Rightarrow$ (I) is clearly true. Suppose that (II) is true. Take a purification of the quantum Markov process as in Eqs. (7.10)-(7.13). Then, the quantum Markov monogamy is given by

$$H(R, S_3)_{\gamma_3} - H(R, E_1, S_3)_{\gamma_3} \geq H(R, S_4)_{\gamma_4} - H(R, E_1, S_4)_{\gamma_4} \tag{7.26}$$

Adding the term $H(R, E_1)_{\gamma_3} - H(E_1)_{\gamma_3} = H(R, E_1)_{\gamma_4} - H(E_1)_{\gamma_4}$ to both sides of this inequality, we obtain

$$I(R:S_3|E_1)_{\gamma_3} \geq I(R:S_4|E_1)_{\gamma_4}. \tag{7.27}$$

Now, note that the subsystem $R \otimes S_4 \otimes E_1$ of γ_4 is obtained by the local action of a quantum channel $\Lambda_3\colon L(S_3) \to L(S_4)$ on the subsystem $R \otimes S_3 \otimes E_1$ of γ_3. Thus, the first

part of the proof is done.

In order to prove (I) $\Rightarrow$ (II), show that any tripartite quantum state $\rho \in L(A \otimes B \otimes C)$ can be written as

$$\rho = (\mathrm{id}_A \otimes \Lambda \otimes \mathrm{id}_C)(\mathrm{id}_A \otimes \mathsf{U})(\psi \otimes \varphi), \qquad (7.28)$$

where ψ is a state of a bipartite system $A \otimes D$, φ is a pure bipartite state of a system E, $\mathsf{U} \colon L(D \otimes E) \to L(F \otimes C)$ is a unitary quantum channel, and $\Lambda \colon L(F) \to L(B)$ is a quantum channel. That is, every ρ can be written as a quantum state with the same form as in Fig. 7.1. This is done as follows.

Let τ be the tripartite quantum state obtained by swapping the systems B and C of ρ. Then, define the marginal state with respect to the bipartite system $A \otimes C$, that is, $\omega = \mathrm{tr}_B[\tau]$. By the same reasoning as before, we know there are a pure bipartite state $\psi \in L(A \otimes D)$ and a quantum channel $\Omega := L(D) \to L(C)$ for which $\omega = (\mathrm{id}_A \otimes \Omega)(\psi)$. Let $U \colon D \otimes E \to C \otimes F$ be a dilation of Ω, such that for some pure state φ of the system E we have $\Omega(\bullet) = \mathrm{tr}_F[U(\bullet \otimes \varphi)U^\dagger]$.

Then, define the unitary quantum channel $\widetilde{\mathsf{U}}(\bullet) = V(\bullet \otimes \varphi)V^\dagger$, where V is the unitary operator obtained by the action of U followed by the swap operation. That is,

$$V = \mathsf{SWAP} \circ U. \qquad (7.29)$$

Define the pure tripartite quantum state $\eta = (\mathrm{id}_A \otimes \widetilde{\mathsf{U}})(\psi \otimes \varphi)$. The state η is a purification of ω. Then, we have $\mathrm{tr}_F[\eta] = \mathrm{tr}_B[\tau]$. Furthermore, there is a quantum channel $\Lambda \colon L(F) \to L(B)$ for which $\tau = (\mathrm{id}_A \otimes \mathrm{id}_C \otimes \Lambda)(\eta)$. Moreover, consider swapping the systems C and B of τ to recover

$$\rho = (\mathrm{id}_A \otimes \Lambda \otimes \mathrm{id}_C)(\mathrm{id}_A \otimes \mathsf{U})(\psi \otimes \varphi), \qquad (7.30)$$

with $\mathsf{U}(\bullet) = U(\bullet)U^\dagger$.

7.2 Witnessing non-Markov behaviour with quantum information inequalities

Here, we consider how quantum information inequalities can be used to witness quantum non-Markovianity. So far, we have properly defined what is meant by quantum Markov processes. Nevertheless, no mention has been made on what is a quantum non-Markov process. So we now examine the distinction between Markov and non-Markov processes.

For instance, let us start with three-time-step quantum Markov processes. Every quantum process $\rho_1 \xrightarrow{\Lambda_1} \rho_2 \xrightarrow{\Lambda_2} \rho_3$ has a system-environment representation provided by the dilation of the quantum operations Λ_1 and Λ_2. Let $U_i \colon L(S_i) \otimes L(F_i) \to L(S_{i+1}) \otimes L(E_i)$ be the unitary representation of the channel Λ_i when the environment F_i is in the state φ_i, with $i \in \{1, 2\}$.

In each step of a quantum Markov process, the system is found uncorrelated with the environment. So in the intermediary stage the environment system E_1 is discarded and replaced with a system F_2 in a fixed quantum state φ_2. That is represented by the replacement channel $\Delta \colon L(E_1) \to L(F_2)$ such that

$$\Delta(\omega) = \mathrm{tr}[\omega]\varphi_2, \tag{7.31}$$

for every $\omega \in L(E_1)$.

Furthermore, the sequence of system-environment states $\{\pi_1, \pi_2, \pi_3\}$ with respect to three-time-step quantum Markov process can always be expressed as

$$\pi_1 = \rho_1 \otimes \varphi_1, \tag{7.32}$$

$$\pi_2 = U_1(\pi_1), \tag{7.33}$$

$$\pi_3 = U_2 \circ (\mathrm{id}_{S_2} \otimes \Delta)(\pi_2). \tag{7.34}$$

The quantum process $\rho_1 \xrightarrow{\Lambda_1} \rho_2 \xrightarrow{\Lambda_2} \rho_3$ is then equivalently defined as the reduced dynamics of $\pi_1 \to \pi_2 \to \pi_3$ in Eqs. (7.32), (7.33) and (7.34). In fact, this is usually adopted as a definition for quantum Markov processes in literature [24]. The top panel of Figure 7.2

presents a diagrammatic system-environment representation of quantum Markov processes.

On the other hand, quantum non-Markov processes allow for non-trivial correlations through system-environment evolution. That is, our initial system-environment bipartite state $\sigma \in L(S_1 \in F_1)$ is possibly not a product state of the form in (7.32). Importantly, instead of resetting the environmental system in a fixed quantum state φ_i every step, non-Markov processes let the system F_{i+1} be fed with the previous environmental system E_i. So our replacement channel Δ in a three-time-step Markov process is substituted by the identity channel id.

The sequence of system-environment states $\{\pi_1, \pi_2, \pi_3\}$ in a non-Markov process is thus of the type

$$\pi_1 \in L(S_1 \otimes F_1), \tag{7.35}$$

$$\pi_2 = U_1(\pi_1), \tag{7.36}$$

$$\pi_3 = (U_2 \circ U_1)(\pi_2), \tag{7.37}$$

where now the environmental systems E_1 and F_2 are identical copies (isomorphic). The bottom panel of Figure 7.2 diagrammatically represents the system-environment quantum non-Markov process in Eqs. (7.35), (7.36) and (7.37).

Now we are ready to define witnesses on quantum non-Markovianity relying on the quantum information inequalities developed so far. Namely, the quantum data processing inequality in Eq. (6.5) and the quantum Markov monogamy inequality in Eq. (7.8). Let $\rho_1 \xrightarrow{\Lambda_1} \rho_2 \xrightarrow{\Lambda_2} \rho_3 \xrightarrow{\Lambda_3} \rho_4$ be any quantum Markov process. Then, it follows the positive

Figure 7.2: **System-environment representation of quantum Markov and quantum non-Markov processes.** The top panel shows a pictorial representation of quantum Markov processes. In any time-step the environmental system is replaced by a new system in a particular fixed state. Therefore, the system and environment are uncorrelated in every stage of the process. The bottom panel considers the distinct situation of quantum non-Markov processes. In fact, any system-environment process violating the conditions in the top panel is cast as a non-Markov process. Here, our example shows how this may be done allowing for environmental memories through evolution.

semi-definite quantities

$$\text{DP}_1^{(q)} := I_c(\rho_1; \Lambda_1) - I_c(\rho_1; \Lambda_2 \circ \Lambda_1), \tag{7.38}$$

$$\text{DP}_2^{(q)} := I_c(\rho_1; \Lambda_1) - I_c(\rho_1; \Lambda_3 \circ \Lambda_2 \circ \Lambda_1), \tag{7.39}$$

$$\text{DP}_3^{(q)} := I_c(\rho_1; \Lambda_2 \circ \Lambda_1) - I_c(\rho_1; \Lambda_3 \circ \Lambda_2 \circ \Lambda_1), \tag{7.40}$$

$$\text{DP}_4^{(q)} := I_c(\rho_2; \Lambda_2) - I_c(\rho_2; \Lambda_3 \circ \Lambda_2), \tag{7.41}$$

$$\text{M4}^{(q)} := I_c(\rho_1; \Lambda_3 \circ \Lambda_2 \circ \Lambda_1) + I_c(\rho_2; \Lambda_2)$$
$$- I_c(\rho_1; \Lambda_2 \circ \Lambda_1) - I_c(\rho_2; \Lambda_3 \circ \Lambda_2). \tag{7.42}$$

Let $\psi \in L(R \otimes S_1)$ be a purification of the initial state $\rho_1 \in L(S_1)$. Then, define a dilation for Λ_1 with unitary channel $U_1 \colon L(S_1 \otimes F_1) \to L(S_2 \otimes E_1)$ and pure quantum state $\varphi_1 \in F_1$. Now we can represent the quantum Markov process with the sequence of reference-system-environment quantum states

$$\gamma_1 = \psi \otimes \varphi_1, \tag{7.43}$$

$$\gamma_2 = \text{id}_R \otimes U_1(\gamma_1), \tag{7.44}$$

$$\gamma_3 = \text{id}_R \otimes \Lambda_2 \otimes \text{id}_{E_1}(\gamma_2), \tag{7.45}$$

$$\gamma_4 = \text{id}_R \otimes \Lambda_3 \otimes \text{id}_{E_1}(\gamma_3), \tag{7.46}$$

Distinctly, quantum non-Markov processes have a non-trivial system-environment interaction through evolution. For instance, consider the following example of qubit reference, system and environment systems with canonical basis $\{|0\rangle, |1\rangle\}$. Let the initial reference-system-environment be in a pure maximally entangled state

$$|\gamma_1\rangle = \frac{1}{\sqrt{3}}(|1,0,0\rangle + |0,1,0\rangle + |0,0,1\rangle). \tag{7.47}$$

70

Then, define the sequence of tripartite reference-system-environment states

$$\gamma_2 = \mathrm{id}_R \otimes \mathrm{U}^{(\lambda)}(\gamma_1), \tag{7.48}$$

$$\gamma_3 = \mathrm{id}_R \otimes \mathrm{U}^{(\lambda)}(\gamma_2), \tag{7.49}$$

$$\gamma_4 = \mathrm{id}_R \otimes \mathrm{U}^{(\lambda)}(\gamma_1), \tag{7.50}$$

where $\mathrm{U}^{(\lambda)} \colon L(S \otimes E) \to L(S \otimes E)$ a single Kraus operator defined in Eq. (6.32).

Figure 7.3 depicts the difference between quantum Markov processes and the example of non-Markov process in Eqs. (7.47)-(7.50). In terms of this tripartite reference-system-environment representation, the witnesses in Eqs. (7.38)-(7.42) take the form

$$\mathrm{DP}_1^{(q)} = [H(S) - H(R,S)]_{\gamma_2} - [H(S) - H(R,S)]_{\gamma_3}, \tag{7.51}$$

$$\mathrm{DP}_2^{(q)} = [H(S) - H(R,S)]_{\gamma_2} - [H(S) - H(R,S)]_{\gamma_4}, \tag{7.52}$$

$$\mathrm{DP}_3^{(q)} = [H(S) - H(R,S)]_{\gamma_3} - [H(S) - H(R,S)]_{\gamma_4}, \tag{7.53}$$

$$\mathrm{DP}_4^{(q)} = [H(S) - H(R,S,E)]_{\gamma_3} - [H(S) - H(R,S,E)]_{\gamma_4}, \tag{7.54}$$

$$\mathrm{M4}^{(q)} = [H(R,S,E) - H(R,S)]_{\gamma_4} - [H(R,S) - H(R,S,E)]_{\gamma_3}. \tag{7.55}$$

Figure 7.4 shows the witnesses of quantum non-Markovianity above as a function of λ. The data processing conditions $\mathrm{DP}_1^{(q)}$ and $\mathrm{DP}_2^{(q)}$ are not violated for any value of λ. On the other hand, the data processing conditions $\mathrm{DP}_3^{(q)}$ and $\mathrm{DP}_4^{(q)}$ are violated for $0.15 \leq \lambda \leq 1$. For the region $0.85 \leq \lambda \leq 1$ the conditions $\mathrm{DP}_3^{(q)}$ and $\mathrm{DP}_4^{(q)}$ are the only information inequalities witnessing the non-Markovianity of the process. The quantum Markov monogamy inequality is violated for $0 \leq \lambda \leq 0.85$. The quantity $\mathrm{M4}^{(q)}$ is the only information inequality witnessing the non-Markov process for $0 \leq \lambda \leq 0.15$. Moreover, for any value of λ there is a quantum information inequality witnessing the non-Markovianity.

It was claimed in chapter 6 that the information inequality $I_c(\Lambda_1(\rho); \Lambda_2) \geq I_c(\Lambda_1(\rho); \Lambda_2)$ hold for some quantum states ρ and quantum channels Λ_1 and Λ_2. Nevertheless, it was not provided a proof of its validity in general. Now, we show that information inequalities of this type are not violated for the non-Markov process considered here.

In this case, four-time-step Markov processes $\rho_1 \xrightarrow{\Lambda_1} \rho_2 \xrightarrow{\Lambda_2} \rho_3 \xrightarrow{\Lambda_3} \rho_4$ are expected to
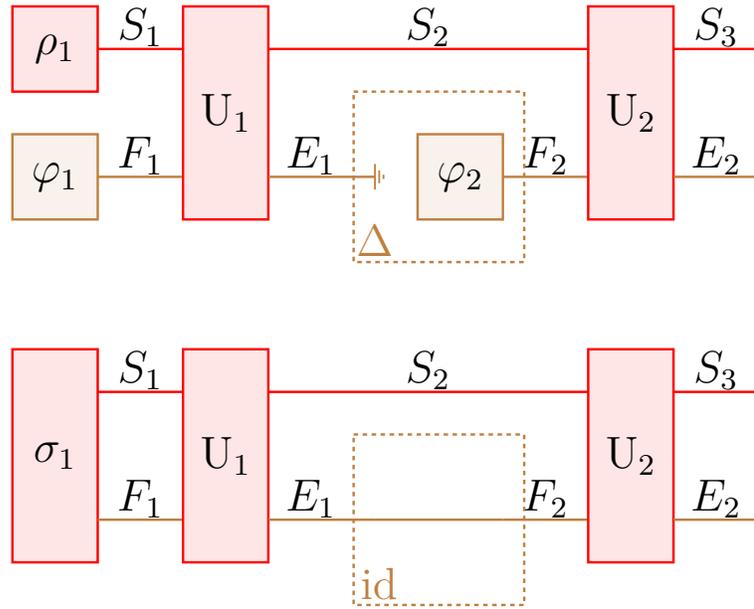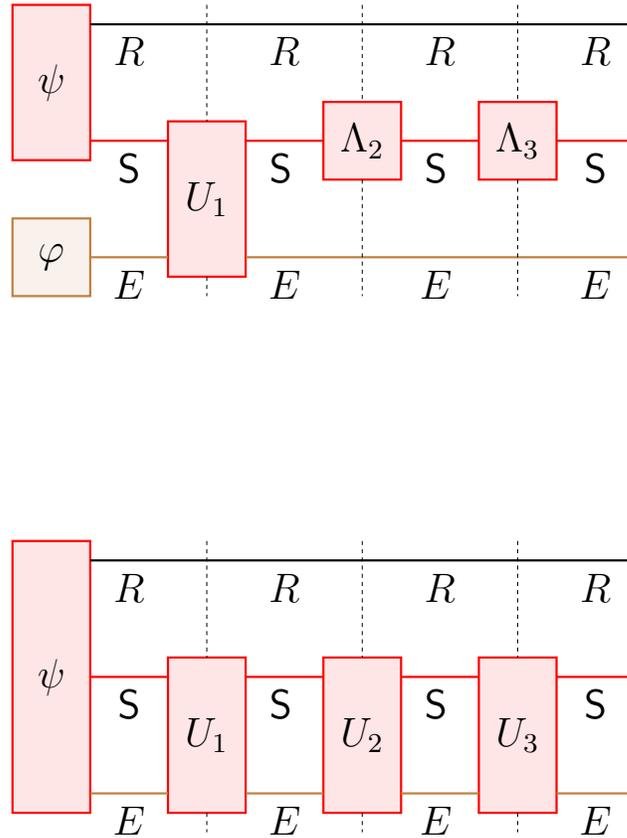
Figure 7.3: **Reference-system-environment representation of quantum Markov and non-Markov processes.** Top panel pictorially represents reference-system-environment quantum Markov processes. The reference system R is a purification system for S. The initial system E works as an environment for the first evolution, thus being uncorrelated from the $R \otimes S$-part. Furthermore, the initial system-environment can go thought arbitrary unitary evolution U. The subsequent evolution maps Λ_2 and Λ_3 can be any quantum channels acting locally on the system-part. Bottom panel shows an example of reference-system-environment process violating this conditions, thus being called quantum non-Markov process. There, the initial system $R \otimes S$ is entangled with the initial environment E. Then, system-environment follow successive global unitary evolutions U_1, U_2 and U_3.
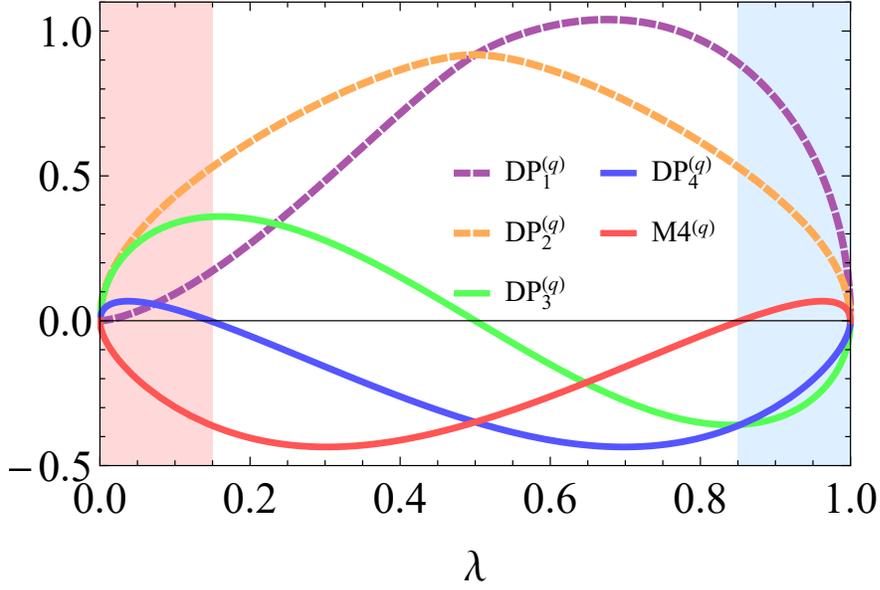
Figure 7.4: **Quantum Markov monogamy violation.** Quantum Markov monogamy is the only information inequality being violated in the region $0 \leq \lambda \leq 0.15$ (shaded in red). Nevertheless, the converse situation is also possible. In the region $0.85 \leq \lambda \leq 1$ (shaded in blue), the quantum Markov monogamy inequality is not violated, while the data processing conditions in (7.54) and (7.55) witness the non-Markov behavior of the process.

define positive semi-definite quantities

$$\mathrm{DP}_5^{(q)} = I_c(\Lambda_1(\rho); \Lambda_2) - I_c(\rho; \Lambda_2 \circ \Lambda_1), \tag{7.56}$$

$$\mathrm{DP}_6^{(q)} = I_c(\Lambda_1(\rho); \Lambda_2) - I_c(\rho; \Lambda_3 \circ \Lambda_2 \circ \Lambda_1) \tag{7.57}$$

$$\mathrm{DP}_7^{(q)} = I_c(\Lambda_1(\rho); \Lambda_3 \circ \Lambda_2) - I_c(\rho; \Lambda_3 \circ \Lambda_2 \circ \Lambda_1), \tag{7.58}$$

$$\mathrm{DP}_8^{(q)} = I_c(\Lambda_2 \circ \Lambda_1(\rho); \Lambda_3) - I_c(\rho; \Lambda_3 \circ \Lambda_2 \circ \Lambda_1), \tag{7.59}$$

$$\mathrm{DP}_9^{(q)} = I_c(\Lambda_2 \circ \Lambda_1(\rho); \Lambda_3) - I_c(\Lambda_1(\rho); \Lambda_3 \circ \Lambda_2). \tag{7.60}$$

Note that $\mathrm{DP}_8^{(q)}$ and $\mathrm{DP}_9^{(q)}$ also depend upon the environmental system of the quantum channel Λ_2. Therefore, those information inequalities cannot be used in our example. In terms of the tripartite reference-system-environment representation, the information inequalities
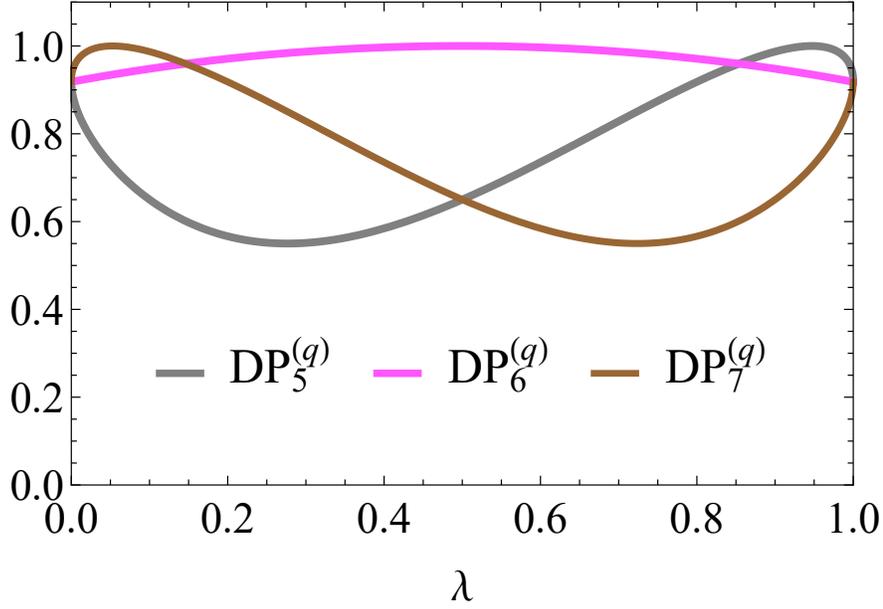
Figure 7.5: **Non-violation of** $\mathrm{DP}_5^{(q)}$, $\mathrm{DP}_6^{(q)}$ **and** $\mathrm{DP}_7^{(q)}$ **for the non-Markov process considered in Eqs. (7.47,7.48,7.49,7.50).** The quantities defined in Eqs. (7.61,7.62,7.63) are positive for the quantum non-Markov processes represented in the bottom panel of Figure 7.3. Picture addapted from [1].

with respect to Eqs. (7.56), (7.57) and (7.58) are written as

$$\mathrm{DP}_5^{(q)} = [H(R,S) - H(R,S,E)]_{\gamma_3}, \tag{7.61}$$

$$\mathrm{DP}_6^{(q)} = [H(S) - H(R,S,E)]_{\gamma_3} - [H(S) - H(R,S)]_{\gamma_4}, \tag{7.62}$$

$$\mathrm{DP}_7^{(q)} = [H(S) - H(R,S,E)]_{\gamma_4} - [H(S) - H(R,S)]_{\gamma_4}. \tag{7.63}$$

Figure 7.5 reveals that the information inequalities $\mathrm{DP}_5^{(q)}, \mathrm{DP}_6^{(q)}, \mathrm{DP}_7^{(q)} \geq 0$ are not violated for the example of non-Markov processes examined here.[1] Therefore, quantum Markov monogamy is the only information inequality witnessing non-Markovianity in the regime $0 \leq \lambda \leq 0.15$ under the tripartite reference-system-environment representation in Figure 7.3. Nevertheless, it is not possible to witness non-Markovianity with the quantum Markov monogamy inequality in the region $0.85 \leq \lambda \leq 1$, case for which only the quantum data processing inequalities are successively violated.

---

[1]Recall that the validity of the inequalities in Eqs. (7.56)-(7.58) was not proved for arbitrary quantum Markov processes. Here, we claim that even if they were valid in general, they would not witness the non-Markov processes considered in this example.

7.3 Six-time-step quantum Markov processes

The information inequalities become more involved as we increase the number of time-steps in the processes. Therefore, from now on we adopt a convention in order to simplify the definition of information inequalities in terms of coherent information. For any Markov process $\rho_1 \xrightarrow{\Lambda_1} \rho_2 \xrightarrow{\Lambda_2} \cdots \xrightarrow{\Lambda_{n-1}} \rho_n$ we use the shorthand notation

$$I_c(\rho_r : \rho_s) := I_c(\Lambda_{r-1} \circ \cdots \circ \Lambda_1(\rho_1); \Lambda_{s-1} \circ \cdots \circ \Lambda_r), \tag{7.64}$$

for $1 \leq r < s \leq n$.

Let us define an arbitrary six-time-step quantum Markov process $\rho_1 \xrightarrow{\Lambda_1} \rho_2 \xrightarrow{\Lambda_2} \cdots \xrightarrow{\Lambda_5} \rho_6$. So let $\rho \in L(S_1)$ be any quantum state. For every quantum channels $\Lambda_i \colon L(S_i) \to L(S_{i+1})$ set $\rho_{i+1} = \Lambda_i(\rho_i)$, with $i \in \{1, \ldots, 5\}$. Then, it holds that [1]

$$I_c(\rho_1 : \rho_6) + I_c(\rho_2 : \rho_5) + I_c(\rho_3 : \rho_4) \geq I_c(\rho_1 : \rho_4) + I_c(\rho_2 : \rho_6) + I_c(\rho_3 : \rho_5); \tag{7.65}$$

$$I_c(\rho_1 : \rho_6) + I_c(\rho_2 : \rho_5) + I_c(\rho_3 : \rho_4) \geq I_c(\rho_1 : \rho_5) + I_c(\rho_2 : \rho_4) + I_c(\rho_3 : \rho_6). \tag{7.66}$$

In order to derive the quantum Markov monogamy inequalities above, let the quantum channels Λ_i have isometric representations $V \colon S_i \to S_{i+1} \otimes E_i$, with $i \in \{1, \ldots, 5\}$. That is, $\Lambda_i(\rho) = \operatorname{tr}_{E_i}[V_i \rho V_i^\dagger]$ for every $\rho \in L(S_i)$. The strategy is completely analogous to before, and we only provide an sketch of the proof here.

Consider any purification of the quantum Markov process as in Figure 7.6. Then, write the relevant coherent information quantities in terms of entropies of the environmental systems E_i, where $i \in \{1, 2, 3, 4, 5\}$. The proof of quantum Markov monogamy inequalities follows adding strong subadditivity inequalities relating the environmental systems only. The interested reader is invited to check this on a more detailed version. Here, we only provide the strong subadditivity inequalities in question.

In order to prove Eq. (7.65), add the strong subadditivity inequalities

$$I(E_1 : E_5 | E_2, E_3, E_4) \geq 0, \tag{7.67}$$

$$I(E_1, E_2 : E_4 | E_3) \geq 0. \tag{7.68}$$
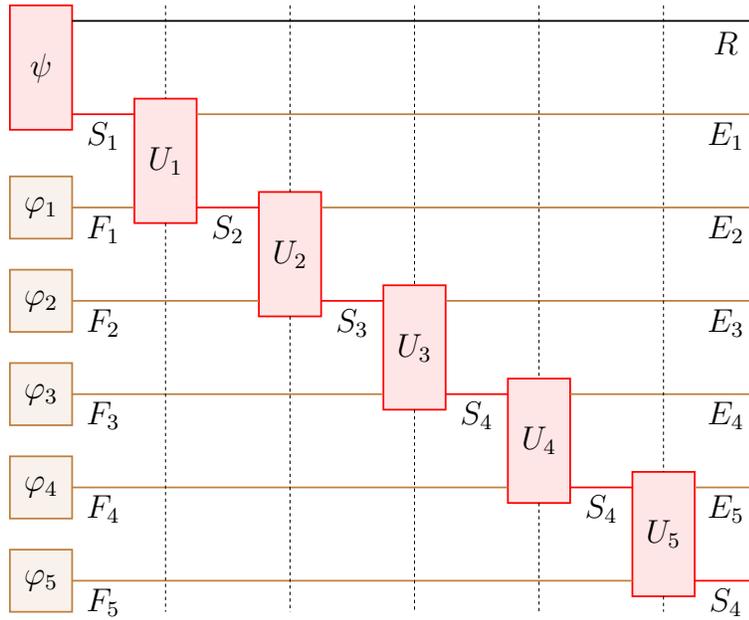
Figure 7.6: **Purification of six-time-step quantum Markov processes.** The purification of every process $\rho_1 \xrightarrow{\Lambda_1} \cdots \xrightarrow{\Lambda_5} \rho_6$ is defined as follows. The quantum state ψ is a purification of ρ_1. The unitary operator $U_i \colon S_i \otimes F_i \to S_{i+1} \otimes E_i$ and the pure state φ_i of F_i specify a unitary representation of the quantum channel Λ_i, with $i \in \{1, 2, 3, 4, 5\}$. An isometric representation is defined with $V_i(\bullet) = U_i(\bullet \otimes \varphi_i)$. The purification process is defined acting the unitary operator U_i on the appropriate system-environment part, and then swapping the resulting output. The marginal quantum states $\{\rho_1, \ldots, \rho_6\}$ are obtained tracing out the appropriate reference-environment subsystems.

To prove Eq. (7.66), consider

$$I(E_1 : E_5|E_2, E_3, E_4) \geq 0, \tag{7.69}$$

$$I(E_1 : E_4, E_5|E_3) \geq 0. \tag{7.70}$$

The quantum Markov monogamy inequalities for six-time-step processes also have an equivalent version in terms of conditional quantum mutual information. The following sentences are equivalent forms of quantum Markov monogamy relations [1]:

(I) Equations. (7.65) and (7.66) hold for any six-time-step quantum Markov process $\rho_1 \xrightarrow{\Lambda_1} \rho_2 \xrightarrow{\Lambda_2} \cdots \xrightarrow{\Lambda_5} \rho_6$.

(II) For any quantum state $\rho \in L(R \otimes E_1 \otimes E_2 \otimes S_4)$, and for any quantum channels $\Lambda_4 \colon L(S_4) \to L(S_5)$ and $\Lambda_5 \colon L(S_5) \to L(S_6)$, it holds that

$$I(R : S_4|E_1, E_2) \geq I(R, E_1 : S_5|E_2) + I(R : S_6|E_1), \tag{7.71}$$

$$I(R, E_1 : S_4|E_2) + I(R : S_5|E_1) \geq I(R : S_6|E_1, E_2). \tag{7.72}$$

The proof of (I) $\Leftrightarrow$ (II) follows the same steps as in the case of four-time-step quantum Markov processes. Therefore, we only discuss a sketch of the proof here. The part (II) $\Rightarrow$ (I) is trivial. The challenge in part (I) $\Rightarrow$ (II) is to show that any quantum state $\rho \in L(R \otimes E_1 \otimes E_2 \otimes S_4)$ can be written as

$$\rho = (\mathrm{id}_{R \otimes E_1 \otimes E_2} \otimes \Lambda)(\mathrm{id}_{R \otimes E_1} \otimes \mathrm{U}_2)(\mathrm{id}_R \otimes \mathrm{U}_1 \otimes \mathrm{id}_{E_2})(\psi \otimes \varphi_1 \otimes \varphi_2), \tag{7.73}$$

with pure quantum states $\psi \in L(R \otimes S_1)$, $\varphi_1 \in L(F_1)$, $\varphi_2 \in L(F_2)$, unitary quantum channels $\mathrm{U}_1 \colon L(S_1 \otimes F_1) \to L(E_1 \otimes S_2)$, $\mathrm{U}_2 \colon L(S_2 \otimes F_2) \to L(E_2 \otimes S_3)$, and a quantum channel $\Lambda \colon L(S_3) \to L(S_4)$. The interested reader is urged to check all the details in this derivation.

7.4 Eight-time-step quantum Markov processes

Now we move on to consider the quantum Markov monogamy inequalities for eight-time-step processes. This will be our last step towards extending the conjecture on Markov monogamy inequalities from classical to quantum processes.

We start by defining an arbitrary eight-time-step quantum Markov process. Let $\rho_1 \in L(S_1)$ be the initial quantum state. Then, define quantum channels $\Lambda_i \colon L(S_i) \to L(S_{i+1})$ from which follows the quantum states $\rho_{i+1} = \Lambda_i(\rho_i)$, with $i \in \{1, \dots, 7\}$. It holds that [1]

$$I_c(\rho_1 : \rho_8) + I_c(\rho_2 : \rho_7) + I_c(\rho_3 : \rho_6) + I_c(\rho_4 : \rho_5) \geq$$
$$I_c(\rho_1 : \rho_5) + I_c(\rho_2 : \rho_8) + I_c(\rho_3 : \rho_7) + I_c(\rho_4 : \rho_6), \quad (7.74)$$

$$I_c(\rho_1 : \rho_8) + I_c(\rho_2 : \rho_7) + I_c(\rho_3 : \rho_6) + I_c(\rho_4 : \rho_5) \geq$$
$$I_c(\rho_1 : \rho_7) + I_c(\rho_2 : \rho_5) + I_c(\rho_3 : \rho_8) + I_c(\rho_4 : \rho_6), \quad (7.75)$$

$$I_c(\rho_1 : \rho_8) + I_c(\rho_2 : \rho_7) + I_c(\rho_3 : \rho_6) + I_c(\rho_4 : \rho_5) \geq$$
$$I_c(\rho_1 : \rho_6) + I_c(\rho_2 : \rho_8) + I_c(\rho_3 : \rho_5) + I_c(\rho_4 : \rho_7), \quad (7.76)$$

$$I_c(\rho_1 : \rho_8) + I_c(\rho_2 : \rho_7) + I_c(\rho_3 : \rho_6) + I_c(\rho_4 : \rho_5) \geq$$
$$I_c(\rho_1 : \rho_5) + I_c(\rho_2 : \rho_6) + I_c(\rho_3 : \rho_8) + I_c(\rho_4 : \rho_7), \quad (7.77)$$

$$I_c(\rho_1 : \rho_8) + I_c(\rho_2 : \rho_7) + I_c(\rho_3 : \rho_6) + I_c(\rho_4 : \rho_5) \geq$$
$$I_c(\rho_1 : \rho_7) + I_c(\rho_2 : \rho_6) + I_c(\rho_3 : \rho_5) + I_c(\rho_4 : \rho_8), \quad (7.78)$$

$$I_c(\rho_1 : \rho_8) + I_c(\rho_2 : \rho_7) + I_c(\rho_3 : \rho_6) + I_c(\rho_4 : \rho_5) \geq$$
$$I_c(\rho_1 : \rho_6) + I_c(\rho_2 : \rho_5) + I_c(\rho_3 : \rho_7) + I_c(\rho_4 : \rho_8), \quad (7.79)$$

$$I_c(\rho_1 : \rho_8) + I_c(\rho_2 : \rho_7) + I_c(\rho_3 : \rho_6) + I_c(\rho_4 : \rho_5) \geq$$
$$I_c(\rho_1 : \rho_5) + I_c(\rho_2 : \rho_6) + I_c(\rho_3 : \rho_7) + I_c(\rho_4 : \rho_8). \quad (7.80)$$

In order to check the validity of the quantum Markov monogamy inequalities above, we need to define a purification for the process as in Figure 7.7. Once again, the detailed proof of the information inequalities above is omitted. Nevertheless, it follows the strong subadditivity inequalities adding up to the referred quantum Markov monogamy inequalities.

In order to prove Eq. (7.74), add

$$I(E_1 : E_7 | E_2, E_3, E_4, E_5, E_6) \geq 0, \quad (7.81)$$
$$I(E_1, E_2 : E_6 | E_3, E_4, E_5) \geq 0, \quad (7.82)$$
$$I(E_1, E_2, E_3 : E_5 | E_4) \geq 0. \quad (7.83)$$

To prove Eq. (7.75), add

$$I(E_1 : E_7 | E_2, E_3, E_4, E_5, E_6) \geq 0, \quad (7.84)$$
$$I(E_2 : E_6, E_7 | E_3, E_4, E_5) \geq 0, \quad (7.85)$$
$$I(E_2, E_3 : E_5 | E_4) \geq 0. \quad (7.86)$$

To prove Eq. (7.76), add

$$I(E_1 : E_7 | E_2, E_3, E_4, E_5, E_6) \geq 0, \quad (7.87)$$
$$I(E_1, E_2 : E_6 | E_3, E_4, E_5) \geq 0, \quad (7.88)$$
$$I(E_3 : E_5, E_6 | E_4) \geq 0. \quad (7.89)$$
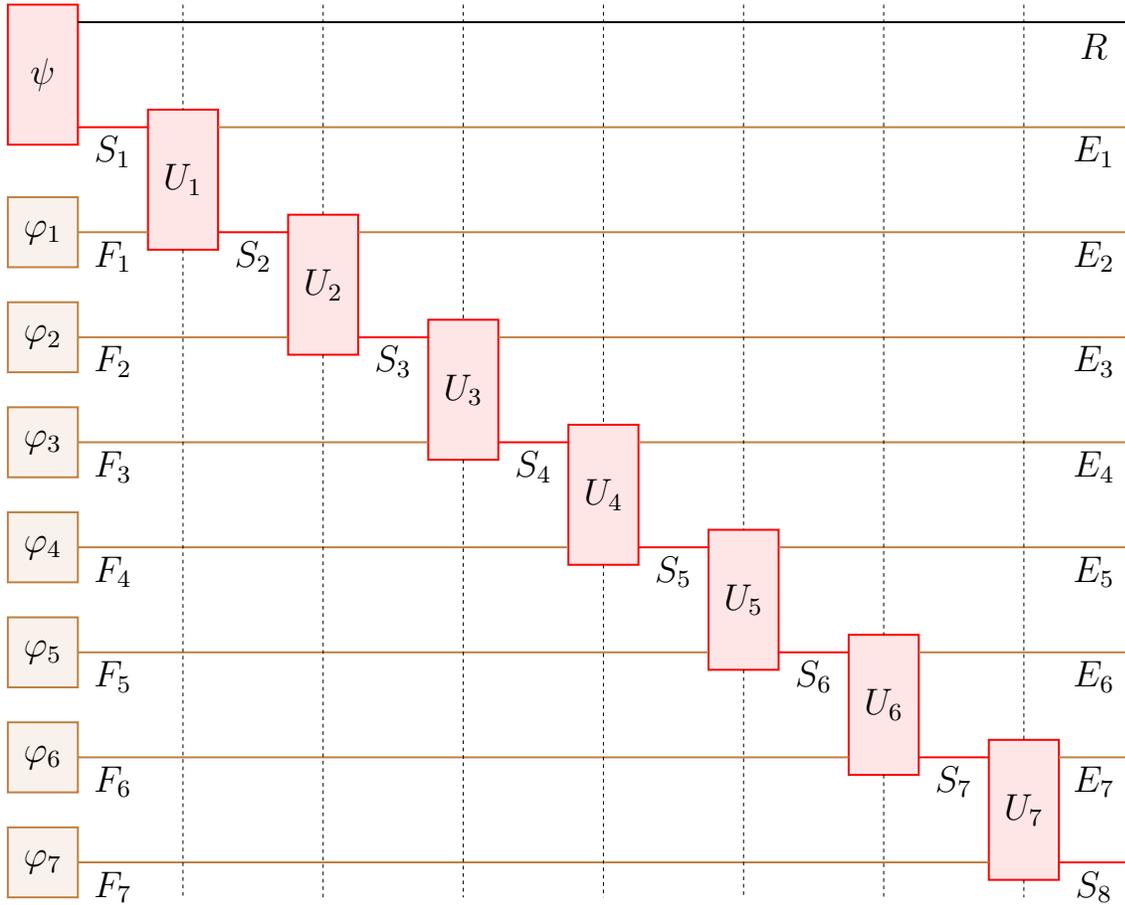
Figure 7.7: **Purification of eight-time-step quantum Markov processes.** The purification of any process $\rho_1 \xrightarrow{\Lambda_1} \cdots \xrightarrow{\Lambda_7} \rho_8$ is given with a purification for the initial system S_1 and unitary representations of the quantum channels $\Lambda_1, \ldots, \Lambda_8$. In the picture, the quantum state ψ is a purification of ρ_1. The unitary operator $U_i \colon S_i \otimes F_i \to S_{i+1} \otimes E_i$ and the pure state φ_i of F_i refers to a dilation of the quantum channel Λ_i, with $i \in \{1, \ldots, 7\}$. Note that the system-environment subsystem is swapped after the action of each unitary operation.

To prove Eq. (7.77), add

$$I(E_1 : E_7|E_2, E_3, E_4, E_5, E_6) \geq 0, \qquad (7.90)$$

$$I(E_2 : E_6, E_7|E_3, E_4, E_5) \geq 0, \qquad (7.91)$$

$$I(E_1, E_2, E_3 : E_5, E_6|E_4) \geq 0. \qquad (7.92)$$

To prove Eq. (7.78), add

$$I(E_1 : E_7|E_2, E_3, E_4, E_5, E_6) \geq 0, \qquad (7.93)$$

$$I(E_2 : E_6, E_7|E_3, E_4, E_5) \geq 0, \qquad (7.94)$$

$$I(E_3 : E_5, E_6, E_7|E_4) \geq 0. \qquad (7.95)$$

To prove Eq. (7.79) add

$$I(E_1 : E_7|E_2, E_3, E_4, E_5, E_6) \geq 0, \qquad (7.96)$$

$$I(E_1, E_2 : E_6|E_3, E_4, E_5) \geq 0, \qquad (7.97)$$

$$I(E_2, E_3 : E_5, E_6, E_7|E_4) \geq 0. \qquad (7.98)$$

To prove Eq. (7.80), add

$$I(E_1 : E_7|E_2, E_3, E_4, E_5, E_6) \geq 0, \qquad (7.99)$$

$$I(E_1, E_2, E_3 : E_5, E_6|E_4) \geq 0, \qquad (7.100)$$

$$I(E_2 : E_6, E_7|E_3, E_4, E_5) \geq 0, \qquad (7.101)$$

$$I(E_3 : E_7|E_4, E_5, E_6) \geq 0. \qquad (7.102)$$

7.5   Conjecture on quantum Markov monogamy inequalities

Markov monogamy inequalities hold for quantum Markov processes $\rho_1 \xrightarrow{\Lambda_1} \rho_2 \xrightarrow{\Lambda_2} \rho_3 \xrightarrow{\Lambda_3}$ $\ldots \xrightarrow{\Lambda_{n-1}} \rho_n$, with at least $n = 4, 6, 8$. In the previous sections of this chapter we have gone

through a detailed examination of the quantum Markov inequalities for these cases. The proof for each one of them has been addressed.

The ultimate goal of this study would be deriving the quantum Markov monogamy inequalities for general quantum Markov processes, that is, for arbitrary even integer n greater than or equal to 4. The natural approach could be addressing the proof with inductive reasoning. Nevertheless, the Markov monogamy inequalities for processes with $(n+2)$-time-steps do not directly follow from the case of n-time-step processes. This is similar to the case of classical processes.

Furthermore, we now conjecture on the validity of Markov monogamy inequalities for general $2m$-time-step Markov quantum processes (with integer $m \geq 2$)

$$\rho_m \xrightarrow{\Gamma_{m-1}} \cdots \xrightarrow{\Gamma_1} \rho_1 \xrightarrow{\Lambda} \sigma_1 \xrightarrow{\Omega_1} \cdots \xrightarrow{\Omega_{m-1}} \sigma_m. \tag{7.103}$$

Let $\rho_1 \in L(S_1^{in})$ and $\sigma_1 \in L(S_1^{out})$ be the input and output states of a quantum channel Λ. Then, define the next pre- and post-processed quantum states $\rho_{i+1} = \Gamma_i(\rho_i)$ and $\sigma_{i+1} = \Omega_i(\sigma_i)$ with encoding and decoding quantum operations $\Gamma_i \colon L(S_{i+1}^{in}) \to L(S_i^{in})$ and $\Omega_i \colon L(S_i^{out}) \to L(S_{i+1}^{out})$ with $i \in \{1, \ldots, m-1\}$, respectively.

Now, for any bijective function

$$f \colon \{1, \ldots, m\} \to \{1, \ldots, m\}, \tag{7.104}$$

it is expect to hold the quantum Markov monogamy inequality

$$\sum_{i=1}^{n} I_c(\rho_i : \sigma_i) \geq \sum_{i=1}^{n} I_c(\rho_i : \sigma_{f(i)}). \tag{7.105}$$

The proof of the information inequality above remains as a major open problem related to this thesis. Note that we have provided the proof of its validity for $m \in \{2, 3, 4\}$. Now, we move to the last chapter in this thesis, extending the information inequalities considered so far to the interventional approach.

Chapter 8

Interventional quantum information inequalities

In chapter 6 and chapter 7 we have defined quantum data processing inequalities and quantum Markov monogamy inequalities, respectively. Those appear as limits on what is possible to achieve under the model of quantum Markov processes. Quantum Markovianity has been introduced as a particular system-environment dynamics for which the environment is reset to a fixed quantum state every stage of the process (cf. Figure 7.2). Moreover, in Eq. (7.1) a Markov process is defined fixing a initial quantum state which goes through the action of fixed quantum channels in a well defined order.

Nevertheless, this is not the most general definition of quantum process one could possibly conceive. For instance, one could allow for an intervention on the system-part in each stage of the process with any valid quantum operation. The process tensor formalism has been introduced in [49] to deal with quantum processes from an operational perspective. See also [50, 51] for reviews on this approach.

In this chapter we consider how to define information inequalities with the process tensor formalism. We first introduce process tensors as higher-order quantum operations. Then, we consider how information inequalities can be defined from a particular interventional scheme. Here, we only discuss in detail the derivation of the interventional quantum Markov monogamy inequality for four-time-step process tensors.

8.1　Process tensor formalism

In general, a process tensor is a linear transformation mapping time-ordered interventions to a quantum state. Much development of the quantum theory of Markovian processes has been achieved since then under this framework [52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68].

Let us start with the case of three-time-step process for simplicity. So define an arbitrary intervention for each time-step $i \in \{1, 2, 3\}$ with quantum channels $A_i \colon L(S_i) \to L(S_i')$. Each intervention A_i can be any valid quantum operation, and therefore, are now cast as variables in the process. Furthermore, the input and output system of A_i do not need to be copies of each other. That is because a valid intervention on the system could consist of adding or removing quantum systems, for instance.

Now, our three-time-step quantum Markov process is realized as a quantum higher-order transformation Υ mapping the sequence interventions $\{A_1, A_2, A_3\}$ to the final quantum state

$$\Upsilon(A_1, A_2, A_3) = A_3 \circ \Lambda_2 \circ A_2 \circ \Lambda_1 \circ A_1(\rho), \tag{8.1}$$

where the initial quantum state $\rho \in L(S_1)$, and quantum channels $\Lambda_1 \colon L(S_1') \to L(S_2)$, $\Lambda_2 \colon L(S_2') \to L(S_3)$ are fixed.
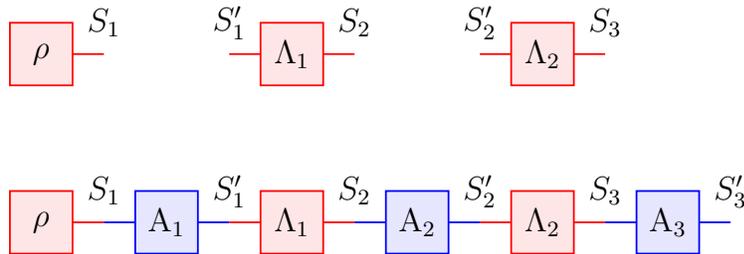
Figure 8.1: **Quantum Markov processes with interventions.** A quantum Markov process allowing for interventions is a higher-order quantum operation (top panel) mapping a sequence of quantum operations to a quantum state (bottom panel).

Figure 8.1 is a pictorial representation of quantum Markov processes allowing for interventions. For instance, within this approach we can recover the definition in Eq. (7.1). For this, consider the case where S_i and S_i' are identical copies, for each $i \in \{1, 2, 3\}$. Now, we show how to get the state of the intermediary system S_2 as the output of the fixed quantum

channel Λ_1.

Perform no action on the first intervention, that is, make $A_1 = \text{id}_{S_1}$. Then, set A_2 to be the preparation of an arbitrary quantum state $\varphi_2 \in L(S_2)$. Thus, we are specifying the intervention A_2 with trivial one-dimensional input system. Also, let A_3 discard the system S_3, that is, define the intervention $A_3 = \text{tr}_{S_3}$ with trivial one-dimensional output system. Then, we have

$$\Upsilon(\text{id}, \varphi_2, \text{tr}_{S_3}) = \Lambda_1(\rho_1) = \rho_2 \tag{8.2}$$

For recovering the state of the initial system S_1 proceed as similarly. Now, let the first intervention A_1 be the preparation of an arbitrary quantum state φ_1 of S_1. Let the second intervention A_2 be an arbitrary quantum channel. Then, discard the output system S_3 making $A_3 = \text{tr}_{S_3}$. Thus, we have

$$\Upsilon(\varphi_1, A_2, \text{tr}_{S_3}) = \rho_1 \tag{8.3}$$

For recovering the state of S_3 as the action of Λ_2 on $\Lambda_1(\rho_1)$, consider the following control strategy. Set $A_1 = \text{id}_{S_1}$, $A_2 = \text{id}_{S_2}$ and $A_3 = \text{id}_{S_3}$. So in each round of the process we let the system go through without acting on it. Then, we must have

$$\Upsilon(\text{id}, \text{id}, \text{id}) = \Lambda_2 \circ \Lambda_1(\rho_1) = \rho_3. \tag{8.4}$$

In turn, quantum non-Markov processes take place in similar way, allowing for non-trivial system-environment interactions. A general three-time-step quantum process maps the sequence of interventions $\{A_1, A_2, A_3\}$ to the final quantum state

$$\Upsilon(A_1, A_2, A_3) = A_3 \circ \Lambda_2 \circ (A_2 \otimes \text{id}_{E_2}) \circ \Lambda_1 \circ (A_1 \otimes \text{id}_{E_1})(\rho), \tag{8.5}$$

where the bipartite quantum state $\rho \in L(S_1 \otimes E_1)$, and bipartite quantum channels $\Lambda_1 \colon L(S_1' \otimes E_1) \to L(S_2 \otimes E_2)$ and $\Lambda_2 \colon L(S_2' \otimes E_2) \to L(S_3)$ are fixed.

The quantum operation Υ above is called process tensor, and has been introduced for dealing with multitime temporal correlations in quantum non-Markov processes. In case the quantum bipartite channels Λ_1 and Λ_2 act non-trivially on environmental systems E_1 and
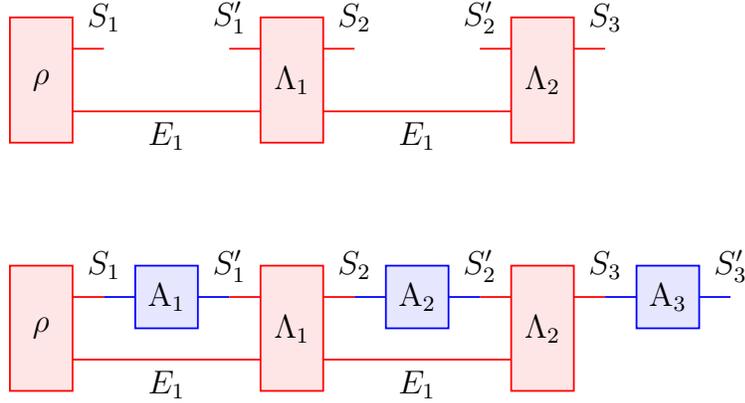
Figure 8.2: **Quantum non-Markov processes with interventions.**

E_2, regarded as non-trivial Hilbert spaces, the quantum operation Υ is called non-Markov process tensor. On the other hand, we recover the case of Markov process tensors in Eq. (8.1) if E_1, E_2 are trivial one-dimensional environmental systems. Figure 8.2 shows a diagrammatic representation of quantum non-Markov processes with interventions.

8.2 Interventional quantum Markov monogamy inequality for four-time-step process tensors

We move to derive the quantum Markov monogamy inequality arising out of an interventional approach to quantum processes. Let Υ be a four-time-step process tensor. The action of Υ on a sequence of interventions $A_i \colon L(S_i) \to L(S_i')$, with $i \in \{1, 2, 3, 4\}$, is therefore defined as

$$\Upsilon(A_1, A_2, A_3, A_4) = A_4 \circ \Lambda_3 \circ (A_3 \otimes \mathrm{id}_{E_3}) \circ \Lambda_2 \circ (A_2 \otimes \mathrm{id}_{E_2}) \circ \Lambda_1 \circ (A_1 \otimes \mathrm{id}_{E_1})(\rho_1), \quad (8.6)$$

where $\rho \in L(S_1 \otimes E_1)$ is quantum state, and $\Lambda_1 \colon L(S_1' \otimes E_1) \to L(S_2 \otimes E_2)$, $\Lambda_1 \colon L(S_2' \otimes E_2) \to L(S_3' \otimes E_3)$, $\Lambda_1 \colon L(S_3' \otimes E_3) \to L(S_4)$ are arbitrary quantum channels. Note we have a four-time-step Markov process tensor whenever the environmental systems E_1, E_2 and E_3 are trivial systems.

Define the control strategy as in Figure 8.3 for arbitrary four-time-step process tensor Υ.

Thus, we have defined the quantum states

$$v_{1:4} = (\mathrm{id}_{R_1 \otimes S_2'} \otimes [\mathrm{A}_4 \circ \Lambda_3 \circ (\mathrm{A}_3 \otimes \mathrm{id}_{E_3}) \circ \Lambda_2 \circ (\mathrm{A}_2 \otimes \mathrm{id}_{E_2}) \circ \Lambda_1])$$
$$\circ (\mathrm{id}_{R_1} \otimes \mathrm{SWAP}_{S_1', S_1'} \otimes \mathrm{id}_{E_2}) \circ (\mathrm{id}_{R_1, S_1'} \otimes \mathrm{A}_1 \otimes \mathrm{id}_{E_1})(\alpha_1 \otimes \rho), \quad (8.7)$$

$$v_{1:3} = \mathrm{tr}_{E_3}[(\mathrm{id}_{R_1 \otimes S_2'} \otimes [(\mathrm{A}_3 \otimes \mathrm{id}_{E_3}) \circ \Lambda_2 \circ (\mathrm{A}_2 \otimes \mathrm{id}_{E_2}) \circ \Lambda_1])$$
$$\circ (\mathrm{id}_{R_1} \otimes \mathrm{SWAP}_{S_1', S_1'} \otimes \mathrm{id}_{E_2}) \circ (\mathrm{id}_{R_1, S_1'} \otimes \mathrm{A}_1 \otimes \mathrm{id}_{E_1})(\alpha_1 \otimes \rho)], \quad (8.8)$$

$$v_{2:4} = (\mathrm{id}_{R_2 \otimes S_2'} \otimes [\mathrm{A}_4 \circ \Lambda_3 \circ (\mathrm{A}_3 \otimes \mathrm{id}_{E_3}) \circ \Lambda_2]) \circ (\mathrm{id}_{R_2} \otimes \mathrm{SWAP}_{S_2', S_2'} \otimes \mathrm{id}_{E_2})$$
$$\circ (\mathrm{id}_{R_1 \otimes S_2'} \otimes [(\mathrm{A}_2 \otimes \mathrm{id}_{E_2}) \circ \Lambda_1 \circ (\mathrm{A}_1 \otimes \mathrm{id}_{E_1})])(\alpha_2 \otimes \rho), \quad (8.9)$$

$$v_{2:3} = \mathrm{tr}_{E_3}[(\mathrm{id}_{R_1 \otimes S_2'} \otimes [(\mathrm{A}_3 \otimes \mathrm{id}_{E_3}) \circ \Lambda_2]) \circ (\mathrm{id}_{R_2} \otimes \mathrm{SWAP}_{S_2', S_2'} \otimes \mathrm{id}_{E_2})$$
$$\circ (\mathrm{id}_{R_1 \otimes S_2'} \otimes [(\mathrm{A}_2 \otimes \mathrm{id}_{E_2}) \circ \Lambda_1 \circ (\mathrm{A}_1 \otimes \mathrm{id}_{E_1})])(\alpha_2 \otimes \rho)], \quad (8.10)$$

We can define the information quantities such that it is possible to make sense of the quantum Markov inequality for the process tensor formalism. Now, we have several quantities playing the same role as coherent information in the quantum information inequalities. Under the protocol in Eqs. (8.7)-(8.10) we can define the information quantities

$$I_{q_1}(j : k) = [H(S_j, R_j) - H(S_j, R_j, S_k)]_{v_{j:k}}, \quad (8.11)$$

$$I_{q_2}(j : k) = [H(S_k) - H(S_j, R_j, S_k)]_{v_{j:k}}, \quad (8.12)$$

$$I_{q_3}(j : k) = [H(S_j, S_k) - H(S_j, R_j, S_k)]_{v_{j:k}}, \quad (8.13)$$

with $1 \leq j < k \leq 4$.

In fact, for any four-time-step Markov process tensor $\Upsilon_{1:4}$ it holds the interventional

quantum Markov monogamy inequality

$$I_{q_i}(1;4) + I_{q_i}(2;3) \geq I_{q_i}(1;3) + I_{q_i}(2;4), \tag{8.14}$$

for $i \in \{1,2,3\}$. Here, the quantum state $\alpha_i \in L(R_i \otimes S_i')$ is the purification of the system S_i, with $i = 1,2$.

We now follow the proof in Ref. [1] of the information inequality above. We focus on the intervantional quantum Markov monogamy for I_{q_1}, nevertheless, the proof for the I_{q_2} and I_{q_3} is completely analogous.

For any four-time-step Markov process tensor $\Upsilon_{1:4}$, Eqs. (8.7)-(8.10) reduce to

$$\upsilon_{1:4} = (\mathrm{id}_{R_1,S_1'} \otimes (\mathrm{A}_4 \circ \Lambda_3 \circ \mathrm{A}_3 \circ \Lambda_2 \circ \mathrm{A}_2 \otimes \Lambda_1)) \circ (\mathrm{id}_{R_1} \otimes \mathrm{SWAP}_{S_1',S_1'})(\alpha_1 \otimes \mathrm{A}_1(\rho)), \tag{8.15}$$

$$\upsilon_{1:3} = (\mathrm{id}_{R_1,S_1'} \otimes (\mathrm{A}_3 \circ \Lambda_2 \circ \mathrm{A}_2 \otimes \Lambda_1)) \circ (\mathrm{id}_{R_1} \otimes \mathrm{SWAP}_{S_1',S_1'})(\alpha_1 \otimes \mathrm{A}_1(\rho)), \tag{8.16}$$

$$\upsilon_{2:4} = (\mathrm{id}_{R_1,S_1'} \otimes (\mathrm{A}_4 \circ \Lambda_3 \circ \mathrm{A}_3 \circ \Lambda_2)) \circ (\mathrm{id}_{R_2} \otimes \mathrm{SWAP}_{S_2',S_2'})(\alpha_1 \otimes (\mathrm{A}_2 \circ \mathrm{A}_1)(\rho)), \tag{8.17}$$

$$\upsilon_{2:3} = (\mathrm{id}_{R_1,S_1'} \otimes (\mathrm{A}_3 \circ \Lambda_2)) \circ (\mathrm{id}_{R_2} \otimes \mathrm{SWAP}_{S_2',S_2'})(\alpha_1 \otimes (\mathrm{A}_2 \circ \Lambda_1 \circ \mathrm{A}_1)(\rho)), \tag{8.18}$$

with quantum state $\rho \in L(S_1)$, and quantum operations $\Lambda_i \colon L(S_i') \to L(S_{i+1})$, where $i \in \{1,2,3\}$.

We start our proof with the case where $S_i \cong S_i'$. Then, define a control scheme as in Fig. 8.3 with identity control operations $\mathrm{A}_i = \mathrm{id}_{S_i}$. Let the quantum channels Λ_i have isometric extension $U \colon S_i \to S_{i+1} \otimes E_i$.

For any Markov process tensor Υ we have $H(S_j, R_j) = H(S_j) + H(R_j)$ and $H(S_j, R_j, S_k) = H(S_j) + H(R_j, S_k)$, with $j < k$. Then, the terms in Eq. (8.14) for I_{q_1} as defined in Eq. (8.11) becomes

$$I_{q_1}(1:4) = H(R_1) - H(E_1, E_2, E_3), \tag{8.19}$$

$$I_{q_1}(2:3) = H(R_2) - H(E_2) \tag{8.20}$$

$$I_{q_1}(1:3) = H(R_1) - H(E_1, E_2), \tag{8.21}$$

$$I_{q_1}(2:4) = H(R_2) - H(E_2, E_3). \tag{8.22}$$

The rest of the derivation follows using strong subadditivity of quantum entropy, as before in the proof of the quantum Markov monogamy inequality in (7.8). This holds since the entropies in the second terms in each of the above equations comes from the same environmental state.

Now, we consider the general case of arbitrary systems S_i and S_i', which are not necessarily isomorphic. The hint is to absorb the action of the interventions into the process, thus generating a Markov process tensor. That is, we define a Markov process with initial state $A_1(\rho_1)$ going through evolution maps $A_2 \circ \Lambda_1$, $A_3 \circ \Lambda_2$ and $A_4 \circ \Lambda_3$.

Let the interventions A_i have isometric representation $V: S_i \to S_i' \otimes A_i$. Then, the second terms in Eqs. (8.19)-(8.22) contain also the systems A_i. We can get back the same type of inequality under the substitution

$$H(E_1, A_2, E_2, A_3, E_3, A_4) \to H(\tilde{E}_1, \tilde{E}_2, \tilde{E}_3), \tag{8.23}$$

$$H(E_2, A_3) \to H(\tilde{E}_2), \tag{8.24}$$

$$H(E_1, A_2, E_2, A_3) \to H(\tilde{E}_1, \tilde{E}_2), \tag{8.25}$$

$$H(E_2, A_3, E_3, A_4) \to H(\tilde{E}_2, \tilde{E}_3), \tag{8.26}$$

with $\tilde{E}_1 := E_1 \otimes A_2$, $\tilde{E}_2 := E_2 \otimes A_3$, and $\tilde{E}_3 := E_3 \otimes A_4$. The proof is concluded using strong subadditivity of quantum entropy. The interested reader is invited to check the same reasoning holds for the quantities I_{q_2} and I_{q_3}.

We show how the interventional approach to information inequalities is relevant in witnessing the non-Markov process considered in chapter 7. Now, the non-Markov process in bottom panel of Figure 7.3 becomes the process tensor in Figure 8.4. We use the same initial state and unitary evolution defined in Eq. (7.47) and Eq. (6.32), respectively.

The interventional Markov monogamy inequality in Eq. (8.14) supports the definition of the interventional witnesses of non-Markovianity

$$\text{M4}_{q_i} := I_{q_i}(1:4) + I_{q_i}(2:3) - I_{q_i}(1:3) - I_{q_i}(2:4), \tag{8.27}$$

with $i = 1, 2, 3$.

The quantity defined above is positive semi-definite for any Markov process tensor $\Upsilon_{1:4}$. Thus, finding a negative value for any $\mathrm{M4}_{q_i}$ (i=1,2,3) implies the process tensor is non-Markovian. In Fig. 8.5 we present the plot of the witnesses in Eq. (8.27) for the process tensor mentioned above.

Interestingly, the quantities $\mathrm{M4}_{q_2}$ and $\mathrm{M4}_{q_3}$ are negative for any value of the parameter λ, thus, efficiently witnessing the non-Markovianity in this example. On the other hand, the quantity $\mathrm{M4}_{q_1}$ is non-negative only in the region $0.30 \leq \lambda \leq 0.55$.
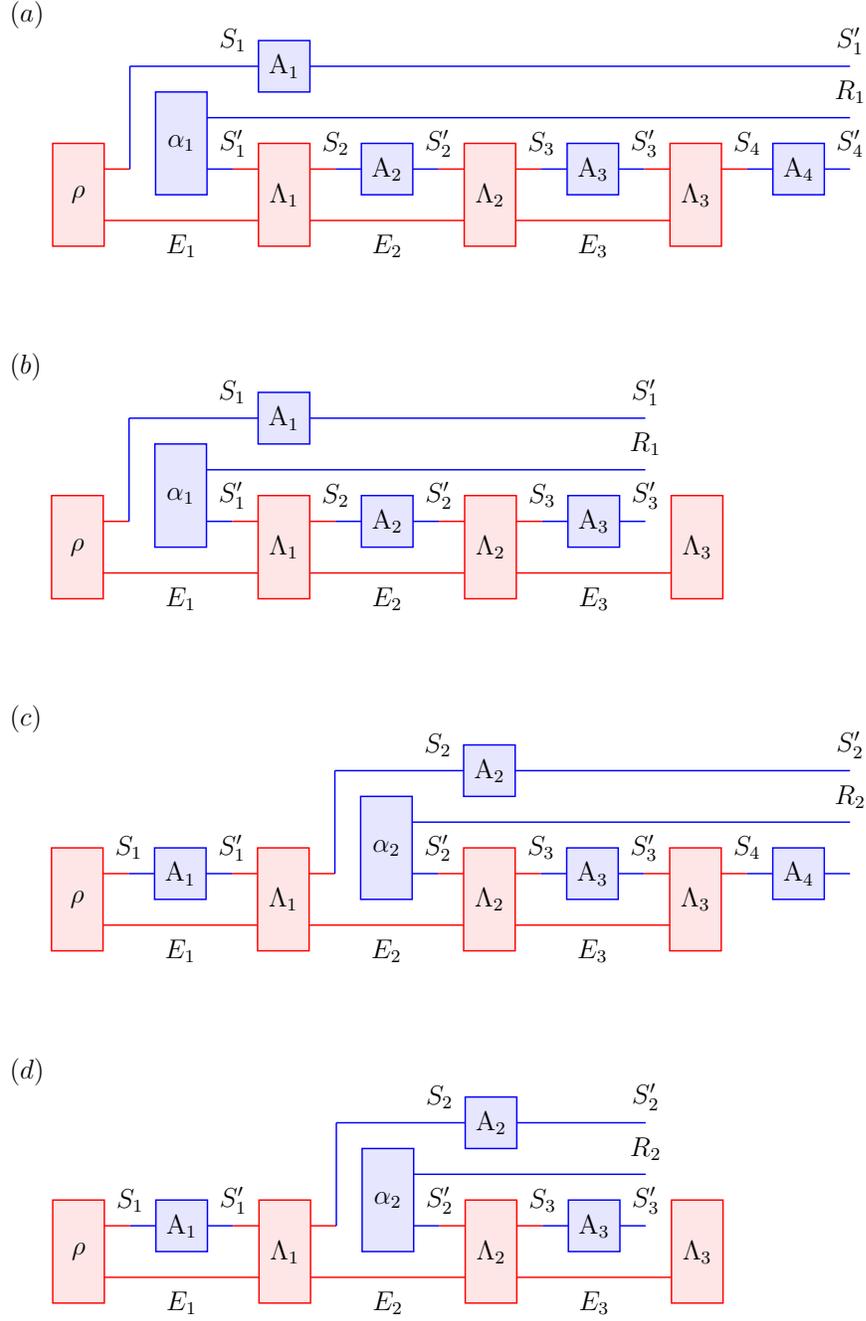
Figure 8.3: **Control strategy for interventional information inequalities.** Here we define an interventional scheme appropriate for defining quantum information inequalities. In panel (a) we feed the system S_1' with part of the state α_1, defined as the purification of the system S_1. The remaining quantum operations $\{A_i : i = 1, 2, 3, 4\}$ are arbitrary. This defines the quantum state $\upsilon_{1:4}$ in Eq. (8.7). In panel (b) we define the quantum state $\upsilon_{1:3}$ in Eq. (8.8) similarly. Nevertheless, we feed the system S_2' with an arbitrary quantum state and trace out its output system S_4. Panel (c) represents $\upsilon_{2:4}$ in Eq. (8.9). This is defined similarly to panel (a), but we feed the system S_2' with part of the state α_2, defined as the purification of S_2. Panel (d) represents $\upsilon_{2:3}$ in Eq. (8.10). This turn we feed S_3' with an arbitrary quantum state and trace out its output S_4.
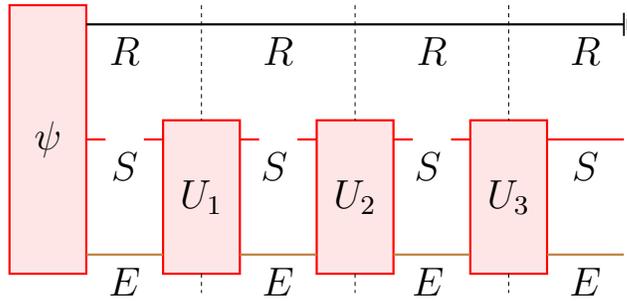
Figure 8.4: **Non-Markov process tensor.** We define the non-Markov process in bottom panel of Figure 7.3, but now allowing for interventions. The state ψ is defined as in Eq. (7.47). The unitary operations U_1, U_2 and U_3 are defined as in Eq. (6.32).
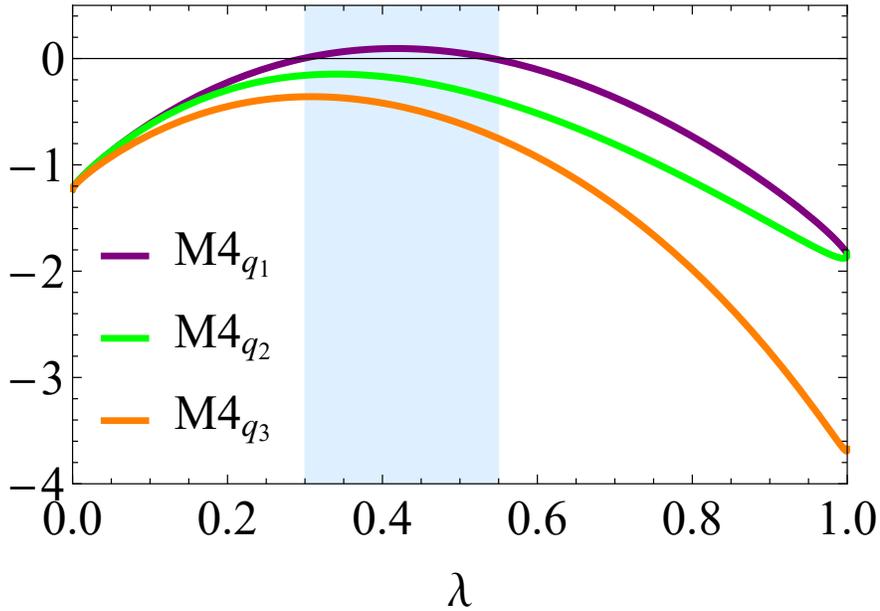
Figure 8.5: **Violation of the interventional Markov monogamy inequalities.** Interventional quantum Markov monogamy with respect to I_{q_1} is not violated only for the region $0.30 \leq \lambda \leq 0.55$ (shadded in blue). The interventional information inequalities with respect to I_{q_2} and I_{q_3} are violated for any value of λ, thus perfectly witnessing the non-Markovianity of the process tensor considered. Figure addapted from [1].

Chapter 9

Discussion

We have considered the development of novel information conditions called Markov monogamy inequalities. Those are limits on classical and quantum information processing which appear to be complementary to the data processing inequalities. Therefore, as argued in chapter 1, this novel conditions have intrinsic value.

Moreover, we have also considered how information inequalities can be useful for witnessing classical and quantum non-Markovianity. Importantly, we have provided examples of non-Markov processes violating the Markov monogamy, but remaining all data processing inequalities still valid. Thus, Markov monogamy conditions have the potential to witness non-Markovianity beyond data processing inequalities in particular cases. Therefore, the novel inequalities were shown to be of practical value as well.

Now, we mention open problems and possible follow-ups related to this thesis. Firstly, the derivation of the general form of Markov monogamy inequalities remains as an open problem.

Techniques similar to the presented here were used in the formulation of novel experiments [69]. Therefore, finding the Markov monogamy inequalities to be useful in practical experimental situations would be of remarkable relevance as well.

Finally, we remark that associating information inequalities to operational results is of main importance in information theory. Therefore, this is left as a key follow-up of this thesis.

Bibliography

[1] M. Capela, L. C. Céleri, R. Chaves, and K. Modi, "Quantum Markov monogamy inequalities," *arXiv preprint arXiv:2108.11533*, 2021.

[2] M. Capela, L. C. Céleri, K. Modi, and R. Chaves, "Monogamy of temporal correlations: Witnessing non-markovianity beyond data processing," *Physical Review Research*, vol. 2, no. 1, p. 013350, 2020.

[3] C. E. Shannon, "A mathematical theory of communication," *The Bell system technical journal*, vol. 27, no. 3, pp. 379–423, 1948.

[4] T. M. Cover and J. A. Thomas, *Elements of Information Theory, second edition.* John Wiley & Sons, 2005.

[5] R. W. Yeung, *Information theory and network coding.* Springer Science & Business Media, 2008.

[6] I. Csiszár and J. Körner, *Information theory: coding theorems for discrete memoryless systems.* Cambridge University Press, 2011.

[7] D.-M. Arnold, "On the data-processing inequality in the mismatch case," in *International Symposium on Information Theory, 2004. ISIT 2004. Proceedings.*, p. 266, IEEE, 2004.

[8] N. Merhav, "Data-processing inequalities based on a certain structured class of information measures with application to estimation theory," *IEEE Transactions on Information Theory*, vol. 58, no. 8, pp. 5287–5301, 2012.

[9] W. Kang and S. Ulukus, "A new data processing inequality and its applications in distributed source and channel coding," *IEEE Transactions on Information Theory*, vol. 57, no. 1, pp. 56–69, 2010.

[10] F. du Pin Calmon, Y. Polyanskiy, and Y. Wu, "Strong data processing inequalities for input constrained additive noise channels," *IEEE Transactions on Information Theory*, vol. 64, no. 3, pp. 1879–1892, 2017.

[11] S. Kamath and C. Nair, "The strong data processing constant for sums of iid random variables," in *2015 IEEE International Symposium on Information Theory (ISIT)*, pp. 2550–2552, IEEE, 2015.

[12] M. Raginsky, "Logarithmic sobolev inequalities and strong data processing theorems for discrete channels," in *2013 IEEE International Symposium on Information Theory*, pp. 419–423, IEEE, 2013.

[13] N. Merhav, "Data processing theorems and the second law of thermodynamics," *IEEE Transactions on Information Theory*, vol. 57, no. 8, pp. 4926–4939, 2011.

[14] V. Anantharam, A. Gohari, S. Kamath, and C. Nair, "On hypercontractivity and a data processing inequality," in *2014 IEEE International Symposium on Information Theory*, pp. 3022–3026, IEEE, 2014.

[15] C. E. Shannon, "A note on a partial ordering for communication channels," *Information and Control*, vol. 1, no. 4, pp. 390–397, 1958.

[16] Y. Zhang and C. Tepedelenlioğlu, "Analytical and numerical characterizations of shannon ordering for discrete memoryless channels," *IEEE Transactions on Information Theory*, vol. 60, no. 1, pp. 72–83, 2013.

[17] R. Nasser, "Characterizations of two channel orderings: Input-degradedness and the shannon ordering," *IEEE Transactions on Information Theory*, vol. 64, no. 10, pp. 6759–6770, 2018.

[18] F. Haddadpour, M. H. Yassaee, S. Beigi, A. Gohari, and M. R. Aref, "Simulation of a channel with another channel," *IEEE Transactions on Information Theory*, vol. 63, no. 5, pp. 2659–2677, 2016.

[19] D. Blackwell, "Equivalent comparisons of experiments," *The Annals of Mathematical Statistics*, pp. 265–272, 1953.

[20] F. Buscemi, "Comparison of noisy channels and reverse data-processing theorems," in *2017 IEEE Information Theory Workshop (ITW)*, pp. 489–493, IEEE, 2017.

[21] B. Schumacher and M. A. Nielsen, "Quantum data processing and error correction," *Physical Review A*, vol. 54, no. 4, p. 2629, 1996.

[22] F. Buscemi and N. Datta, "Equivalence between divisibility and monotonic decrease of information in classical and quantum stochastic processes," *Physical Review A*, vol. 93, no. 1, p. 012101, 2016.

[23] P. Hayden, R. Jozsa, D. Petz, and A. Winter, "Structure of states which satisfy strong subadditivity of quantum entropy with equality," *Communications in Mathematical Physics*, vol. 246, no. 2, pp. 359–374, 2004.

[24] A. Rivas, S. F. Huelga, and M. B. Plenio, "Quantum non-markovianity: characterization, quantification and detection," *Reports on Progress in Physics*, vol. 77, no. 9, p. 094001, 2014.

[25] C. Ferrie, "Data-processing inequalities for quantum metrology," *Physical Review A*, vol. 90, no. 1, p. 014101, 2014.

[26] A. Türkmen, A. Verçin, and S. Yılmaz, "Data processing inequality and open quantum systems: Beyond markov states," *Physical Review A*, vol. 96, no. 4, p. 042112, 2017.

[27] O. Fawzi and R. Renner, "Quantum conditional mutual information and approximate markov chains," *Communications in Mathematical Physics*, vol. 340, no. 2, pp. 575–611, 2015.

[28] D. Sutter, O. Fawzi, and R. Renner, "Universal recovery map for approximate markov chains," *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 472, no. 2186, p. 20150623, 2016.

[29] M. M. Wilde, "Recoverability in quantum information theory," *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 471, no. 2182, p. 20150338, 2015.

[30] P. Taranto, F. A. Pollock, and K. Modi, "Non-markovian memory strength bounds quantum process recoverability," *npj Quantum Information*, vol. 7, no. 1, pp. 1–8, 2021.

[31] L. Koralov and Y. G. Sinai, *Theory of probability and random processes*. Springer Science & Business Media, 2007.

[32] P. J. Fernandez, *Introdução à teoria das probabilidades*. Editora Universidade de Brasília, 1973.

[33] B. R. James, *Probabilidade: um curso em nível intermediário*. IMPA, 2015.

[34] F. Buscemi, "Reverse data-processing theorems and computational second laws," in *Nagoya Winter Workshop: Reality and Measurement in Algebraic Quantum Theory*, pp. 135–159, Springer, 2015.

[35] R. Yeung and Y. Yan, "ITIP (Information Theoretic Inequality Prover)," 2016.

[36] A. Peres, *Quantum theory: concepts and methods*. Kluwer Academic Publishers, 2002.

[37] M. M. Wilde, "From classical to quantum shannon theory," *arXiv preprint arXiv:1106.1445*, 2011.

[38] J. Watrous, *The theory of quantum information*. Cambridge University Press, 2018.

[39] M. Hayashi, *Quantum information theory: mathematical foundation, Second edition*. Springer, 2016.

[40] M. A. Nielsen and I. L. Chuang, *Quantum information and quantum computation*. Cambridge University Press, 2000.

[41] D. Petz, *Quantum information theory and quantum statistics.* Springer Science & Business Media, 2007.

[42] S. Axler, *Linear algebra done right, Third Edition.* Springer, 2015.

[43] E. L. Lima, *Álgebra Linear, Nona Edição.* IMPA, 2016.

[44] E. L. Lima, *Cálculo tensorial.* IMPA, 1965.

[45] W. Greub, *Multilinear Algebra, Second Edition.* Springer-Verlag, 1978.

[46] E. H. Lieb and M. B. Ruskai, "Proof of the strong subadditivity of quantum-mechanical entropy," *Journal of Mathematical Physics*, vol. 14, p. 1938–1941, 1973.

[47] M. B. Ruskai, "Inequalities for quantum entropy: A review with conditions for equality," *Journal of Mathematical Physics*, vol. 43, no. 9, pp. 4358–4375, 2002.

[48] D. Petz, "Monotonicity of quantum relative entropy revisited," *Reviews in Mathematical Physics*, vol. 15, no. 01, pp. 79–91, 2003.

[49] F. A. Pollock, C. Rodríguez-Rosario, T. Frauenheim, M. Paternostro, and K. Modi, "Non-markovian quantum processes: Complete framework and efficient characterization," *Physical Review A*, vol. 97, no. 1, p. 012127, 2018.

[50] S. Milz, F. A. Pollock, and K. Modi, "An introduction to operational quantum dynamics," *Open Systems & Information Dynamics*, vol. 24, no. 04, p. 1740016, 2017.

[51] S. Milz and K. Modi, "Quantum stochastic processes and quantum non-markovian phenomena," *PRX Quantum*, vol. 2, p. 030201, Jul 2021.

[52] F. A. Pollock, C. Rodríguez-Rosario, T. Frauenheim, M. Paternostro, and K. Modi, "Operational markov condition for quantum processes," *Physical Review Letters*, vol. 120, no. 4, p. 040405, 2018.

[53] F. Sakuldee, S. Milz, F. A. Pollock, and K. Modi, "Non-markovian quantum control as coherent stochastic trajectories," *Journal of Physics A: Mathematical and Theoretical*, vol. 51, no. 41, p. 414014, 2018.

[54] P. Taranto, F. A. Pollock, S. Milz, M. Tomamichel, and K. Modi, "Quantum markov order," *Physical Review Letters*, vol. 122, no. 14, p. 140401, 2019.

[55] P. Taranto, S. Milz, F. A. Pollock, and K. Modi, "Structure of quantum stochastic processes with finite markov order," *Physical Review A*, vol. 99, no. 4, p. 042108, 2019.

[56] P. Figueroa-Romero, K. Modi, and F. A. Pollock, "Almost markovian processes from closed dynamics," *Quantum*, vol. 3, p. 136, 2019.

[57] S. Milz, F. A. Pollock, T. P. Le, G. Chiribella, and K. Modi, "Entanglement, non-markovianity, and causal non-separability," *New Journal of Physics*, vol. 20, no. 3, p. 033033, 2018.

[58] S. Milz, F. Sakuldee, F. A. Pollock, and K. Modi, "Kolmogorov extension theorem for (quantum) causal modelling and general probabilistic theories," *Quantum*, vol. 4, p. 255, Apr. 2020.

[59] P. Strasberg, "Operational approach to quantum stochastic thermodynamics," *Physical Review E*, vol. 100, no. 2, p. 022127, 2019.

[60] P. Strasberg, "Repeated interactions and quantum stochastic thermodynamics at strong coupling," *Physical Review Letters*, vol. 123, no. 18, p. 180604, 2019.

[61] P. Strasberg, "Thermodynamics of quantum causal models: An inclusive, hamiltonian approach," *Quantum*, vol. 4, p. 240, 2020.

[62] S. Milz, M. S. Kim, F. A. Pollock, and K. Modi, "Completely positive divisibility does not mean markovianity," *Physical Review Letters*, vol. 123, p. 040401, Jul 2019.

[63] K. Modi, "Operational approach to open dynamics and quantifying initial correlations," *Scientific Reports*, vol. 2, pp. 581 EP –, Aug 2012. Article.

[64] S. Vinjanampathy and K. Modi, "Entropy bounds for quantum processes with initial correlations," *Physical Review A*, vol. 92, p. 052310, Nov 2015.

[65] P. Strasberg and M. G. Díaz, "Classical quantum stochastic processes," *Physical Review A*, vol. 100, p. 022120, Aug 2019.

[66] S. Milz, D. Egloff, P. Taranto, T. Theurer, M. B. Plenio, A. Smirne, and S. F. Huelga, "When is a non-markovian quantum process classical?," *Physical Review X*, vol. 10, p. 041049, Dec 2020.

[67] G. A. L. White, C. D. Hill, F. A. Pollock, L. C. L. Hollenberg, and K. Modi, "Experimental non-Markovian process characterisation and control on a quantum processor," *Nature Communications*, vol. 11, p. 6301, 2020.

[68] G. A. L. White, F. A. Pollock, L. C. L. Hollenberg, C. D. Hill, and K. Modi, "Non-markovian quantum process tomography," *arXiv preprint arXiv:2106.11722*, 2021.

[69] Y. Guo, P. Taranto, B.-H. Liu, X.-M. Hu, Y.-F. Huang, C.-F. Li, and G.-C. Guo, "Experimental demonstration of instrument-specific quantum memory effects and non-markovian process recovery for common-cause processes," *Physical Review Letters*, vol. 126, no. 23, p. 230401, 2021.