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LUCAS AURÉLIO DE OLIVEIRA BORGES

Thermodynamics and the Unruh Effect

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Lucas Aurélio de Oliveira Borges

Thermodynamics and the Unruh Effect

A dissertation submitted in partial fulfillment of the requirements for the degree of Master of Physics.

Advisor: *Prof. Dr. Lucas Chibebe Céleri*

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ATA DE DEFESA DE DISSERTAÇÃO

Ata nº 182 da sessão de Defesa de Dissertação de Lucas Aurélio de Oliveira Borges, que confere o título de Mestre em Física, na área de concentração em Física.

Aos 13 dias do mês de abril de 2020, a partir das 13h00min, por meio de videoconferência, realizou-se a sessão pública de Defesa de Dissertação intitulada “Thermodynamics and the Unruh effect”. Os trabalhos foram instalados pelo Orientador, Professor Doutor Lucas Chibebe Céleri (IF/UFG), com a participação dos demais membros da Banca Examinadora: Professor Doutor Daniel Augusto Turolla Vanzella (IFSC/USP), membro titular externo; e Professor Doutor Fábio Luis Braghin (IF/UFG), membro titular interno. Durante a arguição, os membros da banca não fizeram sugestão de alteração do título do trabalho. A Banca Examinadora reuniu-se em sessão secreta a fim de concluir o julgamento da Dissertação, tendo sido o candidato aprovado pelos seus membros. Proclamados os resultados pelo Professor Doutor Lucas Chibebe Céleri, Presidente da Banca Examinadora, foram encerrados os trabalhos e, para constar, lavrou-se a presente ata que é assinada pelos Membros da Banca Examinadora, aos 13 dias do mês de abril de 2020.

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Abstract

A uniformly accelerated detector perceives the Minkowski vacuum as a thermal state. In particular, for a massless scalar field, one cannot distinguish these two scenarios -accelerated detector in the vacuum and an inertial detector in contact with a thermal reservoir- just by using a point-like detector. In this work we show that one can probe the existence of acceleration by using an extended detector with dimension perpendicular to the acceleration direction. Also, we verify that the internal degrees of freedom of the detector may get entangled by a process of spontaneous emission, and as such is dependent on the initial state of the detector. It is found that the entropy production and entropy fluxes from out the detector also allow one to tell the two scenarios apart: more internal entangled detectors produce less entropy and vice-versa; the wavelength of the sought particle is the critical length at which the behavior of the two system is inverted.

Keywords: Quantum information. The Unruh effect. Thermodynamics. Entanglement. Quantum Field Theory.

Resumo

Um detector uniformemente acelerado percebe o vácuo de Minkowski como um estado térmico. Em particular, para um campo escalar sem massa, os dois cenários -detector acelerado no vácuo e detector inercial em contato com um reservatório térmico- não podem ser diferenciados utilizando-se apenas um detector adimensional. Neste trabalho mostramos que os dois cenários podem ser diferenciados com o uso de um detector estendido de dimensão perpendicular à direção de aceleração. Verificamos também que os graus de liberdade internos do detector podem se correlacionar por meio de emissão espontânea, e portanto, dependem do estado inicial do detector. Foi observado que a produção e fluxos de entropia do detector também permitem distinguir os dois cenários: detectores com graus de liberdade mais emaranhados produzem menos entropia e vice-versa; o comprimento de onda da partícula buscada é uma distância crítica para a qual o comportamento dos dois sistemas se inverte.

Palavras-chave: Informação Quântica. O efeito Unruh. Termodinâmica. Emaranhamento. Teoria Quântica de Campos.

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Contents

Introduction

In 1905, and perhaps in a much more profound way, in 1915, Albert Einstein came up with an astounding idea concerning the structure of what we used to call space and time. Up to that point, it was taken for granted that there existed a three-dimensional space at each instant of time; further, this whole pile of space parameterized by an absolute and universal parameter called time, was but a stage where physics were to take place. Things were getting completely different; with more and more facts supporting the predictions of the General Theory of Relativity [1][2][3], it became clear that, whether we liked it or not, the universe behaves in a mind-bending way. As time goes on, physicists grow more and more used to these newly introduced ideas, to the point that they become natural. Eventually, however, one realizes that the otherwise almost completely accepted scenario implies some phenomena that seems to defies the current understanding of physics -for instance, particle creation by a non-stationary background, and the Unruh effect, which is the subject of this work.

Perhaps it was L. Parker [4] one of the first to dedicate a great deal to the study of particle creation by an expanding universe. Certainly heavily influenced by his work, a new entity, if it can be called this way, entered the set of things physics could study and try to understand: the stage itself, a now dynamical stage. When mixed with quantum mechanics, a mix that must be made with care, this dynamical behavior of spacetime gives rise to some truly fruitful results: some of them quite challenging to the mind, one of these being the so called Unruh effect [11]. However, we would like to look at it in a slightly different manner than is usually done. Given the current paradigm in science, some authors find it hard to accept that a phenomenon such as the Unruh effect should be considered physical, some of

them even pointing to mathematical inconsistencies in the derivation.

Thus, contrary to saying that an uniformly accelerated observer would measure temperature in a otherwise empty space -a quite vague statement, since a lot of thing are not well specified there- we found it more interesting to start from the most basic possible structure and grow it little by little by endowing it with more and more properties, thus putting upon it more restrictions, but in a way that unnecessary concepts would not arise -and were one given concept to appear here or there, it should be clear the specific conditions under which this happens. The line of work is then try to formulate, following R. Wald [12], in the most general and natural possible way, a theory that would reduce to the known results in very specific regimes where symmetries that are not strictly necessary to the construction might be available, and then, if that is the case, new concepts introduced -such as the concept of particle, that appears due to the existence of a symmetry that is not at all necessary. With this in mind, even though the primary objective of this dissertation might be to study information-theoretical and thermodynamic properties of the Unruh effect, the elegance of the concepts involved were too big not to give them much of the attention. So in chapter one, the presentation is quite general, and much attention is given to the appearance of the aforementioned phenomenon from a purely mathematical point of view. Thus, in the first chapter, we are not talking about the Unruh effect particularly, but rather, about the formulation of quantum field theory in curved manifolds.

Being quantum field theory in curved spacetimes one of our only semiclassical description of gravity, it calls our attention that more prospects can be made towards a better understanding of quantum gravity by further analyzing properties of phenomena that arise due to attempts in trying to mix quantum mechanics and general relativity together in the same framework. In particular, since the Unruh effect predicts that a uniformly accelerated observer perceives the Minkowski vacuum as a thermal state, connection with thermodynamics seems natural.

It is worth noting that it was shown [5] already that, in the case of a massless scalar field, one cannot tell apart if a system is moving with uniform acceleration a, or if it is inertial

but in contact with a heat bath in thermal equilibrium with temperature $T = a/2\pi$, that is, there is no difference in the correlation functions derived in both case, and thus there is no difference in dynamics as well -since we will be referring to these two systems very often, we call the first one the Unruh setting, and the second one, with equivalent temperatures, the Heat bath setting. So it seems that in studying the Unruh effect by means of a massless scalar field, we are basically analysing the dynamics of a system in contact with a heat reservoir, which is not the desire. However, it was later found that if our system is composed of two parties, then correlations between these two parties do provide ways of differentiating these two settings [7][8][9]. In Ref. [8], the authors shows that the concurrence between these parties shows a behavior that allows us to distinguished the Unruh effect from a ordinary heat bath, in particular, there are suitable conditions under which those two detectors get entangled by spontaneous emission [25] in one setting while no entanglement is verified in the other. This is why we work with a system composed of two parties rather than just one.

In Ref. [10], the authors point to the role of coherence in the study of nonequilibrium thermodynamics. Since the loss of coherence implies the loss of quantum correlation between two quantum systems, a question arise: can one tell the difference between a Unruh and a Heat bath setting just by looking at thermodynamical properties of the system, in particular, if the answers is positive, how does this difference in behavior in the irreversibility of the system relates to the quantum correlation among its parties? By studying how this behavior in the Unruh setting differs from that of a static heat bath setting, we hope some light can be shed about semiclassical properties of entropy and entanglement, since a description of both quantities in a relativistic regime is still lacking.

The outline of this work is as follows. The first chapter is primarily concerned with the causal structure of a general manifold, and what is needed in order for one to formulate an initial value problem, also known as Cauchy problem. In the second chapter, the Unruh effect is derived, and how it is related to a static thermal bath is discussed. The third chapter is the one that ultimately justifies all the need of the previous ones, even though one might say that

chapters one and two stand on their own, being much more fundamental in nature. In this chapter we introduce an operational definition of particle by introducing a particle detector model, we then present our set up, and follow on with the query for distinct behavior in the irreversibility of a uniformly accelerated system in contact with the Minkowski vacuum of a massless scalar field theory; when compared to an inertial system in contact with a heat reservoir, as mentioned.

Chapter 1

Quantum Field Theory on Manifolds

In this chapter we develop the main ideas concerning Quantum Field Theory in curved manifolds and explore the difficulties related to the usual decomposition of field operators in positive and negative frequency in this more general scenario. It is worth noting again that unlike many treatments of this subject, we are not going to start from a Minkowski spacetime - assuming all its symmetries - and try to generalize the ideas to curved manifolds by means of some naive substitutions, rather, we will construct the theory directly from a more general metric so to speak, be its curvature zero or not - of course, we do so in close analogy with flat spacetime physics. From this perspective, we believe, many misconceptions will not show up and we can later derive Minkowski physics as a special case where the metric is a rather symmetric object; although we might not even work with nonzero Ricci scalar, the machinery will not be wasted; it is, in fact, of paramount importance to the subject. So, from here on, we ask our readers to give up on the idea of a privileged/global inertial frame of reference - and therefore, as we shall soon see, to a privileged notion of particle - since such a thing does not exist according to Einstein's General Theory of Relativity.

1.1 Future, Past and Global Hyperbolicity

The first problem with this approach appears right at the beginning, where we try to define a notion of time direction, which is crucial to state an initial value problem - and also

an "instant of time" as to make sense of the "initial" in initial value - and there is no obvious way, at first glance, on how to achieve such a goal in our scenario: it might not even be possible to do so.

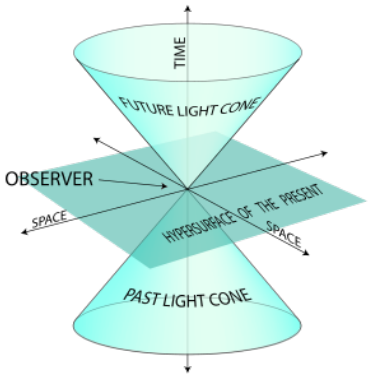

Figure 1.1: Illustration of a light cone

To proceed, let $(M, g_{\alpha\beta})$ be a manifold with Lorentzian metric $g_{\alpha\beta}$ [1] and $p \in M$, then, by the very definition of manifold A.1 the tangent vector space V_p at p is isomorphic to Minkowski space; we can thus consider a light cone -the path taken by a light signal travelling in all directions and emanating from a single event- passing through the origin of V_p, as in Fig. 1.1, where half of this light cone is the "future" and the other half the "past" of the event p. Timelike vectors lying on each side will be correspondingly called "future" or "past" directed just as in ordinary flat spacetime. Note, however, that in general this can be done only at the point p as there is no *a priori* reason to believe that the light cones defined at p and q, with $p \neq q$, will have a common notion of future and past. If one varies p throughout the manifold M and a continuous choice of such light cones can be made, $(M, g_{\alpha\beta})$ is said to be time-orientable and, from here on, we consider only time-orientable manifolds. This basically implies that all observers on the manifold agree on the causal connection of timelike separeted events, a smooth timelike vector field can be defined over $(M, g_{\alpha\beta})$ [13] and we have thus succeed in defining some sort of time direction common to all $p \in M$.

To address the issue of time direction, we need some definitions. We will call causal curve

[1] The signature choosen here is the conventional one used in General Relativity, i.e, there is a basis, not necessarily a coordinate basis, in which the metric takes the form $g = \text{diag}(-1, +1, +1, +1, ...)$

any curve that is either timelike or null everywhere, that is, a non spacelike curve. Now, let $S \subset M$ be a subset of M. For each $p \in S$ we define a subset $J^+(p)|_S \subset M$ called the causal future of p, composed of all point that can be connected to p by a future-directed causal curve; the chronological future, similarly, is defined as the set of points connected to p by a future-directed timelike curve and is denoted by $I^+(p)|_S$. The causal and chronological past is defined analogously and denoted as $J^-(p)|_S$ and $I^-(p)|_S$, respectively[2]. Subsets S with no points connected by timelike curves, $S \cap J^+(p)|_S = \emptyset$, are called achronal sets. Note that any event $q \in J^+(p)|_S$ can be influenced by a causal signal from p, what we cannot yet say is that an inextendible, past-directed causal curve starting from any event in $J^+(p)|_S$ will intersect S, that is, even though we can reach any event in $J^+(p)|_S$ starting from an event in S, there might be events in $J^+(p)|_S$ that is influenced by different events other than those in S, and as such, cannot have the dynamics completely characterized by information given on S only.

To fix this we consider an achronal, closed[3] subset $S \subset M$ and construct its future domain of dependence $D^+(S)$ to be composed of all events $p \in D^+(S)$ such that *any* inextendible, i.e, that goes on to infinity rather than ending in some finite point, past-directed causal curves from p must intersect S. By definition, all events in the future domain of dependence is completely determined by data given on the set S. Using the same protocol we define the past domain of dependence denoted $D^-(S)$ and, likewise, all points on S, and only points on S, are reached by some future-directed timelike curve emanating from $p \in D^-(S)$. Therefore the whole domain of dependence, or simply domain of dependence, is given by $D(S) = D^+(S) \cup D^-(S)$. When $D(S) = M$ we say that S is a *Cauchy surface*; the manifold M is, in turn, said to be *Globally Hyperbolic* and all events on M can be *completely* determined by the events on the surface S; this last being the closest we have of a "instant of time" in general, curved manifolds where we can define "initial" value problem, since from data on S all, say, matter fields evolution are uniquely determined.

We have started our analysis with an arbitrary Lorentzian manifold which, after some more

[2]It is an immediate fact that $I^a \subset J^a$, $a = +, -$.

[3]whose complement is an open set

than reasonable requirements, was constrained to fulfill some conditions in order to serve as a background for a well-posed initial value formulation of matter fields; namely, $(M, g_{\alpha\beta})$ must be globally hyperbolic, or equivalently have a *Cauchy surface*, a property we hereafter assume it to attain.

1.2 Flow of Time and Hilbert Space Construction

Now that we have a way to characterize a well behaved background in the sense of being possible to formulate physical laws on it, we are still left with the problem of finding a realization of the concepts described in the last section. To start, we seek a derivative operator, which we will denote ∇_μ, that can be used unambiguously at any point of M and thus must be independent of any coordinates. It turns out that the operator ∇_μ is highly non unique, requiring us to impose some condition on it to pick one, and only one, of them. Since we are working with a manifold with a metric tensor defined everywhere, there is a natural choice: we can impose the condition of metric compatibility, namely, by requiring that $\nabla_\mu g_{\alpha\beta} = 0$, a theorem [13] states that we are able to filter out all possibilities of such derivative operator being left with a unique option. This is the one we choose to work with.

As already mentioned, a globally hyperbolic spacetime $(\mathcal{M}, g_{\alpha\beta})$ with Cauchy surface Σ has some resemblance to the idea of a space at a fix instant of time, in fact the topology of $\mathcal{M}$ is $\mathbb{R} \times \Sigma$, allowing us to "slice", or "foliate" the whole manifold with spacelike surfaces Σ_t parameterized by some continuous parameter $t \in \mathbb{R}$; by running t throughout the real numbers, and being each Σ_t Cauchy, we can reconstruct $\mathcal{M}$. Given a smooth scalar field $\hat{t} : \mathcal{M} \longrightarrow \mathbb{R}$, we have

$$\Sigma_t = \left\{ p \in \mathcal{M} | \hat{t}(p) = t \right\}. \tag{1.1}$$

Since if $\mathcal{M}$ is n-dimensional, for each $t \in \mathbb{R}$, $\Sigma_t \subset \mathcal{M}$ is an embedded manifold of dimension $(n-1)$, the tangent bundle[4] $\mathcal{T}_p(\Sigma_t)$ is a subspace of $\mathcal{T}_p(\mathcal{M})$, and there will be a 1-form [5]

[4] Collection of all tangent vector spaces.
[5] Unique up to scaling

$d\hat{t} \in \mathcal{T}_p^*(\mathcal{M})$[6], the gradient of $\hat{t}$, that maps all vectors tangent to Σ_t to 0. The vector $g^{\mu\nu}(d\hat{t})_\nu$ is said to be normal to the surface Σ_t.

Consider, then, a "time-direction" vector field t^μ on $\mathcal{M}$ satisfying $t^\mu(d\hat{t})_\mu = 1$. We can decompose t^μ as

$$t^\mu = Nn^\mu + T^\mu, \tag{1.2}$$

where N is a function on M, n^μ is a unit timelike vector normal to Σ_t, and in the same sense, T^μ is tangential to Σ_t.

From (1.2) we can define a projector operator $P^\mu{}_\nu$ whose action is to remove components parallel to n^μ from any vector it acts upon, i.e., $P^\mu{}_\nu$ projects vectors into Σ_t. It can easily be written as

$$\begin{aligned} P^\mu{}_\nu &= \delta^\mu{}_\nu + n^\mu n_\nu \\ \Rightarrow g^{\mu\sigma}P_{\sigma\nu} &= g^{\mu\sigma}g_{\sigma\nu} + g^{\mu\sigma}n_\sigma n_\nu \\ \Rightarrow P_{\mu\nu} &= g_{\mu\nu} + n_\mu n_\nu; \end{aligned} \tag{1.3}$$

and since

$$P_{\mu\nu}V^\mu V^\nu = g_{\mu\nu}V^\mu V^\nu \tag{1.4}$$

if V^μ, W^μ are already tangent to Σ_t, $P_{\mu\nu}$ is also called the spatial metric, even though it still is a 2-form acting on the tangent spaces of $\mathcal{M}$. It is worth noting that $g_{\mu\nu}$ can, in fact, be thought as a true metric acting on $T_p(\Sigma_t)$ if we use a map $\phi_t : \mathcal{M} \longrightarrow \Sigma_t$ to pull it back to the sub manifold Σ_t; the result is called induced metric, γ_{ij}, and we will refer to it later [15]; details can be found in A. We now have the basic ingredients to start our formulation.

In what follow, we work in close analogy to Ashtekar and Magnon [14], as their formulation does not rely on the concept of particles -which seems to be present only in very specific cases, and our aim is to be as general as possible. For simplicity, we will work with a real scalar

[6]This is the cotangent bundle, the collection of all cotangent vector spaces.

matter field[7], more specifically, an action of the form

$$S = -\frac{1}{2}\int_{\mathcal{M}} \left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + m^2\phi^2\right)\epsilon, \tag{1.5}$$

where ϵ is the n-form volume element associated with $(\mathcal{M}, g_{\mu\nu})$. The equation of motion obtained from extremization of (1.5) is the coordinate independent Klein-Gordon equation [19]

$$\nabla^\mu\nabla_\mu\phi - m^2\phi = 0. \tag{1.6}$$

Note that up to this point everything was done truly coordinate independently -in the sense that all of the above formulae are valid regardless of the coordinate system defined on the manifold. We need to be more specific now. Let us define local coordinates (x^1, x^2, x^3)[8] on the hypersurface Σ_t. If the just mentioned set of coordinates changes smoothly from surface to surface, this implies that (t, x^1, x^2, x^3), with $t = \hat{t}(p)$, is a coordinate system at each $p \in \mathcal{M}$ with associated basis vectors $(\partial_{(t)}, \partial_{(1)}, \partial_{(2)}, \partial_{(3)})$[9]. Considering the vector field discussed before, and requiring $t^\mu(dx^{(\alpha)})_\mu = 0$ for $\alpha = 1, 2, 3$, it follows that $t^\mu = (\partial_{(t)})^\mu$, i.e, t^μ is a coordinate basis in this system and orthogonal to all other axis. To see explicitly that the basis coordinate 1-form dt maps vectors in Σ_t to 0, note that given a family of maps $\Phi_t : \Sigma_t \longrightarrow \mathcal{M}$, $t \in \mathbb{R}$, and $v \in \mathcal{T}_p(\Sigma_t)$, it is immediate that

$$dt((\Phi_t)_*(v)) = 0, \tag{1.7}$$

where $(\Phi_t)_* : \mathcal{T}_p(\Sigma_t) \longrightarrow \mathcal{T}_p(\mathcal{M})$ is the pushforward map, and the unfamiliar reader is again advised to read Appendix A

Finally, rewriting (1.5) in terms of our newly defined coordinate system (t, x^1, x^2, x^3)

$$S = \frac{1}{2}\int_{\mathcal{M}} \left[(n^\mu\nabla_\mu\phi)^2 - P^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - m^2\phi^2\right]\epsilon, \tag{1.8}$$

[7] In fact, there seems to be no natural generalization of arbitrary spin fields to curved spacetimes [13]

[8] We restrict ourselves to 4 dimensions as well. The generalization to higher dimension is straightforward.

[9] The parenthesis on the indices indicate we are talking about vectors, and not their components; so $\partial_{(\alpha)} = \delta^\sigma{}_\alpha\partial_{(\sigma)}$ and $dx^{(\alpha)} = \delta^\alpha{}_\sigma dx^{(\sigma)}$

with $\epsilon = N\sqrt{\gamma}\, dt \wedge dx^1 \wedge dx^2 \wedge dx^3$ -being $\wedge$ the wedge product- where $\gamma > 0$ is the determinant of the induced Riemannian metric γ_{ij} on Σ_t, and $P^{\mu\nu}$ the projector discussed above. To put (1.8) in a more "conventional" form for the eye of one who works with flat spacetime physics, we write

$$S = \int_{\mathbb{R}} L(t)dt, \tag{1.9}$$

with the Lagrangian given by

$$L(t) = \frac{1}{2}\int_{\Sigma_t} \left[(n^\mu \nabla_\mu \phi)^2 - P^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - m^2\phi^2\right] N\sqrt{\gamma}\, d^3x. \tag{1.10}$$

Expressing n^μ in terms of t^μ and T^μ by use of (1.2) one gets, from the definition of conjugate momentum π,

$$\pi = \frac{\delta S}{\delta \dot{\phi}} = \sqrt{\gamma}(n^\mu \nabla_\mu \phi) \tag{1.11}$$

is the momentum density [6]. At this point we are almost in position to canonically quantize our theory since we already have a phase space $\mathfrak{M}$ defined by a given pair $[\phi, \pi]$ of smooth functions specified on a Cauchy surface, say $\Sigma_{t=0}$[10], both mapping Σ_0 to the real numbers[11].

$$\mathfrak{M} \equiv \{[\phi,\pi] | \phi : \Sigma_0 \longrightarrow \mathbb{R}, \pi : \Sigma_0 \longrightarrow \mathbb{R}; \phi, \pi \in C_0^\infty(\Sigma_0)\}\,. \tag{1.12}$$

Let $\mathcal{S}$ be the vector-space of real-valued solutions to the Klein-Gordon equation (1.6). One can prove that given a pair of smooth functions $[\phi_0, \pi_0]$ specified on a Cauchy surface Σ, then there is a unique solution $\phi \in \mathcal{S}$ such that $\phi = \phi_0$ and $\sqrt{\gamma} n^\mu \nabla_\mu \phi = \pi_0$ on Σ. Due to this fact we can identify both spaces, and work with the space of solution $\mathcal{S}$, also independent of Σ, as an alternative to the phase space $\mathfrak{M}$ [12].

Keeping in mind our theory is still a classical one, and taking into account the sympletic nature of the space of solution of Hamiltonian systems [16], we define, by analogy, a non

[10] In fact, the space of solutions will be independent of this Cauchy surface, as long as it is Cauchy [13]

[11] C_0^∞ is a function that can be differentiated infinitely many times, with all derivatives being continuous.

degenerate 2-form: a bilinear map[12] $\Omega : \mathcal{S} \times \mathcal{S} \longrightarrow \mathbb{R}$

$$\begin{aligned}\Omega(\phi_1, \phi_2) &= \int_{\Sigma_0} (\pi_1 \phi_2 - \pi_2 \phi_1) d^3x \\ &= \int_{\Sigma_0} (\phi_2 n^\mu \nabla_\mu \phi_1 - \phi_1 n^\mu \nabla_\mu \phi_2) \sqrt{\gamma} d^3x,\end{aligned} \tag{1.13}$$

with $\phi_1, \phi_2 \in \mathcal{S}$, where in the last line we have used the definition (1.11). Using Stoke's theorem it is easy to check that the above statement is independent of the chosen slice Σ_0. We will follow a somewhat inverted approach where we first define the operators and then construct the space of quantum solutions. With this goal in mind, to each element $\phi \in \mathcal{S}$ we associate an operator $F(\phi)$ that generates a $*$-algebra, the algebra of quantum mechanical observables. Besides the properties of a $*$-algebra, we further require these operators to satisfy

self-adjointness:	$F(\phi) = F^*(\phi)$	(1.14a)
linearity:	$F(\phi_1 + r\phi_2) = F(\phi_1) + rF(\phi_2)$	(1.14b)
commutation relations:	$[F(\phi_1), F(\phi_2)] = -i\Omega(\phi_1, \phi_2)\mathbb{1}$	(1.14c)

with $\phi, \phi_1, \phi_2 \in \mathcal{S}$, and $r \in \mathbb{R}$. Note how we are essentially creating a map from operations done on the classical solutions to operations on the operators that generates the $*$-algebra, which if identified with the quantum observables position and momentum, respectively, then condition (1.14c) is nothing but the usual canonical quantization scheme, $\{\cdot, \cdot\} \to -i[\cdot, \cdot]$, where one relates the commutator to the Poisson bracket. Indeed, let $\Phi \in \mathcal{S}$ be a solution to the classical Klein-Gordon equation, choose any Cauchy hypersurface, say Σ_t, with fixed t, and let $\phi_1, \phi_2 \in \mathcal{S}$ be such that the point $(\phi_2, \pi_2) = (f, 0)$ and $(\phi_1, \pi_1) = (0, g)$, with $f = \Phi|_{\Sigma_t}$, and $g = \sqrt{\gamma} n^\mu \nabla_\mu \Phi|_{\Sigma_t}$ just as discussed below (1.12). Denoting $F(\phi_2)$ by $\Pi(f)$ and $F(\phi_1)$ by

[12] To be precise, this operator acts on $\mathcal{M} \times \mathcal{M}$. Thus, for a given point $[x, y] \in \mathfrak{M}$, $\Omega([x, y], \cdot)$ is an arbitrary linear combination of the dynamical variables.

$\Phi(g)$ equation (1.14c) yields

$$[\Phi(g), \Pi(f)] = i\Omega(\phi_1, \phi_2) = i \int_{\Sigma_t} f(\vec{x}) g(\vec{x}) d^3x. \tag{1.15}$$

Here f, g are just test functions[13]. This is just the mathematically well-defined distribution version of the more familiar equal-time commutation relation to which it gives meaning:

$$[\Phi(t, \vec{x}_1), \Pi(t, \vec{x}_2)] = i\delta^3(\vec{x}_1 - \vec{x}_2). \tag{1.16}$$

To see why, note that $\Phi(t, \vec{x})$ is just a shorthand notation to $\Phi(g) = \int_{\Sigma_t} \Phi(t, \vec{x}) g(\vec{x}) d^3x$, the same goes with $\Pi(t, \vec{x})$, that is, $\Phi(g)$ as well as $\Pi(f)$ are true operators, while the former ones are operator-valued distribution and must, as such, be integrated against appropriate test functions. Thus

$$\begin{aligned} [\Phi(g), \Pi(f)] &= \int_{\Sigma_t} [\Phi(t, \vec{x}), \Pi(t, \vec{y})] f(\vec{y}) g(\vec{x}) d^3x d^3y \\ &= \int_{\Sigma_t} \left(i\delta^3(\vec{x} - \vec{y}) \right) f(\vec{y}) g(\vec{x}) d^3x d^3y \\ &= i \int_{\Sigma_t} f(\vec{x}) g(\vec{x}) d^3x, \end{aligned} \tag{1.17}$$

where in the second step we have plugged in equation (1.16). It is worth emphasizing that the above is not a derivation, but rather a justification of why equation (1.16) is used in place of the more formal (1.15). The other commutators is similarly written as

$$[\Phi(g), \Phi(g')] = 0 \ \text{ and } \ [\Pi(f), \Pi(f')] = 0, \tag{1.18}$$

or yet

$$[\Phi(t, x_1), \Phi(t, x_2)] = 0 \ \text{ and } \ [\Pi(t, x_1), \Pi(t, x_2)] = 0. \tag{1.19}$$

We have constructed what really looks like a set of observables out of some very formal procedure starting from the classical theory. We now construct the so awaited Hilbert space $\mathcal{H}$ of states, and we do so by equipping the space of classical solutions with some extra structures.

[13] See equation (1.12)

Even though we do not talk about particles, being that concept ill-defined, we will consider, as it is standard, that our field-operators are a sum of creating and annihilation operators acting on some Fock space; what they create or annihilate is not of interest right now, these are just names we give them. This Fock space, in turn, is build out of some *first* Hilbert space, on which we focus now.

To begin with, we demand $\mathcal{S}^{\mathbb{C}}$, the complexification of $\mathcal{S}$, as a real vector space, to be an identical copy of $\mathcal{S}$. Not forgetting that $\mathcal{S}$ is a space with real scalar multiplication only, we must first introduce in it a complex structure; let this structure be named $J : \mathcal{S} \longrightarrow \mathcal{S}$. For J to be a complex structure means that $J^2 = -\mathbb{1}$. Note how this allows us to add complex scalar multiplication to our otherwise real vector space. If $x \in \mathbb{R}$ then, since the scalar multiplication by a real number is defined, $\Phi \in \mathcal{S} \Rightarrow x\Phi \in \mathcal{S}$. Now, let i be the imaginary unit, $y \in \mathbb{R}$ and consider the vector space endowed with our complex structure $(\mathcal{S}, J)$. We define multiplication by a complex scalar through

$$(x + iy)\Phi \longmapsto x\Phi + yJ(\Phi) \in (\mathcal{S}, J). \tag{1.20}$$

It is consistent for if we multiply the above by the conjugate of $x + iy$ we get

$$\begin{aligned}(x - iy)(x + iy)\Phi &\longmapsto (x - iy)(x\Phi + yJ(\Phi)) \\ &\longmapsto x^2\Phi + xyJ(\Phi) - xyJ(\Phi) - y^2J^2(\Phi) \\ &\longmapsto (x^2 + y^2)\Phi \in (\mathcal{S}, J),\end{aligned} \tag{1.21}$$

and we see that J is some sort of imaginary unit in our vector space. By further endowing this space with a Hermitian inner product $\langle\cdot|\cdot\rangle$ and Cauchy completing it, i.e, for any sequence $\{a_n\}_{n\in\mathbb{N}}$ with elements in $(\mathcal{S}, J, \langle\cdot|\cdot\rangle)$ if

$$\forall\varepsilon > 0\ \exists\ell \in \mathbb{N};\ m, n > \ell \Rightarrow \langle a_m - a_n | a_m - a_n\rangle < \varepsilon \tag{1.22}$$

then $a_n \to a$ converges and we demand that $a \in \left(\mathcal{S}^{\mathbb{C}}, J, \langle\cdot|\cdot\rangle\right)$. One can check that this space now enjoys all the properties of a Hilbert space. As we will see later, $J^\dagger = -J$, implying

$iJ : \mathcal{S}^{\mathbb{C}} \longrightarrow \mathcal{S}^{\mathbb{C}}$ is self-adjoint with eigenvalues $\pm i$. We can thus decompose $\mathcal{S}^{\mathbb{C}}$ into two orthogonal eigensubspaces, one associated with the eigenvalue $+i$ and the other with $-i$. Finally, the Hilbert space $\mathcal{H}$ is identified with the eigensubspace of eigenvalue $+i$. And a natural linear map $K : \mathcal{S}^{\mathbb{C}} \longrightarrow \mathcal{H}$ that is the orthogonal projector of $\mathcal{S}^{\mathbb{C}}$ into its eigensubspace $\mathcal{H} \subset \mathcal{S}^{\mathbb{C}}$, is naturally defined. One might also say this is the "one-particle space"; to whom we ask: what is it that you call "particle"? A Fock-space $\mathcal{F}(\mathcal{H})$ can be constructed from this *first* Hilbert space $\mathcal{H}$; it is given by

$$\mathcal{F}(\mathcal{H}) = \overline{\bigoplus_{k=0}^{\infty} S\mathcal{H}^{\otimes k}}, \tag{1.23}$$

where $\mathcal{H}^{\otimes 0} = \mathbb{C}$ is understood, $S\mathcal{H}^{\otimes k}$ is the symmetrised tensor product taken of $\mathcal{H}$ with itself k times, and the overbar indicates Cauchy completion just as discussed around (1.22).

As with general relativity we will use the index notation, with indices summation running from 0 to $\dim(\mathcal{H})-1$ following the usual Einstein notation, as it turns out to be clearer. Hence, the components of elements belonging to $\mathcal{H}$ and to the dual $\mathcal{H}^*$ will be denoted with one upper and one lower index, respectively, and the components of a map from tensor products of these to the complex numbers will be denoted correspondingly by a mix of upper and lower indices. An arbitrary element of the Fock space $\mathcal{F}$ is some object with finite nonzero entries and is thus expressed as $\Psi = (\xi^0, \xi^1, \xi^2, ..., \xi^k, ...)$, with $\xi^k \in S\mathcal{H}^{\otimes k}$, i.e, $\xi^0 \in \mathbb{C}$, $\xi^1 \in \mathcal{H}$, $\xi^2 \in \mathcal{H} \otimes_S \mathcal{H}$[14] and so on; the j in ξ^j is not a component index, but rather a label telling us to which $S\mathcal{H}^{\otimes j}$ it belongs - as such, its placement is not important, and we shall change its location at will not to coincide with component indices whose placement are of great importance. Define the operators $\mathcal{A}_\sigma, \mathcal{B}_\sigma : \mathcal{F} \longrightarrow \mathcal{F}$, for each $\sigma \in \mathcal{H}$, whose actions on elements $\Psi \in \mathcal{F}$ is demanded to satisfy

$$\mathcal{A}_\sigma \Psi = (\sigma_a \xi_1^a, \sqrt{2}\sigma_a \xi_2^{ab}, \sqrt{3}\sigma_a \xi_3^{abc}, ...). \tag{1.24}$$

Just as a side note, an element of $\eta \in \mathcal{H}^*$ can be contracted with an element $\omega \in \mathcal{H}$ using the Hilbert metric by means of $\eta_a^* \omega^a \in \mathbb{C}$ which is just the inner product, so we can use the

[14] $\otimes_S$ is just the symmetrised tensor product.

metric to lower or rise indices[15]. In the same fashion, the action of $\mathcal{B}_\sigma$ on Ψ is to obey

$$\mathcal{B}_\sigma \Psi = (0, \xi^0\sigma, \sqrt{2}\sigma \otimes_S \xi^1, \sqrt{3}\sigma \otimes_S \xi^2, \sqrt{4}\sigma \otimes_S \xi^3, ...). \tag{1.25}$$

It is clear now why this operators receive the names they do; $\mathcal{B}_\sigma$ maps a state from $\bigoplus_{k=0}^{j} S\mathcal{H}^{\otimes k}$ to another state in $\bigoplus_{k=0}^{j+1} S\mathcal{H}^{\otimes k}$, and the action of $\mathcal{A}_\sigma$ is the other way around. This is what they create or annihilate: the number of $\mathcal{H}$'s necessary to describe the state. From the very own definitions it follows that the operators are adjoint of one another, i.e $\mathcal{A}_\sigma = \mathcal{B}_\sigma^\dagger$, or yet $\langle \mathcal{B}_\sigma \cdot | \cdot \rangle = \langle \cdot | \mathcal{A}_\sigma \cdot \rangle$; also

$$[\mathcal{A}_\sigma, \mathcal{A}_\tau] = 0, \quad [\mathcal{B}_\sigma, \mathcal{B}_\tau] = 0, \quad \text{and} \quad [\mathcal{A}_\sigma, \mathcal{B}_\tau] = \langle \sigma | \tau \rangle \mathbb{1}. \tag{1.26}$$

To see that the above definition does indeed recover the notion of creation and annihilation operators on the Fock space of a single harmonic oscillator, let $a^\dagger$ be the creation operator associated with the state $\Psi_1 \in \mathcal{H}$, and $1 \in \mathbb{C}$ the vacuum state of the system, i.e, $\Psi_0 = (1, 0, 0, ...)$ in $\mathcal{F}$. From ordinary quantum mechanics we have that $\Psi_n = \frac{(a^\dagger)^n}{\sqrt{n!}}\Psi_0$, which from (1.25) results in

$$\begin{aligned}
&\Psi_0 = (1, 0, 0, ...) \coloneqq |0\rangle; \\
&\Psi_1 = \mathcal{B}_{\Psi_1}\Psi_0 = (0, \Psi_1, 0, 0, ...) \coloneqq |1\rangle; \\
&\Psi_2 = \frac{1}{\sqrt{2!}}(\mathcal{B}_{\Psi_1})^2\Psi_0 = 2^{-1/2}\mathcal{B}_{\Psi_1}\Psi_1 = (0, 0, \Psi_1 \otimes \Psi_1, 0, 0, ...) \coloneqq |2\rangle; \\
&\Psi_3 = \frac{1}{\sqrt{3!}}(\mathcal{B}_{\Psi_1})^3\Psi_0 = 3^{-1/2}\mathcal{B}_{\Psi_1}\Psi_2 = (0, 0, 0, \Psi_1 \otimes \Psi_1 \otimes \Psi_1, 0, 0, ...) \coloneqq |3\rangle
\end{aligned} \tag{1.27}$$

and so on. Of course, in this case we have only one type of state in $\mathcal{H}$ so the construction is trivially symmetric, although one can show that it generalizes in an immediate manner to the case where $\mathcal{H}$ is more generally populated, as it were.

Having defined these operators, we now make the last demand upon the whole structure we are building, namely, that the *field-operators* $F(\phi)$ associated with the classical solution

[15]This action will obviously also complex conjugate the object it acts upon, so that $\eta_a\omega^a = (\eta^a\omega_a)^*$ holds, as it must.

$\phi \in \mathcal{S}$ must be a sum of the operators $\mathcal{A}$ and $\mathcal{B}$ just discussed, specifically, we demand that $F(\phi) = \mathcal{A}_{\hat{\phi}} + \mathcal{B}_{\hat{\varphi}}$, being $\hat{\phi} \in \mathcal{H}$ the element associated with the classical counterpart ϕ. This seemingly soft requirement impose severe restrictions on the relation between the complex structure J and the 2-form Ω defined by (1.13). From the commutation relation satisfied by $F(\phi)$, equation (1.14c), we have

$$\begin{aligned}
[F(\phi), F(\varphi)] &= [\mathcal{A}_{\hat{\phi}} + \mathcal{B}_{\hat{\phi}}, \mathcal{A}_{\hat{\varphi}} + \mathcal{B}_{\hat{\varphi}}] \\
&= [\mathcal{A}_{\hat{\phi}}, \mathcal{B}_{\hat{\varphi}}] + [\mathcal{B}_{\hat{\phi}}, \mathcal{A}_{\hat{\varphi}}] \\
&= (\langle\hat{\phi}|\hat{\varphi}\rangle - \langle\hat{\varphi}|\hat{\phi}\rangle)\mathbb{1} \\
&= 2i\mathrm{Im}\{\langle\hat{\phi}|\hat{\varphi}\rangle\} = -i\Omega(\phi, \varphi).
\end{aligned} \tag{1.28}$$

here $\mathrm{Im}\{\cdot\}$ indicates the imaginary part. This allows us to decompose the inner product

$$\langle\hat{\alpha}|\hat{\beta}\rangle = \frac{1}{2}\omega(\alpha, \beta) - \frac{1}{2}i\Omega(\alpha, \beta) \qquad \forall\alpha, \beta \in \mathcal{S} \tag{1.29}$$

where $\omega(\cdot, \cdot)$ is a yet undefined real peace, and the multiplicative $1/2$ is just because "why not?". Interestingly enough, the result (1.28) -together with the fact that (1.29) is to satisfy the axioms of a Hilbert space inner product- completely fixes the inner product, and hence the real part can be determined. Just to remember, Ω is antisymmetric and ω is real, linear, and symmetric by definition. Linearity in the second slot of $\langle\cdot|\cdot\rangle$ yields

$$\omega(\alpha, J(\beta)) - i\Omega(\alpha, J(\beta)) = i\omega(\alpha, \beta) + \Omega(\alpha, \beta), \tag{1.30}$$

since ω is real, by comparing real and imaginary parts we find

$$\omega(\alpha, \beta) = -\Omega(\alpha, J(\beta)) - \Omega(\beta, J(\alpha)) \tag{1.31}$$

The last equality is due to the symmetry of ω. Finally

$$\langle\hat{\alpha}|\hat{\beta}\rangle = -\frac{1}{2}\Omega(\alpha, J(\beta)) - \frac{1}{2}i\Omega(\alpha, \beta) \qquad \forall\alpha, \beta \in \mathcal{S} \tag{1.32}$$

In terms of the map K, with $\hat{\alpha} = K\alpha \in \mathcal{H}$, and similarly for β, one defines

$$\langle K\alpha | K\beta \rangle = -\Omega((K\alpha)^*, K\beta) \qquad \forall \alpha, \beta \in \mathcal{S}, \tag{1.33}$$

where the restriction on J is incorporated in the definition of K itself.

The above result is of great importance. For one it shows that the Hermitian inner product defined on $\mathcal{H}$ is completely determined by the complex structure defined on $\mathcal{S}$. Secondly, since $\Omega(\cdot, J(\cdot))$ must be real, symmetric and positive definite, the complex structure itself is tied up to the 2-form Ω as to satisfy these conditions. Therefore, our freedom in choosing a Hilbert space related to the space of classical solutions and to the $*$-algebra we have constructed is the freedom we have upon the choice of J, and there is no reasons whatsoever to believe such a choice is unique. Looking carefully at the whole construction we can also see that the definition of the operators $\mathcal{A}$ and $\mathcal{B}$ are also somewhat fixed by J via the last equation in (1.26). In the next section we look at a specific case where the choice of a J can be said to be *natural*, of course, given our definition of "natural".

We emphasize that we have not considered our manifold to have any symmetry at all besides the existence of the smooth vector field (1.2) - indeed, the conserved map (1.13) arose due to symmetries of the Lagrangian and not of the spacetime itself, since it is valid on-shell only[16]. In special cases where the manifold enjoys the privilege of a specific family of isometries, the subspace $\mathcal{H}$ just mentioned may acquire a well known meaning: the space of positive frequency solutions. Enough of our digression. Now that we have clarified the irrelevance of the particle concept for the construction of the theory, we will turn to a more practical discussion: that concerning stationary spacetimes, where, ironically, a particle concept is available - even though not all observers will agree upon the definition.

[16] On-shell just means where the equation of motion (1.6) are satisfied.

1.3 Stationary Spacetimes and Positive-frequency Solutions

Well, in the last section a rather obscure formulation of quantum field theory was said to be well posed even in the absence of the simplest of all possible symmetries: time translation. That was done to clarify what is really fundamental in the construction, and therefore necessary and unambiguous, and what might be interpreted in one way or another given different procedures of observation. In this section we discuss how the concept of particle *may* arise if the spacetime possess some sort of time translation symmetry, and more importantly, even when this is the case, the concept is not unique and, as such, not fundamental. The so called Unruh effect, named after W. Unruh(1976) is a realization of the aforementioned fact, i.e., the arisen of the concept of particle.

Our starting point is now to assume that our spacetime has a somewhat more symmetrical structure, specifically, we require $(\mathcal{M}, g_{\mu\nu})$ to be endowed with a timelike Killing vector field χ^μ,

$$\mathcal{L}_\chi g = 0, \tag{1.34}$$

where $\mathcal{L}_\chi$ stands for the Lie derivative along χ^μ. If this is the case, then, by its very own definition, the metric tensor will not change if we move along orbits of the vector field χ^μ, that is to say that if we identify a parameter, say t, of the family of isometries $\Upsilon_t : \mathcal{M} \longrightarrow \mathcal{M}$, $t \in \mathbb{R}$, generated by χ^μ to be our time coordinate, then one cannot tell the difference between "moments" of time, and time evolution will only depend on the interval $\Delta t = t - t'$, independent of t, t'. What a symmetrical spacetime!

Related to internal symmetries of the action (1.8), and to the aforementioned Killing symmetry, there is a symmetric conserved tensor that goes by the name of stress-energy-momentum tensor, or simply energy-momentum tensor $T_{\mu\nu} = T_{\nu\mu}$,

$$T_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}g_{\mu\nu}\left(\nabla^\sigma\phi\nabla_\sigma\phi + m^2\phi^2\right) \qquad \forall\phi \in \mathcal{S}, \tag{1.35}$$

with $\nabla_\mu T_{\mu\nu} = 0$. Given this quantity we can calculate the energy of a given classical solution by means of

$$E(\phi) = \int_\Sigma \chi^\mu T_{\mu\nu} n^\nu \sqrt{\gamma} d^3x, \tag{1.36}$$

where the coordinate setup is as discussed in the previous section, n^ν is a unit timelike vector orthogonal to the hypersurface Σ as before, and $T_{\mu\nu}$ carries all dependence on the space $\mathcal{S}$. From the symmetry of $T_{\mu\nu}$, and the fact that χ^μ is Killing, it follows, by means of Stoke's theorem, that (1.36) is independent of the Cauchy surface Σ as it should be.

We go on and follow the whole process outlined in the last section in order to construct the Hilbert space $\mathcal{H}$ by endowing $\mathcal{S}$ with a complex structure J and an inner product $\langle\cdot|\cdot\rangle$. The difference now is that we can use the new existent symmetry to try and choose some physical relevant J, some structure that, as pointed in the last paragraph of the previous section, can be regarded as a "natural" choice. Note that, related to the any classical solution ϕ there is an element $\hat{\phi} \in \mathcal{H}$, and thus, related to the Lie derivative $\mathcal{L}_\chi \phi \in \mathcal{S}$ along χ^μ, by the Stone's theorem on representation of one-parameter groups [12]

$$H\hat{\phi} = -\hat{J}(\mathcal{L}_\chi \phi) \tag{1.37}$$

where $H : \mathcal{H} \longrightarrow \mathcal{H}$ is the generator of t translation along orbits of χ^μ, and $\hat{J}(\mathcal{L}_\chi\phi)$ is the Hilbert space version of i times $\mathcal{L}_\chi\phi$. The above equation is just like Schrödinger equation, since they arise in the same context of time translation, in fact, it is precisely the same equation, so it seems quite natural to call our time translation generator H the energy of the system, just as is the case in ordinary quantum mechanics. With this identification, we can regard $\langle\hat{\phi}|H\hat{\phi}\rangle$ as the energy of the state $\hat{\phi} \in \mathcal{H}$. This is still very ambiguous since $\langle\cdot|\cdot\rangle$ depends on Ω and J. To fix this ambiguity we require further the very "natural" condition

$$E(\phi) = \langle\hat{\phi}|H\hat{\phi}\rangle \qquad\qquad \forall\phi \in \mathcal{S}, \tag{1.38}$$

i.e, we want the expectation value of our "energy operator" in the state $\hat{\phi}$ to always equals the

energy of the classical field $E(\phi)$. To see how this imposes a complex structure, let us define the inner product on $\mathcal{S}$ by

$$(\phi_1, \phi_2) = \int_\Sigma \chi^\mu T_{(\mu\nu)}(\phi_1, \phi_2) n^\nu \sqrt{\gamma} d^3x. \tag{1.39}$$

This is just a generalization of the energy-momentum tensor (1.35) that can be evaluated in two different solutions, it is a good candidate for a conserved inner product and is defined for the sole reason of yielding us a complex structure fulfilling all our demands; that is why we symmetrise it, the first term in the original expression might fail to be symmetric since in general $\nabla_\mu\phi_1\nabla_\nu\phi_2 \neq \nabla_\mu\phi_2\nabla_\nu\phi_1$, and we need T to be symmetric for the integral to be independent of the chosen hypersurface and conserved. Since only the first term in (1.35) has a potentially nonsymmetrical part, we write

$$(\phi_1, \phi_2) = \frac{1}{2}\int_\Sigma \left[\nabla_\mu\phi_1\nabla_\nu\phi_2 + \nabla_\mu\phi_2\nabla_\nu\phi_1 - g_{\mu\nu}\left(\nabla^\sigma\phi_1\nabla_\sigma\phi_2 + m^2\phi_1\phi_2\right)\right]\chi^\mu n^\nu\sqrt{\gamma}d^3x, \tag{1.40}$$

where we have symmetrised the first term of the integrand.

Let us take a closer look at the first term only, and see how it behaves if we apply the Lie derivative to one of its arguments. Remembering that $\mathcal{L}_\chi = \chi^\mu\nabla_\mu$ we obtain

$$\begin{aligned}\left[\nabla_\mu\phi_1\nabla_\nu(\mathcal{L}_\chi\phi_2)\right]\chi^\mu &= (\chi^\mu\nabla_\mu\phi_1)\nabla_\nu(\mathcal{L}_\chi\phi_2)\\ &= (\mathcal{L}_\chi\phi_1)\nabla_\nu(\mathcal{L}_\chi\phi_2)\\ &= -(\mathcal{L}_\chi\phi_2)\nabla_\nu(\mathcal{L}_\chi\phi_1) + \nabla_\nu[(\mathcal{L}_\chi\phi_1)(\mathcal{L}_\chi\phi_2)],\end{aligned} \tag{1.41}$$

and upon integration

$$\int_\Sigma \left[\nabla_\mu\phi_1\nabla_\nu(\mathcal{L}_\chi\phi_2)\right]\chi^\mu n^\nu\sqrt{\gamma}d^3x = \int_\Sigma \left[\nabla_\mu\phi_2\nabla_\nu(\mathcal{L}_\chi\phi_1)\right]\chi^\mu n^\nu\sqrt{\gamma}d^3x, \tag{1.42}$$

the last term vanishing because the derivatives fall-off at spatial infinity; we literally integrated by parts and discarded the surface terms. If we repeat a similar calculation to the other terms in (1.40) we find

$$(\phi_1, \mathcal{L}_\chi\phi_2) = -\left(\mathcal{L}_\chi\phi_1, \phi_2\right). \tag{1.43}$$

This means $\mathcal{L}_\chi$ is an antisymmetrical operator with respect to the inner product $(\cdot,\cdot)$ and therefore $\Theta = -\mathcal{L}_\chi\mathcal{L}_\chi$ is self-adjoint by immediate use of equation (1.43) twice. At last, by Cauchy completing the space $(\mathcal{S},(\cdot,\cdot))$ we have a natural complex structure defined on it, namely,

$$J = -\Theta^{-1/2}\mathcal{L}_\chi \tag{1.44}$$

The above can be show to be the *only* complex structure compatible with Ω in the sense of (1.32) and with equation (1.44) [14]. To see it is indeed a complex structure, note that $[\Theta,\mathcal{L}_\chi]=0$ and therefore $[\Theta^{-1/2},\mathcal{L}_\chi]=0$ also holds, then

$$\begin{aligned} J^2 &= \Theta^{-1/2}\mathcal{L}_\chi\Theta^{-1/2}\mathcal{L}_\chi = \Theta^{-1/2}\Theta^{-1/2}\mathcal{L}_\chi\mathcal{L}_\chi \\ &= \Theta^{-1}\mathcal{L}_\chi\mathcal{L}_\chi = -\mathbb{1}, \end{aligned} \tag{1.45}$$

in the last line we just employed the definition of Θ. The final observation we make is the obvious dependence of J on the Killing vector field χ^μ, therefore, if more than one option of such symmetries are available, more than one J can be chosen, and a different creation and annihilation operators defined. Anyway, given one vector field, the choice is unique and the vacuum expectation value stable, i.e, it will not evolve under evolution along orbits of the chosen Killing timelike vector, even though it might disagree with different choices. Moreover, if $(\mathcal{M},g_{\mu\nu},\chi^\mu)$ is static in addition to stationary [17], then there is a decomposition of any classical solution in positive and negative-frequency parts, i.e, $\phi=\phi^+ + \phi^-$, with $\phi^- = (\phi^+)^*$, where

$$\mathcal{L}_\chi\phi^+ = -i\omega\phi^+, \tag{1.46}$$

is a positive-frequency solution of frequency ω. Back to the definition (1.44), we see that $\phi^\pm$ are eigenfunctions of J, thus $J\phi^+ = -(-i\omega)/[-(-i\omega)^2]^{1/2}\phi^+ = i\phi^+$ and $J\phi^- = -(i\omega/|\omega|)\phi^- =$

[17] Static meaning the metric is independent of the time variable.

$-i\phi^-$. Denoting $K\phi = \hat{\phi}^+ \in \mathcal{H}$,

$$\begin{aligned}\langle\hat{\alpha}|\hat{\beta}\rangle &= -\frac{1}{2}\Omega(\alpha, J(\beta^+ + \beta^-)) - \frac{1}{2}i\Omega(\alpha, \beta^+ + \beta^-) \\ &= -\frac{1}{2}\Omega(\alpha, i\beta^+ - i\beta^-) - \frac{1}{2}i\Omega(\alpha, \beta^+ + \beta^-) \\ &= -i\Omega(\alpha, K\beta) = -i\Omega((K\alpha)^*, K\beta),\end{aligned} \tag{1.47}$$

we have dropped the term $-i\Omega(K\alpha, K\beta)$ because of the property $\langle\hat{\alpha}|\hat{\beta}\rangle = \langle\hat{\beta}|\hat{\alpha}\rangle^*$. Finally, in a static spacetime

$$\langle\hat{\alpha}|\hat{\beta}\rangle = \langle K\alpha|K\beta\rangle = -i\Omega((K\alpha)^*, K\beta) \tag{1.48}$$

is just the usual Klein-Gordon inner product. If $\alpha = \beta$ is a mode whose positive-frequency is given by $\sigma > 0$, then

$$\begin{aligned}\langle\hat{\alpha}|\hat{\alpha}\rangle &= -i\Omega((K\alpha)^*, K\alpha) = -i\Omega(\alpha^-, \alpha^+) \\ &= \int_\Sigma \left[\sigma|\alpha^+|^2\right]\sqrt{\gamma}d^3x \geq 0\end{aligned} \tag{1.49}$$

Note that $\langle\hat{\alpha}|\hat{\alpha}\rangle = 0 \Leftrightarrow \alpha = 0$.

We end this section, and the chapter, with a disclaimer: the theory developed up to this point is a mathematical view on why such phenomena as the Unruh effect arises. It is, therefore, not obliged in anyway to match experimental data. What we have done were just to construct quantum mechanics in the most general way possible, without relying on any specific symmetry that might be present in special cases alone, and in a way that would recover that formulation in the correspondingly limiting cases we already have a good grasp of, since we must recover from ours any description already known to be in consistent agreement with experiments. By doing so we avoid unnecessary assumptions that in general lead to great misconceptions, such as the notion that the particle concept is absolute, and can be more certain of the final result, even though the whole concept might, in the future, be shown to lack physical meaning, i.e, we are, at this point, given the absence of a good theory to work in the regimes we are proposing to work, worried with mathematical consistency, since in a sense, due to the technological difficulty of getting measurement data from some predicted

phenomena, it is our sole guide in this journey.

Chapter 2

The Unruh Effect

Given all the very carefully constructed theory of the last chapter to outline the fundamental quantities in a possible descriptions of matter fields in curved spacetimes, we now turn to a case where a slightly different situation occurs: we are given a time-orientable globally hyperbolic spacetime $(\mathcal{M}, g_{\mu\nu}, t^\mu)$ where not only different choices of Killing fields t^μ are available, but different choices might lead to different spacetimes via Killing horizon. No pragmatical concept of particle, such as something detected by a "particle detector", will be introduced here. We will just analyze the observables behavior under different choices of a everywhere timelike Killing vector field, and as a consequence, we restrict ourselves to the study of matter field whose background is stationary, in fact, static. In those cases there is a natural choice of inner product, as discussed in the previous section, and to avoid redundant information, $\langle\cdot|\cdot\rangle$ in this chapter stands for

$$\langle\phi_1|\phi_2\rangle = -i\int_{\Sigma_0}(\pi_1\phi_2 - \pi_2\phi_1)d^3x \tag{2.1}$$

for any two solutions $\phi_1, \phi_2 : \mathcal{F} \longrightarrow \mathcal{F}$, where $\mathcal{F}$ is the Fock space being discussed, and $\pi = \sqrt{\gamma}n^\mu\nabla_\mu\phi$ the conjugate momentum.

2.1 Minkowski Observers

Consider then the spacetime $(\mathcal{M}, g, t)$, with coordinates (t, x, y, z), such that the metric is given by

$$g = -dt \otimes dt + dx \otimes dx + dy \otimes dy + dz \otimes dz. \tag{2.2}$$

Note that $t^\mu = (\partial_{(t)})^\mu$ is an everywhere timelike[1] Killing vector field; simply due to the fact the above metric has no dependence on t. We can thus write the equation of motion for the theory (1.5),

$$\left[-\partial_t^2 + \partial_x^2 + \partial_y^2 + \partial_z^2 - m^2\right] \phi = 0, \tag{2.3}$$

choose an orthonormal basis of positive-frequency solutions $\{F_\ell\}$, i.e,

$$\mathcal{L}_t F_\ell = \partial_t F_\ell = -i\omega_\ell F_\ell, \tag{2.4}$$

satisfying

$$\begin{aligned} \langle F_i | F_j \rangle &= -\langle f_i^* | f_j^* \rangle = \delta_{ij} \\ \langle f_i | f_j^* \rangle &= 0 \end{aligned} \tag{2.5}$$

where $\omega_\ell > 0$ is the frequency with which F_ℓ oscillates in time, and finally, expand any solution to the equation (2.3) as

$$\phi = \sum_i \left[F_i a_M(F_i) + F_i^* a_M^\dagger(F_i) \right]. \tag{2.6}$$

Here the coefficients $a_M(F_i)$ and $a_M^\dagger(F_i)$ are the well known annihilation and creation operators associated with the mode F_i, respectively. From equation (2.5) one gets

$$\begin{aligned} a_M(F_i) &= \langle F_i | \phi \rangle \\ a_M^\dagger(F_i) &= -\langle F_i^* | \phi \rangle. \end{aligned} \tag{2.7}$$

[1] Indeed $g(\partial_{(t)}, \partial_{(t)}) = -1 < 0$.

One example of normalized positive-frequency solution is given by [19]

$$F_i = \frac{1}{\sqrt{(2\pi)^3 2\omega_i}} e^{-i\omega_i t + i\vec{k}.\vec{x}} \tag{2.8}$$

with $\vec{k}.\vec{x} = xk_x + yk_y + zk_z$.

It is worth noting that $\phi(t, \vec{x})$ is a Heisenberg operator-valued distribution, and its corresponding Schrödinger picture operator is obtained by integration against a test function as in the last chapter. This is pretty standard and nothing alarming has yet appeared.

2.2 Rindler Observers

We now consider the very same spacetime as in the last section but choose a different chart of coordinates. Set local coordinates (η, ξ, x, y) so that

$$g = -e^{2a\xi}(a^{-2} d\eta \otimes d\eta - d\xi \otimes d\xi) + dx \otimes dy + dz \otimes dy, \tag{2.9}$$

with $a > 0$ constant, $-\infty < \eta, \xi, x, y < \infty$. In terms of the previous coordinates we have

$$t = a^{-1} e^{a\xi} \sinh(\eta) \qquad z = a^{-1} e^{a\xi} \cosh(\eta) \qquad x = x \quad \text{and} \quad y = y. \tag{2.10}$$

This set of coordinates only cover one-quarter of the whole $\mathcal{M}$, the so called right Rindler wedge $I = \{z \in \mathcal{M}; z > |t|\}$. We can use very similar coordinates, with $t \to -t, z \to -z$, and identical ranges to cover the left Rindler wedge $II = \{z \in \mathcal{M}; z < |t|\}$[2]. The proper time

[2]The only difference being the Killing field direction, which is opposite in the regions I and II.

along a trajectory of constant ξ, x and y, say, $\gamma(\lambda') = (\lambda', \xi_0, x_0, y_0)$, is just

$$\begin{aligned}
\tau &= \int_0^\eta \left(-g_{\mu\nu}\frac{dx^\mu}{d\lambda'}\frac{dx^\nu}{d\lambda'}\right)^{1/2} d\lambda' \\
&= \int_0^\eta \left(-g_{\eta\eta}\frac{d\eta}{d\lambda'}\frac{d\eta}{d\lambda'}\right)^{1/2} d\lambda' \\
&= \int_0^\eta \left(\frac{e^{2a\xi_0}}{a^2}\right)^{1/2} d\lambda' \\
&= e^{a\xi_0}\frac{\eta}{a} \\
&\Rightarrow \eta(\tau) = \tau a e^{-a\xi_0},
\end{aligned} \tag{2.11}$$

that is, η is the proper time along the trajectory γ up to a linear factor. Tangent to this trajectory is the vector

$$\frac{d\gamma(\lambda')}{d\lambda'} = (1,0,0,0) = \partial_\eta, \tag{2.12}$$

that is, γ are orbits of the field $b = \partial_\eta$, which is also a Killing vector by the immediate absence of η in the metric (2.9)[3]. Its squared norm is[4]

$$\begin{aligned}
g(b,b) &= -a^{-2}e^{2a\xi}\left(d\eta\left(\frac{\partial}{\partial\eta}\right)\right)^2 + g_{ii}\left(dx^i\left(\frac{\partial}{\partial\eta}\right)\right)\left(dx^i\left(\frac{\partial}{\partial\eta}\right)\right) \\
&= -a^{-2}e^{2a\xi} < 0,
\end{aligned} \tag{2.13}$$

i.e, b^μ is a everywhere timelike Killing vector field in our manifold, meaning our spacetime is static and we can we use b^μ as the t^μ of the last section. Trajectories like the ones in (2.12) are orbits of the vector b^μ, thus, we can think of an acceleration related to the orbits, whose only non vanishing component reads

$$\begin{aligned}
A^\xi &= u^\nu\nabla_\nu u = (ae^{-a\xi})b^\nu\nabla_\nu[(ae^{-a\xi})b^\xi] \\
&= (a^2e^{-2a\xi})(\partial_\eta b^\xi + \Gamma^\xi_{\eta\mu}b^\mu) \\
&= a^2e^{-2a\xi}\Gamma^\xi_{\eta\eta} = ae^{-2a\xi} \\
&\Rightarrow |A(\xi)| = \sqrt{g(A,A)} = ae^{-2a\xi}.
\end{aligned} \tag{2.14}$$

[3]It is important not to mix things up; $\partial_{(\eta)}$ is a vector, not its components, and were it to be expanded in the coordinate basis of the tangent space at some point of our manifold we would write $\partial_{(\eta)} = \delta^\mu{}_\nu\partial_{(\nu)}$.

[4]As is standard, Greek indices run from 0 to 3 and Latin indices from 1 to 3.

here $\Gamma^{\xi}_{\eta\mu} = a^{-1}$ is the Christoffel symbol associated with the metric (2.9), and $u = (ae^{-a\xi})b$ is the four-velocity, $g(u,u) = -1$. This very simple result tell us that every observer with constant ξ experiment constant acceleration, moreover, were ξ to equal 0 then a is but the magnitude of the proper acceleration, and the convenience of the coordinates chosen is revealed. We are then led to work with trajectories like $\gamma(\tau) = (\tau a, 0, x_0, y_0)$, parameterized by the proper time τ, which describe the motion of a uniformly accelerated observer with proper acceleration of magnitude a, and as before, a, x_0 and y_0 are constants. Note also that $g(a,u) = 0$, meaning the acceleration vector is orthogonal to the velocity.

Interestingly enough, even though the right Rindler wedge is just one-fourth of the Minkowski spacetime, it is a static, globally hyperbolic spacetime, and as such, (1.5) is well defined here as well. Let us quantize the Klein-Gordon field (1.6) in this portion of $\mathcal{M}$; it is worth noting that the metric (2.9) has no curvature whatsoever, with the Ricci scalar $R = 0$ vanishing: we are, as we should, in flat spacetime. Equation (1.6) in the coordinates (η, ξ, x, y) reads

$$a^2\partial_\eta^2\phi = \left[\partial_\xi^2 + e^{2a\xi}\left(\partial_x^2 + \partial_y^2\right) + e^{2a\xi}m^2\right]\phi \tag{2.15}$$

or yet, in terms of τ

$$\partial_\tau^2\phi = \left[\partial_\xi^2 + e^{2a\xi}\left(\partial_x^2 + \partial_y^2 + m^2\right)\right]\phi. \tag{2.16}$$

From now on we use the rescaled coordinates (τ, ξ, x, y) whose convenience was already clarified. The metric is just

$$g = -e^{2a\xi}(d\tau \otimes d\tau - d\xi \otimes d\xi) + dx \otimes dy + dz \otimes dy. \tag{2.17}$$

Note that (2.16) is separable, meaning the complete solution can be expressed as a product of a spatial only and time only dependent parts, we make use of this feature below. To decompose the field in terms of positive and negative frequencies in the right Rindler wedge, we use the

coordinate independent definition of positive-frequency along a Killing field to be given by

$$\mathcal{L}_b f_{I\omega} = b^\mu \nabla_\mu f_{I\omega} = -i\omega f_{I\omega}. \tag{2.18}$$

In our coordinates, since $b = (1, 0, 0, 0)$, $b^\mu \nabla_\mu = \partial_\tau$. Thus

$$\partial_\tau f_{I\omega} = -i\omega f_{I\omega}, \tag{2.19}$$

$\{f_{I\omega}\}$ represents our basis of positive-frequency solution in region I[5]. This basically defines how the solution will generally depend on time; we go further and guess the dependence on x, y to be of the same type, namely, $e^{i\mathbf{x}_\perp . \mathbf{k}_\perp}$, where $\mathbf{k}_\perp = (k_x, k_y)$ and $\mathbf{x}_\perp = (x, y)$. The general positive frequency solution is then

$$f_{I\omega\mathbf{k}_\perp} = \frac{1}{2\pi\sqrt{2\omega}} g_{\omega\mathbf{k}_\perp}(\xi) e^{-i\omega\tau + i\mathbf{k}_\perp . \mathbf{x}_\perp}, \tag{2.20}$$

where the ξ dependence is yet to be determined. The subscript I indicates the above solution has support only on the right Rindler wedge. Plugging the solution (2.20) back in (2.16) result in the following condition on the function $g_{\omega\mathbf{k}_\perp}(\xi)$

$$\left[-\partial_\xi^2 + e^{2a\xi}\left(\mathbf{k}_\perp^2 + m^2\right)\right] g_{\omega\mathbf{k}_\perp}(\xi) = \left[-\partial_\xi^2 + V_{eff}\right] g_{\omega\mathbf{k}_\perp}(\xi) = \omega^2 g_{\omega\mathbf{k}_\perp}(\xi), \tag{2.21}$$

which is but a Schrödinger equation with effective potential $V_{eff} = e^{2a\xi}\left(\mathbf{k}_\perp^2 + m^2\right)$. This is interesting because we can analyze (2.21) in light of our knowledge from wave mechanics. Indeed, in the limit $\xi \to -\infty$, we have $V_{eff} \to 0$, that is, the above equation asymptotically approach

$$\partial_\xi^2 g_{\omega\mathbf{k}_\perp}(\xi) = -\omega^2 g_{\omega\mathbf{k}_\perp}(\xi), \tag{2.22}$$

[5]We consider only the positive-frequency solutions because the field is actually decomposed in terms of $\{f_{I\omega}\}$ and $\{f^*_{I\omega}\}$, and from (2.19) above we can see that if $f_{I\omega}$is a positive-frequency solution, then $f^*_{I\omega}$ is a negative-frequency solution.

as $\xi \to -\infty$. The solution in this limit is straightforwardly given by [5]

$$g_{\omega \mathbf{k}_\perp}(\xi) \sim \frac{1}{\sqrt{2\pi}} A_{\mathbf{k}_\perp} \left(e^{i\omega\xi} + R_{\mathbf{k}_\perp} e^{-i\omega\xi} \right), \tag{2.23}$$

where $R_{\mathbf{k}_\perp}$ is just the phase shift factor due to reflection; note however that as $\xi \to +\infty$, V_{eff} also goes to infinity, i.e, there is an infinity barrier of potential ahead, and we know from elementary quantum mechanics that such scenario allows no transmission of wave, everything being reflected, $|R_{\mathbf{k}_\perp}| = 1$. We choose $A_{\mathbf{k}_\perp}$ as to normalize these solutions, i.e,

$$\begin{aligned} &\langle g_{\omega \mathbf{k}_\perp} | g_{\omega' \mathbf{k}_\perp} \rangle = -\langle g^*_{\omega \mathbf{k}_\perp} | g^*_{\omega' \mathbf{k}_\perp} \rangle = \delta_{\omega\omega'} \\ &\langle g_{\omega \mathbf{k}_\perp} | g^*_{\omega' \mathbf{k}_\perp} \rangle = 0. \end{aligned} \tag{2.24}$$

Therefore

$$\begin{aligned} &\langle f_{I\omega \mathbf{k}_\perp} | f_{I\omega' \mathbf{k}'_\perp} \rangle = -\langle f^*_{I\omega \mathbf{k}_\perp} | f^*_{I\omega' \mathbf{k}'_\perp} \rangle = \delta_{\omega\omega'} \delta_{\mathbf{k}_\perp \mathbf{k}'_\perp} \\ &\langle f_{I\omega \mathbf{k}_\perp} | f^*_{I\omega' \mathbf{k}'_\perp} \rangle = 0. \end{aligned} \tag{2.25}$$

It was already mentioned that the Killing fields have opposite orientation in the two Rindler wedges. To see this, we write $\partial_{(\tau)}$ in the basis $(\partial_{(t)}, \partial_{(x)}, \partial_{(y)}, \partial_{(z)})$ in both regions. In region I

$$\partial_{(\tau)} = \frac{\partial z}{\partial \tau} \partial_{(z)} + \frac{\partial t}{\partial \tau} \partial_{(t)} = \left| \frac{\partial z}{\partial \tau} \right| \partial_{(z)} + \left| \frac{\partial t}{\partial \tau} \right| \partial_{(t)} \tag{2.26}$$

giving

$$g\left(\partial_{(\tau)}, \partial_{(t)}\right) = -\left| \frac{\partial t}{\partial \tau} \right| < 0. \tag{2.27}$$

By convention, we say that any vector a satisfying $g\left(\partial_{(t)}, a\right) < 0$ is future-directed, and as a consequence, if $g\left(\partial_{(t)}, a\right) > 0$, then a is past-directed. Now, the left Rindler wedge is obtained from the right one by means of $t \to -t$ and $z \to -z$, as already mentioned, then, in region II

$$g\left(\partial_{(\tau)}, \partial_{(t)}\right) = \left| \frac{\partial t}{\partial \tau} \right| > 0, \tag{2.28}$$

and $\partial_{(\tau)}$ is past-directed in this region. Therefore, here we must have positive-frequency

solutions defined by the Killing field $\partial_{(-\tau)}$, then

$$f_{II\omega\mathbf{k}_\perp} = f^*_{I\omega\mathbf{k}_\perp}(t \to -t, z \to -z), \tag{2.29}$$

with support in region II only, is a positive-frequency solution in the left wedge for each $\omega > 0$, everything else being the same. Since it is already clear, we will just drop the subscript $\mathbf{k}_\perp$ from these.

Now we have a completely different, but for all reasons valid, description of the same field in terms of another set of modes. The Rindler analogue to the expansion (2.5) reads

$$\phi = \begin{cases} \sum_\ell \left[f_{I\ell}\, a_{RI}(f_{I\ell}) + f^*_{I\ell}\, a^\dagger_{RI}(f_{I\ell}) \right] & \text{Region I} \\ \sum_\ell \left[f_{II\ell}\, a_{RII}(f_{II\ell}) + f^*_{II\ell}\, a^\dagger_{RII}(f_{I\ell}) \right] & \text{Region II} \end{cases} \tag{2.30}$$

where ℓ runs over all $\omega > 0$, $a_{RI}, a^\dagger_{RI}$ are the annihilation and creation operator of Rindler quanta in region I, and $a_{RII}, a^\dagger_{RII}$ are analogously defined for region II [18].

For a reason that will become clearer in a second, we would like to find a complete set of Minkowski positive-frequency solution whose Rindler positive and negative-frequency parts are known. We accomplish this by analytically extending both $f_{I\omega}$ and $f_{II\omega}$ to the whole Minkowski spacetime. First, let us introduce null coordinates defined by

$$\begin{gathered} U = t - z, \quad V = t + z \\ u = \tau - \xi \quad v = \tau + \xi \end{gathered} \tag{2.31}$$

Then, by the relation between Rindler and Minkowski coordinates (2.17), we can write

$$\begin{aligned} e^{-au} &= \begin{cases} a(-U) & \text{Region I} \\ a(+U) & \text{Region II} \end{cases} \\ e^{av} &= \begin{cases} a(+V) & \text{Region I} \\ a(-V) & \text{Region II} \end{cases} \end{aligned} \tag{2.32}$$

the relations for region II follow from those for region I simply by letting $t \to -t$ and $z \to -z$, as already mentioned.

Some comments are in order here. Due to the very own definition of U and V it follows that a solution is positive-frequency with respect to t if, and only if, it is also positive-frequency solution with respect to U and V; the same is true if we exchange $U \to u, V \to v$ and $t \to \tau$. And, most importantly, it can be shown that any solution to the Klein-Gordon equation in Minkowski spacetime is completely determined by its restriction to $\mathsf{h}_1 \cup \mathsf{h}_2$, where h_1 is the hypersuface defined by $V = 0$, and h_2 the one where $U = 0$. It is only at this point that we understand why we did not bother to obtain the function $g_{\omega \mathbf{k}_\perp}(\xi)$ everywhere, but only its asymptotic behavior at $\xi \to -\infty$. Indeed, from the (2.32) above we see that at h_1 and h_2 we have $\xi \to -\infty$, bring together the fact that it is enough to know the restriction of a solution on those surfaces, and we see why the asymptotic form is sufficient. We are then left with the task of finding the restriction of a positive-frequency solution with respect to the inertial time, or equivalently, with respect to U or V. From equations (2.20) and (2.23)

$$\begin{aligned} f_{I\omega} &= \frac{A}{\sqrt{(2\pi)^3 2\omega}}(e^{i\omega\xi} + Re^{-i\omega\xi})e^{-i\omega\tau + i\mathbf{k}_\perp.\mathbf{x}_\perp} \\ &= \frac{1}{\sqrt{(2\pi)^3 2\omega}}(e^{-i\omega u} + \alpha e^{-i\omega v})e^{i\mathbf{k}_\perp.\mathbf{x}_\perp} \\ &= \frac{1}{\sqrt{(2\pi)^3 2\omega}}e^{-i\omega u}e^{i\mathbf{k}_\perp.\mathbf{x}_\perp}, \end{aligned} \tag{2.33}$$

where α is just a phase, i.e, $|\alpha| = 1$. It is worth noting that just as plane-waves in ordinary quantum mechanics, the modes discussed so far are not normalizable, and thus, one is to consider that wave packets will be constructed when needed for some quantities to converge, and in this scenario $e^{-i\omega v}$ is to be understood as a wave packet sharply concentrated around the frequency ω, and as such is ignored, since we are in the regime $v \to -\infty$ [11]. Using (2.32) we can rewrite the above as

$$\begin{aligned} \sqrt{(2\pi)^3 2\omega} f_{I\omega} = e^{-i\omega u}e^{i\mathbf{k}_\perp.\mathbf{x}_\perp} &= \left(e^{-au}\right)^{i\omega/a} e^{i\mathbf{k}_\perp.\mathbf{x}_\perp} \\ &= a^{i\omega/a}\left(-U\right)^{i\omega/a} e^{i\mathbf{k}_\perp.\mathbf{x}_\perp} \end{aligned} \tag{2.34}$$

And from (2.29) we have

$$\begin{aligned}\sqrt{(2\pi)^3 2\omega} f^*_{II(-\omega)} &= \sqrt{(2\pi)^3 2\omega} f_{I(-\omega)}(u \to -u) \\ &= e^{-i\omega u} e^{i\mathbf{k}_\perp.\mathbf{x}_\perp} \\ &= \left(e^{-au}\right)^{i\omega/a} e^{i\mathbf{k}_\perp.\mathbf{x}_\perp} \\ &= (aU)^{i\omega/a} e^{i\mathbf{k}_\perp.\mathbf{x}_\perp} \\ &= a^{i\omega}\left(-e^{-i\pi}U\right)^{i\omega/a} e^{i\mathbf{k}_\perp.\mathbf{x}_\perp} \\ &= e^{\pi\omega/a} a^{i\omega/a} (-U)^{i\omega/a} e^{i\mathbf{k}_\perp.\mathbf{x}_\perp},\end{aligned} \tag{2.35}$$

where in the last line we have just used[6] $-e^{-i\pi} = 1$. Therefore, the combination

$$\sqrt{(2\pi)^3 2\omega}\left(f_{I\omega} + e^{-\pi\omega/a} f^*_{II(-\omega)}\right) = a^{i\omega/a} (-U)^{i\omega/a} e^{i\mathbf{k}_\perp.\mathbf{x}_\perp}, \tag{2.36}$$

is the analytic continuation of $f_{I\omega}$ across $U = 0$ and is well-defined, i.e, has support in the whole surface $V = 0$. We now follow Unruh analytic argument to shows that the mode

$$F_{1\omega} \propto \left(f_{I\omega} + e^{-\pi\omega/a} f^*_{II(-\omega)}\right) \tag{2.37}$$

has no negative-frequency part with respect to the inertial time. A positive-frequency Minkowski mode on this surface is written as

$$\begin{aligned}F_i(U, V = 0, x, y) &= \frac{1}{\sqrt{(2\pi)^3 2\omega}} e^{-i\omega t + i\vec{k}.\vec{x}} \\ &= \frac{1}{\sqrt{(2\pi)^3 2\omega}} e^{-i(\omega + k_z)U/2 + i\mathbf{k}_\perp.\mathbf{x}_\perp},\end{aligned} \tag{2.38}$$

which basically means that the positive-frequency Minkowski solutions are characterized by being analytic and bounded in the lower half of the V-U complex plane at $V = 0$, since we must have $\mathrm{Im}(U) < 0$ for (2.38) to be bounded; negative-frequency solutions will obviously be analytic and bounded in the half $\mathrm{Im}(U) > 0$ on $V = 0$. Now, back to the equations (2.34)

[6]The reason why we use $-e^{-i\pi} = 1$ instead of $-e^{i\pi} = 1$ is because, as we will see, $f(z) = e^{iz}$ is taken to be analytic in the lower-half plane $\mathrm{Im}(U) < 0$. Thus we must have, for $\mathbb{R} \ni x > 0$, $e^{ix} = \cos(x) - i|\sin(x)|$, which means $x \in [-\pi, 0]$

and (2.35), we have that

$$F_{1\omega} \propto |U|^{i\omega/a}, \tag{2.39}$$

which is obviously bounded. Furthermore, if we take the branch cut of the power to lie in the upper-half of the V-U complex plane, then $F_{1\omega}$ is bounded and analytic in the lower-half V-U plane, i.e, it is positive frequency with respect to the inertial time. If we normalize it, finally

$$\begin{aligned} F_{1\omega}(U, V=0, x, y) &= \frac{1}{\sqrt{1-e^{-2\pi\omega/a}}}\left(f_{I\omega} + e^{-\pi\omega/a} f^*_{II(-\omega)}\right) \\ &= \frac{1}{\sqrt{2\sinh(\pi\omega/a)}}\left(e^{\pi\omega/2a} f_{I\omega} + e^{-\pi\omega/2a} f^*_{II(-\omega)}\right). \end{aligned} \tag{2.40}$$

If we follow the same procedure and analytically extend the modes $f_{II\omega}$ across $U=0$ as we have just done with $f_{I\omega}$, we obtain

$$F_{2\omega}(U, V=0, x, y) = \frac{1}{\sqrt{2\sinh(\pi\omega/a)}}\left(e^{\pi\omega/2a} f_{II\omega} + e^{-\pi\omega/2a} f^*_{I(-\omega)}\right), \tag{2.41}$$

which is also purely positive-frequency with respect to the inertial time. Note that it is consistent with (2.29), since $F_{2\omega}(u) = F_{1\omega}(-u)$

The relation between the Minkowski and Rindler modes are obtained by means of a Bogoliubov transformation, and is given by

$$a_M(F_{1\omega}) = \frac{1}{N_\omega}\left[a_{RI}(f_{I\omega}) - e^{-\pi\omega/a} a^\dagger_{RII}(f_{II(-\omega)})\right] \tag{2.42}$$

and since $F_{2\omega}(u) = F_{1\omega}(-u)$

$$a_M(F_{2\omega}) = \frac{1}{N_\omega}\left[a_{RII}(f_{II(-\omega)}) - e^{-\pi\omega/a} a^\dagger_{RI}(f_{I\omega})\right]. \tag{2.43}$$

where $N_\omega = (1-e^{-2\pi\omega/a})^{1/2}$.

If $|0_M\rangle$ is such that $a_M(F_{1\omega})|0_M\rangle = a_M(F_{2\omega})|0_M\rangle = 0$, then one can show that

$$|0_M\rangle = \prod_j \left\{ N_j \sum_{n_j} e^{-\pi n_j \omega_j/a} |n_j, I\rangle \otimes |n_j, II\rangle \right\}, \tag{2.44}$$

where j runs over a complete set of Rindler modes, $|n_j, I\rangle$ is a state containing n_j quanta in

the mode $(f_{I\omega})_j$ and similarly for $|n_j, II\rangle$.

This is a very interesting result. For one it tells us that there is a strong correlation between the Rindler wedges, but the most surprising fact is that a Rindler observer restricted to one of the wedges will see the pure state $|O_M\rangle$ as mixed, just because, due to the causal structure of the wedges, you cannot have access to the whole state, hence, what one see is but an incomplete reality. That is, an observer in region I see $|0_M\rangle$ with region II traced out, and vice-versa.

Note the main relevance to the phenomenon, one uniformly accelerated observer will see a Minkowski vacuum as a mixed state due to the causal structure, i.e, due to the geometry, of the spacetime. This effect will occur whenever the manifold is divided into four parts, by Killing horizons - as was the case here, where the spacetime was divided by the surfaces $U = 0$ and $V = 0$. The intersection of such surfaces is the two dimensional surface $(U = 0, V = 0, x, y)$, and, as it turns out, whenever, in a n-dimensional spacetime, there is a $(n-2)$-dimensional surface to which a timelike Killing vector field is normal, this division will happen; the Killing horizons whose intersection give rise to this surface is called a *bifurcate* Killing horizon, and might obviously happen in curved, as well as in flat, spacetimes.

Chapter 3

Nonequilibrium Thermodynamics

Throughout the text we have been extremely careful when talking about particles. Besides examples, the first appearance of such concept was to describe the Unruh effect, that is, to show how the concept itself seems to be disputable. This is obviously a problem: we observe particles, and they seem to be quite real in the sense that everyone agrees on their properties, for instance, that the vacuum state of a field is devoided of particles. The goal of the present chapter is to take the concept away from that positive-frequency stuff, which looks quite abstract, and try and build a more pragmatical way of seeing particles, after all, independently of what the math might say, a detector will either click or not. After doing so, we briefly discuss the system whose properties we are going to investigate - and since we are looking for aspects of the Unruh effect besides those of a simple thermal bath, we constantly compare them.

3.1 Point-like Detector Model and Setup

Since we are much more interested in the event of the detection itself than anything else, a simple boolean answer is enough, hence we use as detector a two-level fermionic system, known as the Unruh-Dewitt detector model [18]. Let the basis be $\{|0\rangle, |1\rangle\}$, where $|0\rangle$ is the ground state of our system. The Hamiltonian of the system is taken to be

$$H_D = \Omega A^\dagger A, \tag{3.1}$$

where $A^\dagger,\ A$ are ladder operators satisfying

$$A|0\rangle = A^\dagger|1\rangle = 0, \qquad A^\dagger|0\rangle = |1\rangle \qquad \text{and} \qquad A|1\rangle = |0\rangle, \tag{3.2}$$

a consequence of the anticommutation relation $\{A^\dagger, A\} = \mathbb{1}$. The Hamiltonian is understood to generate time translation along a giving Killing vector field χ, that is, the Schrödinger equation we are interested in is,

$$H_D\varphi = i\mathcal{L}_\chi\varphi, \tag{3.3}$$

where $\varphi \in \mathcal{H}_D$ is an arbitrary state of the system, and $\mathcal{H}_D$ the Hilbert space of states spanned by $\{|0\rangle, |1\rangle\}$.

The idea is that this system is immersed in a background scalar field Φ, inasmuch as the definition of the interaction is basically the definition of the whole detector. The interaction Hamiltonian is taken in analogy with [18] and is given by

$$H_I(t) = \varepsilon(t)\int_{\Sigma_t} \Phi(t,\vec{x})[\Psi(\vec{x})A + \Psi^*(\vec{x})A^\dagger]\sqrt{-g}d^3x, \tag{3.4}$$

where t is the parametrization of the isometries generated by the Killing vector χ, $\varepsilon(t)$ is a time-dependent coupling constant, and $\Psi(\vec{y})$ is a smooth function defining the spacial profile of our detector, which must be of compact support for we are going to use it as a test function to integrate the fields against [18]; later we will see that this function can also be used to give rise to the $i\epsilon$-prescription of regularization [23]. Note also that $\sqrt{-g}$ is the determinant of the metric itself, as opposed to the induced metric on Σ_t which is Riemannian, and thus positive. We need time-dependent coupling to filter out any unwanted excitation of the detector due to the fact that we are turning it on instantaneously, we will discuss more about this bellow.

Since at this point our sole aim is to model the point-like detector, we will not bother with the explicit form for the Hamiltonian H_F of the Field. Therefore, for the total system, we have

$$H = H_I + H_D + H_F, \tag{3.5}$$

as the generator of t translation.

Thus, if we take our initial state to be $|s_{t\to-\infty}\rangle$, then to first order in ε, the evolved state is

$$|s_t\rangle = \left\{\mathbb{1} - i\int_{-\infty}^{t} H_I(\ell)d\ell\right\}|s,-\infty\rangle, \tag{3.6}$$

$|s_t\rangle$ is to be understood as a interaction picture operator, as should be clear from (3.6).

There are some problems related to turning on and off the detector, namely, divergences and unwanted excitation due to sudden switch on [21] [22]. It is of our interest then, to keep the detector on for a finite amount of proper time and do the switching process adiabatically, i.e, very slowly. Thus, for time scales much greater than that characteristic of our system, $|t| > T >> \Omega^{-1}$, $\varepsilon(t)$ should vanish and we can extend the upper limit in the integral to $+\infty$ in a good approximation, further, we require $\varepsilon(t)$ to be essentially constant for $|t| < T$, since it is indeed, just to avoid the aforementioned problems. Let the initial state be $|s,-\infty\rangle = |n\rangle\otimes|0\rangle = |n;0\rangle$, i.e, the field is in a state with n quanta of some state χ, and the detector is in its ground state. Basically we slowly turn on the detector, leave it on for an interval of time T, and then slowly turn it off again. The result is

$$\begin{aligned}
|s_{t>T}\rangle &= |n;0\rangle - i\int e^{i\Omega\ell}\varepsilon(\ell)\Phi(\ell,\vec{y})\Psi^*(\vec{y})\sqrt{-g}d^3x|n;1\rangle \\
&= |n;0\rangle - i\int f\Phi(\ell,\vec{y})\sqrt{-g}d^4x|n;1\rangle \\
&= |n;0\rangle - i\Phi(f)|n;1\rangle \\
&= |n;0\rangle + \left[a(\Gamma_-^*) - a^\dagger(\Gamma_+)\right]|n;1\rangle
\end{aligned} \tag{3.7}$$

with the test function $f = e^{i\Omega\ell}\varepsilon(\ell)\Psi^*(\vec{y})$[1], and $\Gamma_+(\Gamma_-)$ is the positive(negative) part of the solution to the classical equation associated with the source f. Since ε is almost constant, f is almost purely negative-frequency, i.e, $\Gamma_+ \approx 0$ and $\gamma = \Gamma_- \approx f$:

$$\begin{aligned}
|s_{t>T}\rangle &= |n;0\rangle - i|1\rangle\otimes a(\gamma)|n\rangle \\
&= |n;0\rangle - i\sqrt{n}\langle\gamma|\chi\rangle|1\rangle\otimes|n-1\rangle,
\end{aligned} \tag{3.8}$$

[1] $\Psi^*(\vec{y})$ guarantees that f has compact support.

in accordance with the definition (1.24). What (3.8) is saying is basically that, to first-order, there is a probability $n|\langle\gamma|\chi\rangle|^2$ of the detector getting excited by absorbing one quantum, and thus lowering the energy of the field; it is this phenomenon that one would call "detection of a particle", and therefore justify our detector model[2].

3.1.1 Two-qubit System; Extended detector

This particle detector concept, known as Unruh-Dewitt detector model, although being able to probe the existence of excitations for trajectories of constant proper acceleration, it cannot tell, in the massless, real scalar field case, the difference between an uniformly accelerated observer and a static one in contact with a heat bath at the corresponding temperature[3]; it can be shown that the two-point Wightman correlation functions associated with both situations are indeed equal [5]. This renders any attempt to study properties related to the irreversibility of the phenomenon being discussed fruitless. It was pointed out in Ref. [8] that in the case of two detectors with uniform proper acceleration perpendicular to their separation, there exists circumstances, i.e, a set of parameters, for which differences in the Unruh and heat bath cases can be seen, in particular, they observed such variances studying the amount of entanglement between the two detectors, showing that sometimes the accelerated detectors would get entangled while the static ones remain uncorrelated; the opposite can also be achieved.

Inspired by the aforementioned work, our goal in this chapter, and one of the main goals of the work as a whole, is to verify if any distinction can be made when one is concerned with nonequilibrium thermodynamic properties such as the total entropy change in the equilibration process, and how much of this entropy is due to irreversibility. Hence, our system will consist of two independent Unruh-Dewitt detectors in four-dimensions, separated by a distance L along the z-axis, both with uniform and identical proper acceleration a along the x-axis. The

[2]This treatment is due to Wald, of course we could just workout the integral in the usual manner, and we might do so in the next sections, however, the elegance of this procedure speaks for itself.

[3]By this we mean that an uniform proper acceleration a has an associated temperature given by $T = a/2\pi$. We use a and T interchangeably.

Hamiltonian for the two detectors system follows simply by adding the individual ones. As an example, to expose the notation, the free Hamiltonian of the two independent detectors reads

$$\begin{aligned} H_D &= \Omega \left(A^\dagger \otimes \mathbb{1}\right)(A \otimes \mathbb{1}) + \Omega \left(\mathbb{1} \otimes A^\dagger\right)(\mathbb{1} \otimes A) \\ &= \Omega A^\dagger_{(A)} A_{(A)} + \Omega A^\dagger_{(B)} A_{(B)}, \end{aligned} \tag{3.9}$$

as can be seen we use a subscript to indicate the system upon which the operator acts; we choose to arbitrarly place the system A at $z = 0$ and the B system at $z = L$, since the distance of separation is perpendicular to the acceleration, in the end what matter is the difference $\Delta z = L$ between the detectors only. In Ref. [29] the authors put forward a model of extended detector to investigate the inhomogeneous nature of the temperature at equilibrium. Following their convention, we call this two-qubit system an extended detector with size L -note this size is perpendicular to the acceleration direction, being the detector dimensionless in the direction of the acceleration as to avoid the complexity of inhomogeneous temperature.

3.2 Entropy, System Dynamics and Entanglement

At this point we have discussed the Unruh effect and formulated some sort of extended particle detector. Here the basic ingredients we will be using to study the behavior of our system are outlined. Our main goal is to study nonequilibrium thermodynamics, i.e, the change in the entropy of an open system, and what are the contributions from entropy production and flux. Among some interesting aspects to be investigating is how the quantum correlation between such detectors relates to the irreversibility of the process.

3.2.1 Degree of Irreversibility

To begin

$$\frac{dS}{dt}(t) = \Pi(t) - \Phi(t), \tag{3.10}$$

is the rate of change of the entropy of the system, Π is the entropy production and Φ is the flux of entropy from the system to the environment. The second law of thermodynamics is

thus expressed as $\Pi \geq 0$, being zero only for reversible processes.

The first problem arise right when we try to define entropy quantitatively, since no single definition is avaible in quantum mechanics; in fact there are even different, inequivalent ways of obtaining the quantum entropy of a system [4] and, in general, they represent different quantities. As is usual, we will work with the entropy of a density matrix given by von Neumann [30].

$$S(\rho) = \mathrm{tr}(\rho \ln \rho), \tag{3.11}$$

where ρ is the density matrix. This definition is nothing but a generalization of the the Shannon entropy of usual classical information theory, that is, it is a measure of the level of ignorace about the state of the system; for instance, if the state of the syste is pure, the above definition vanish. However, for a system in thermal equilibrium the above definition reaches a maximum, meaning a thermal state is a state where the information held about the system is minimum. This is interesting, because even though the von Neumann entropy is not directly related to the thermodynamical entropy, just like the latter, the former is also maximum for a system in thermodynamic equilibrium [31][27]. Since the density matrix ρ is time-dependent, so is $S(\rho)$, in contrast with the definition of entropy from thermodynamics, since thermodynamic potentials are time-independent, being defined for equilibrium states only. Also, using this form for S we can derive fluctuation theorems [33]. For this reason, we may call the above definition thermodynamics entropy whenever ρ is a Gibbs state [32].

Back to our problem, to describe its dynamics, we need an evolution map that maps density matrices into density matrices. The most general map consistent with this requirement and also with all the postulates of quantum-mechanics is the generator of a dynamical semigroup [24], in general, presented as a master equation, which, in the interaction picture reads

$$\frac{d}{dt}\rho(t) = \mathcal{L}(\rho(t)), \tag{3.12}$$

[4] Rényi entropy and Tsallis entropy are two examples.

here ρ is the reduced density matrix of the system weakly coupled to an environment[5], and $\mathcal{L}$ is a superoperator that satisfy, among other properties, (i) there is an invariant state ρ_0 such that $\mathcal{L}(\rho_0) = 0$ and (ii) $\mathrm{tr}(\mathcal{L}(\rho)) = 0, \mathrm{tr}(\rho) = 1$, where $\mathrm{tr}(\cdot)$ stands for the trace in the space of the system alone - this is much weaker than the unitary condition of closed quantum systems. One can show that [34]

$$S(\mathcal{L}(\rho)||\rho_0) \leq S(\rho||\rho_0), \tag{3.13}$$

where $S(\rho||\sigma) = \mathrm{tr}(\rho \ln \rho - \rho \ln \sigma)$ is the relative entropy. Put together the above result with the fact that $\Pi \geq 0$ and we can define

$$\Pi(\rho) = -\frac{d}{dt} S(\rho||\rho_0) \geq 0, \tag{3.14}$$

being positive due to (3.13) -to show that this is indeed an entropy production in the sense that it satisfies fluctuation theorems well known from classical statistical mechanics is a bit more complicated [33]. Using the above definition we can separate the two contributions shown in (3.10) that, together with the definition of $S(t)$ given by (3.11), yields

$$\Pi(\rho) = \mathrm{tr}(-\mathcal{L}(\rho) \ln \rho + \mathcal{L}(\rho) \ln \rho_0) \tag{3.15a}$$

$$\Phi(\rho) = \mathrm{tr}(\mathcal{L}(\rho) \ln \rho_0), \tag{3.15b}$$

where we have used (3.12). Those are rates of change. To obtain the total entropy produced up to time t we just integrate the first equation in (3.15), let us call this quantity Σ,

$$\Sigma(t) = \int_0^t \Pi(t')dt'. \tag{3.16}$$

Similarly, for the flux of entropy from the system to the environment

$$\Upsilon(t) = \int_0^t \Phi(t')dt'. \tag{3.17}$$

Since at equilibrium both the entropy production and flux vanish, so as long as we give

[5]As in (3.5) of the last section, the system here is described by H_D and the environment by H_F

enough time for the system to reach equilibration, we might get the total entropy in such a process by extending the integration limit to infinity. In those cases we drop the time-dependence indication and denote the total entropy produced in a equilibration process as $\Sigma = \Sigma(t \to = +\infty)$, and similarly for total flux.

3.2.2 Lindblad Markovian Master Equation

In this chapter we will be investigating the behavior of the above quantities when the system in question is two uniformly accelerated atoms interacting with a massless scalar field in its ground state as seen from an inertial perspective. To chive this goal we need the explicit form of the generator $\mathcal{L}$ for our specific case. The derivation is pretty standard and we just discuss briefly the physical assumptions we are making -a more thorough derivation can be found in Appendix B. The starting point is the von Neumann equation written for two independent systems described by (3.5), as already discussed. In the interaction picture

$$\frac{d}{dt}\rho_D(t) = -\int_0^t ds \mathrm{tr}[H_I(t), [H_I(s), \rho(s)]], \tag{3.18}$$

with $H_I(t)$ being the interaction picture version of the Hamiltonian H_I. In the weak-coupling limit the system affect the environment only negligibly, allowing us to separate, at all times, $\rho(t) \approx \rho_D(t) \otimes \rho_0$, being ρ_0 the state of the bath. Thus

$$\frac{d}{dt}\rho_D(t) = -\int_0^t ds \mathrm{tr}[H_I(t), [H_I(t-s), \rho_D(t) \otimes \rho_0]], \tag{3.19}$$

which is local in time, but non-Markovian.

Born-Markov approximation: the time scale at which our system varies, τ_D, is too long compared to that of the environment, τ_F, i.e, $\tau_D \gg \tau_F$ and we can discard memory effects. We can thus take the limit $t \to +\infty$ in the above equation since the integrand will drop to zero for $s \gg \tau_F$; because the Wightman function is very concentrated and $\varepsilon(t)$ almost constant, we can also set $\varepsilon(t) = \varepsilon_0$, given that any possible deviation would be erased by the vanishing of the Wightman function.

After the Born-Markov approximation, $\mathcal{L}$ is still not guaranteed to generate a semigroup. To this end we perform a further approximation in which we neglect rapid oscillating terms in (3.19). The final result is of the form

$$\frac{d}{dt}\rho_D(t) = \mathcal{L}(\rho_D) = -i[H_{eff}, \rho_D] + \mathcal{D}(\rho_D), \tag{3.20}$$

with $H_{eff} = H_{eff}^\dagger$ being the effective Hamiltonian of the system. The first term is related to the unitary evolution while the second takes care of dissipation and is the sole responsible for equilibration; we are only interested in the second one and should disconsider the first. The explicit form of $\mathcal{D}(\rho)$ in the interaction picture [6] reads

$$\begin{aligned}\mathcal{D}(\rho) = \varepsilon_0^2 \sum_{i,j=A,B} \Big\{ &\gamma_{ij}(\Omega)\Big(A_{(i)}\rho A_{(j)}^\dagger - \frac{1}{2}\big\{A_{(j)}^\dagger A_{(i)}, \rho\big\}\Big) \\ &+ \gamma_{ij}(-\Omega)\Big(A_{(i)}^\dagger \rho A_{(j)} - \frac{1}{2}\big\{A_{(j)} A_{(i)}^\dagger, \rho\big\}\Big)\Big\},\end{aligned} \tag{3.21}$$

where A, B label the two systems, and $\gamma(\Omega)$ is a positive matrix obtained from the two-point correlation function of the field. As shown in the Appendix B,

$$\gamma(\Omega) = \frac{1}{2\pi}\frac{\Omega}{1-e^{-2\pi\Omega/a}} \begin{pmatrix} 1 & \frac{\sin(\frac{2\Omega}{a}\sinh^{-1}\frac{aL}{\Omega})}{\Omega L\sqrt{1+a^2b^2/4}} \\ \frac{\sin(\frac{2\Omega}{a}\sinh^{-1}\frac{aL}{\Omega})}{\Omega L\sqrt{1+a^2b^2/4}} & 1 \end{pmatrix}. \tag{3.22}$$

Looking at the above equation we can see that the off-diagonal terms are the ones responsible for the interaction between the systems A and B; in fact, as the separation L becomes larger and larger compared to Ω^{-1}, the multiplying matrix above approaches the identity

$$\gamma(\Omega) = \frac{\mathbb{1}}{2\pi}\frac{\Omega}{1-e^{-2\pi\Omega/a}}, \tag{3.23}$$

and the whole dynamics is reduced to that of two independent detectors, consequently, in the large separation limit case, no distinction can be made between a bath of Rindler particles and a heat bath at same temperature since the matrix $\gamma(\Omega)$ is precisely the same for both cases, again, at least for a real, massless scalar field.

[6] The form in the Schrödinger picture is the same.

3.2.3 Entanglement Profile

Going back to the equations (3.15a) and (3.15b) of Subsection 3.2.1, we can now obtain the sought after quantities by use of the explicit form of $\mathcal{L}$ given by (3.21). However, before looking into those, and since one of our final aims is to compare them with the entanglement dynamics of the system, we will brifly look into the latter. In [25] Z. Ficek shows that two inertial subsystems A and B, in the same configuration as ours, can get entangled by spontaneous emission during equilibration when the correlation functions are those from a heat bath in a Gibbs state. Because of this, both detectors are taken to be initially in the uncorrelated, excited state $\rho_0 = |00\rangle\langle 00|$. We use the entanglement monotone concurrence, $\mathcal{C}[\rho]$, as a quantifier of the amout of entanglement present in a state; it ranges from 0, for pure, separable states, to 1, for maximally entangled states. For a two-qubit system it is defined as [26]

$$\mathcal{C}(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{3.24}$$

where λ_i, $i \in \{1, 2, 3, 4\}$, are the square roots, in descending order, of each eigenvalue of the non-hermitian matrix $\rho\tilde{\rho}$, and the matrix

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y), \tag{3.25}$$

is known as the spin-flipped version of ρ; σ_y is the usual Pauli matrix.

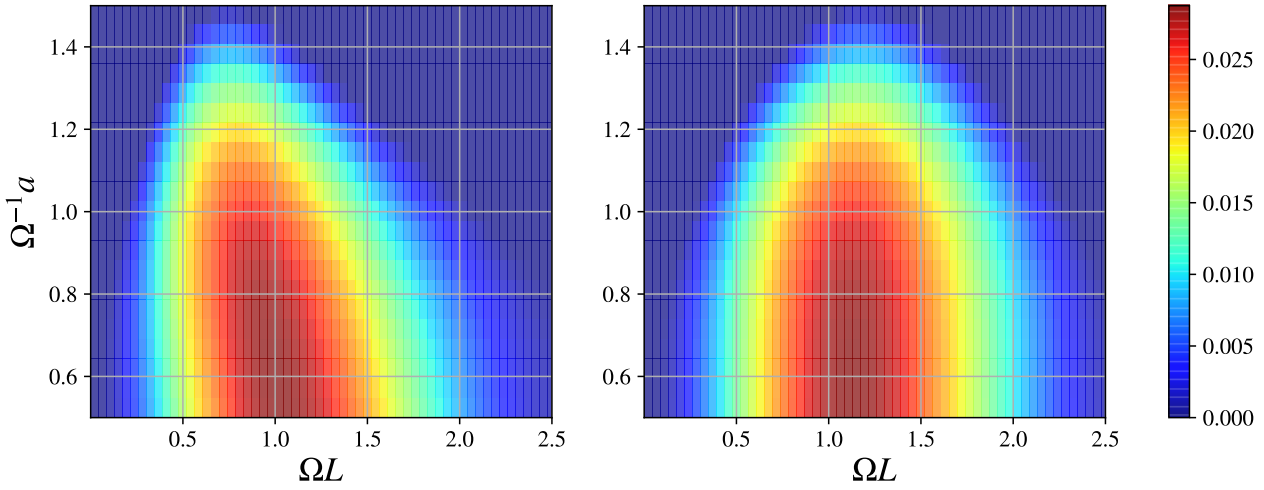

Figure 3.1: Maximum of concurrence attained during time evolution. On the left we have the profile of concurrence for the Unruh effect, $C_{UN}(\rho)$, and on the right the heat bath case, $C_{HB}(\rho)$, with temperature $T = a/2\pi$.

Obtaining an analytical solution for the general case proved far too tricky and a numerical analysis was required. With this in mind, it is worth noting that the energy scale of our system is fixed by the frequency Ω defining the Hamiltonian, thus, when we talk about a large acceleration a, this is to be understood as measured in units of Ω. Similarly, we measure the separation distance, or detector size, in units of Ω^{-1}, the characteristic wavelength. Finally, whenever reference is done to the proper time, we are talking about the dimensionless rescaled proper time variable $t \to (\Omega\varepsilon_0^2/2\pi)t$.

Remember that the initial state of our two-qubit system is taken to be $\rho_0 = |00\rangle\langle 00|$, an uncorrelated state, $\mathcal{C}[\rho_0] = 0$. Now, our final state is reached at equilibrium with a thermal bath of particle, after complete decoherence in the energy eigenbasis, that is, the concurrence of the final state is also zero. Since the concurrence can attain no negative values, we see that it will certanly reach a maximum value, even if this is zero. Thus, in Fig. 3.1 we have plotted the maximum value of the concurrence, i.e, the maximum amount of entanglement present in our density matrix reached during the relaxation process to equilibrium with the given parameters. One can easily see that indeed, as was found in [26], for certain parameters, the atoms do get entangled by the process of spontaneous emission, also, the profile for the Unruh

setting is pretty similar to the heat bath case as one would expect; however it is clear that some differences do exit. Even though there is sudden birth of entanglement, due to the loss of coherence the system is undergoing, the concurrence does not reach high values, making it even harder to quantify the difference just by looking at Fig. 3.1. As a final comment, note how the properties of the two setups become more and more indistinguishable as L increases; this holds true for any property of the system.

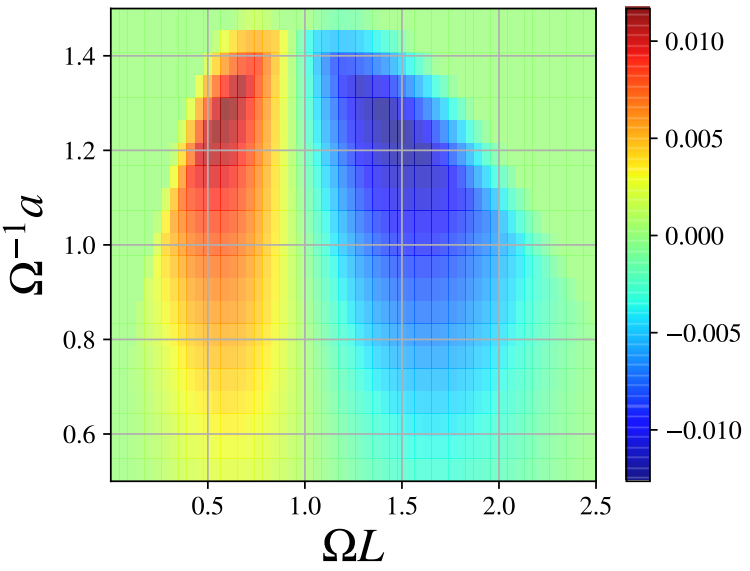

Figure 3.2: This shows the difference in the maximum concurrence between the Unruh and heat bath cases, $C_{UN}(\rho) - C_{HB}(\rho)$. Thus, the red (blue), or positive (negative), region, indicates the set of parameters for which in the Unruh case more (less) entanglement is generated than the heat bath counterpart. Finally, the green area indicates parameters for which the behavior of the two setups are identical. The maximum difference happens around the point $a \approx 1.25\Omega$, for both cases.

Fig. 3.2 conveys the information of the previous figure in a rather neat way. While in Fig. 3.1 there seems to be no critical values or alike, we can now clearly see that the $\Omega L = 1$ line sort of divide the image in two. This is the case where the separation of the detectors precisely match the wavelength of the particle they where designed to detect, meaning the emission of a quanta by one of the systems is more likely to interact with the other detector in a coherent manner, rather than dissipatively. If the separation distance is shorter than the characteristic wavelength of the system, $\Omega L \gtrsim 1$, then, given any temperature, the accelerated

system will always attain a higher degree of entanglement than the static case. However, if $\Omega L \lesssim 1$, than the static case will always produce more entanglement than the accelerated counterpart. When it comes to entanglement, the separation is thus at the heart of the difference between accelerated and static observers, i.e., given a detector, one cannot move between the red and blue regions of the figure just by changing the acceleration parameter. What is more striking about the above is that when $\Omega L \approx 1$ there is no temperature for which one can distinguish the two cases under study. So, in general, if a detector interacts with frequencies lower (higher) than its energy gap, than the accelerated (static) setup will hold more coherence than its counterpart, or at least as much as, for any value of acceleration.

3.2.4 Entropy Production and Flux Profiles

Now we employ equations (3.15a) and (3.15b) to calculate the production and flux rates. However, we are interested in studying thermodynamics, and as discussed in subsection 3.2.1, we cannot say that Equation (3.11) gives us the thermodynamic entropy of the system unless it is in an equilibrium state, which in our case, will happen only as $t \to +\infty$, that is, the initial state of the system, as well as the evolving state during all times, are not equilibrium states, so that (3.11) is not a thermodynamical quantity -it is the information entropy. To fix this, consider for now that both the initial and final states of a given system are equilibrium states, say ρ_i and ρ_f respectively. Then the quantity

$$S(t) = \Sigma(t) - \Upsilon(t) \tag{3.26}$$

where $S = \int S(t)dt$, Υ is the flux of entropy from the system to the reservoir and Σ the entropy production, is a well defined thermodynamic entropy difference during the process, even if nonequilibrium states where accessed. This is obtained from Equation (3.10) by integration.

In our case, even though the final state is a equilibrium state, the initial state is not, so that Equation (3.26) is still not applicable. What is usually done is to consider the relative entropy between the initial nonequilibrium state ρ_0 and the initial equilibrium state ρ_i defined

as the Gibbs state with the initial Hamiltonian, which in our case is H_D. That is, first we take into account the entropy produced in the process of taking the system from the state ρ_i to the state ρ_0, and then we subtract the result from $\Sigma(t)$

$$\Sigma(t) = \int_0^t \Pi(t')dt' - S(\rho_0||\rho_i) \tag{3.27}$$

with $\rho_i = exp(-\beta H_D)/tr[exp(-\beta H_D)]$; since when calculating Σ the system was considered to be initially in equilibrium. However, after all that, we see that the relative entropy $S(\rho_0||\rho_i)$ is a constant, and because we will be working mainly with differences in these quantities, we will just drop the constant $S(\rho_0||\rho_i)$ without any change in the behavior of the system, and call Σ and Υ thermodynamic quantities. To finish this discussion, note that our detector is uniformly accelerated, that is, there is a constant injection of energy into the system. But we are not taking into account the change it causes to the Hamiltonian, and therefore, we are ignoring the work done on the system by this force. We do this because our system is accelerated eternally, which would imply the divergence of this work. Moreover, by imposing the trajectory of a uniformly accelerated observer on our system in the way we did, we are only considering the kinematics effects associated with it, and not the dynamical properties of the entity causing this acceleration. Thus, in what follow, we are not accounting for the energy involved in the acceleration process; in fact, this energy is precisely the energy of the particles measured by the accelerated observer -as well as the energy of the emitted particle as seen by a inertial observer-, so that in essence, this energy is being considered implicitly.

Following our protocol, we will subtract the heat bath total entropy from the accelerated system and plot the difference, looking for deviances from 0, which would otherwise indicate that there is no difference at all.

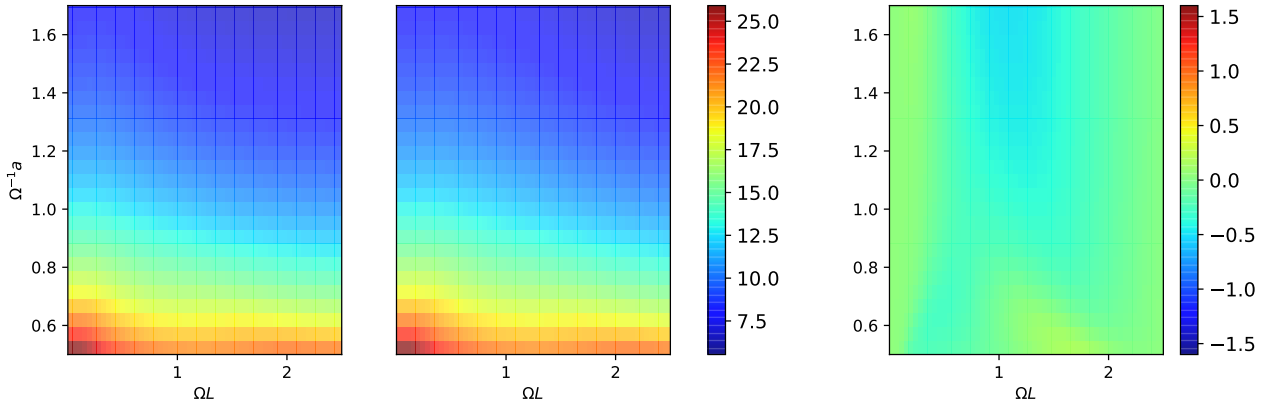

Figure 3.3: Total entropy as shown in Equation (3.26). In the left we have the profile for both accelerated and static systems; the inner left accounts for the Unruh effect, while the inner right, for the static system case. On the far right we have the difference of the first two, showing how much the total entropy variation of the two system are similar.

Entropy variation of a system is closely related to the loss of coherence in quantum systems. Coherences are a basis dependent quantities, but since we are working with thermodynamics, there are arguments to choose the energy eigenbasis of (3.9) as a preffered basis to this discussion. In fact, for a Gibbs state -whose entropy is maximum[27]- the coherences in this energy eigenbasis is inexistent. Giving the discussion of the last subsection one would therefore expect to see a difference in entropy variation in the two regions of Fig. 3.1, i.e, the accelerated two-qubit system should have an inferior increasing of total entropy whenever the entanglement is stronger, and vice-versa. However, Fig. 3.3 seems to tell another story. Indeed, what we see is that even though very small variations can be seen, the entropy variation of an accelerated observer and a static one in contact with a heat bath at corresponding temperature seems to bear no resemblance to 3.1, i.e, it does not seems to matter if a system get entangled while the other does not: the variation in entropy will be roughly the same. This is not that alarming if we remember that what is shown in the figure is the total entropy of the whole extended detector system, while what is shown in Fig 3.1 are correlations internal to the detector.

The data in the above image appears in the left hand side of Equation (3.26), and can therefore be separeted into the two contribuitions. According to a recent work [28] quantum coherence plays an important role in the entropy production of an nonequilibrium system. Let

us then look at the entropy production of the two settings.

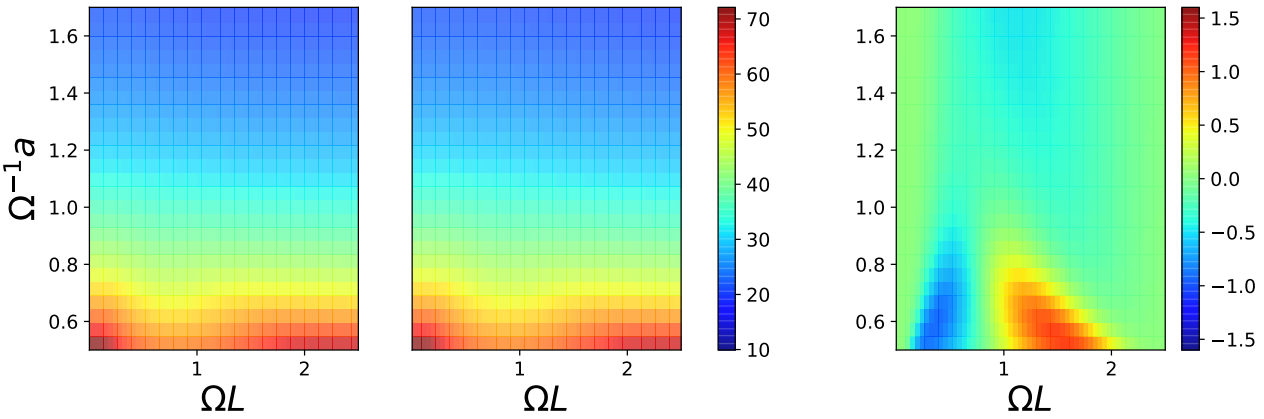

Figure 3.4: Total entropy production. In the inner left (right) we have the accelerated (static) system. On the far right we have the difference of the first two, that is, $\Sigma = \Sigma_{UN} - \Sigma_{HB}$

.

The entropy production of the two settings indeed differ in a manner consistent with the Fig. 3.1. To start with, by comparing the two images we can see that indeed less entropy is produced by the system capable of getting entangled. We still see a critical separation distance very close to the wavelengh, $\Omega L \approx 1$, but somewhat transladed to the left a bit, meaning that separations comparable to Ω^{-1} will have the accelerated system producing more entropy. But in general we observe what we would expect if we were analysing the entropy produced by each qubit, i.e., a behavior in constrast with the concurrence. Now, if $\Omega L < 1$, then then accelerated detector will produce more entropy and will not get entangled, as one should expect; if $\Omega L > 1$, then the opposite is true. It is interesting to point that if we consider our two-qubit system to be just one extended detector, wich it is, in fact, than we can say that we are using a single detector to probe differences between the Unruh setting and the Heat bath one. This is stricking because we have already commented that a single point-like Unruh-Dewitt detector cannot probe any differences in the setup under discussion, that is, just by changing the internal structure of the detector, specifically, by allowing it to have a dimension perpendicular to the acceleration direction[7], we are able to distinguish the

[7]It is important not to have dimensions in the direction of the acceleration for it would lead to an inhomogeneous temperature across this dimension [29].

two setting just by looking at the entropy of the detector. Moreover, this extended detector shows an entropy production that reflects the correlation among its internal parts.

Knowing S and Σ one can obviously determine Υ by use of (3.26). In our case, the fact that the entropy production does behaves differently for the two system and the total entropy does not, implies that the entropy flux should also behave in a different manner as to compensate for the excess of produced entropy, but for completeness we present the entropy flux below. Thus one can distinguish between an accelerated observer and a static one in contact with a heat bath by looking at the entropy production or the entropy flux, but not buy looking at the total entropy variation only. Expliciting the role of coherence in entropy production and flux when distinguishing an accelerated from an ordinary static observer.

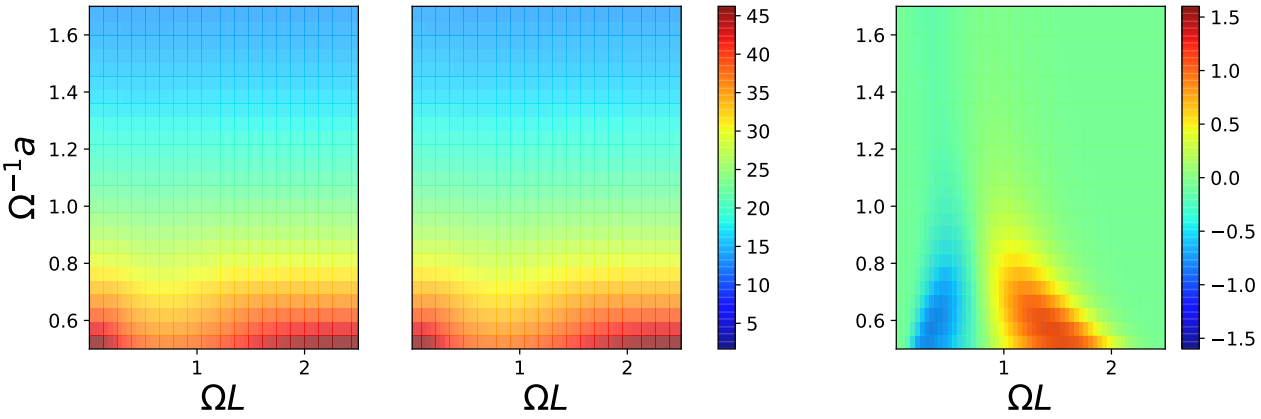

Figure 3.5: Total entropy flux from the system to the environment. In the inner left (right) we have the accelerated (static) system. On the far right we have the difference of the first two, that is, $\Upsilon = \Upsilon_{UN} - \Upsilon_{HB}$

.

Chapter 4

Concluding Remarks

The Unruh effect is one of those phenomena that seems to be inexplicable with our current understanding of physics, in fact, there is no consensus in the scientific community regarding the physical meaning of the calculation leading to the effect. It must be clear, as discussed with detail in Chapter 1, that no mathematical inconsistencies exists in the derivation, and that all, if any, claim against the effect must rely on the physical interpretation of the obtained result, i.e., even though the math is saying that a uniformly accelerated observer does heat up where an inertial observer sees zero temperature, how this connects to the physical world is still an open question. Without the aid of direct observation of the effect, the physicists are left with the task of studying the implications of such phenomenon in other quantities they are already able to observe on the lab, and it is in this direction this thesis walks.

The verification that indeed the Unruh and Heat bath setting are not equivalent, giving us some way to separate what is really due to the vacuum entanglement and what is just due to a thermal bath, open a new door on trying to find specific behavior that would differ from known results, and as such, shed light into the physics involved in phenomena that lies in the edge of quantum mechanics and the classical theory of spacetime, perhaps, extending our understanding of this intersection. Guided by this line of thought, this work dives into the relation of entropy of entanglement between the vacuum and the two probes and the entanglement between the probes itself.

Interestingly enough, even though a Unruh-Dewitt detector alone cannot distinguish be-

tween the to settings discussed in this thesis, an extended detector formed by two such detectors separated perpendicularly to the acceleration does distinguish the two scenarios, even for a massless scalar field. Moreover, and perhaps what is most striking is that using a single extended detector, we can expose differences in the two scenarios if we analyze the entanglement between the system and the environment. One of the most interesting results obtained here is perhaps the fact that the entropy production carries with it what appear to be a signature of entanglement within the extended detector, i.e., we can probe internal properties of the extended detector just by looking at his interaction with the field. Also, more entropy is produced by the two-qubit system when internal correlation is not verified, precisely what one would expect from the a genuine thermodynamic entropy of either one of the qubits, since more entropy produced implies more loss of coherence, which renders the creation of entanglement by spontaneous emission harder. As such, the interaction of the extended detector with the field, seems to reflects the correlations among its internal degrees of freedom.

A room for improvement in this work comes from the fact that we have only studied the case where both initial probes are in their excited states, since we were interested in the entanglement that would be created from the interaction alone. A interesting point to extend the finding here would be to check if one can probe differences in the two settings, having the probes a initial state other than $\rho_0 = |00\rangle\langle 00|$. This is nice because if no difference can be observed, than we have more evidences, that the internal correlation of our extended detector is of paramount importance to the very existence of any difference. For example, if no difference in entropy of entanglement can be observed between the two setting for initial states that do not allow for internal quantum correlation, then one is led to the conclusion that in the energy eigenbasis, coherence is at the core of the Unruh effect, as opposed to an ordinary thermal heat bath. Another point to be exploring is the actual thermodynamic entropy of each qubit, and see how this influences the correlation between them. It would be nice to study driven systems as well.

Appendix A

Basic Differential Geometry Concepts

Differential Geometry is at the core of this work. For those not so familiar with the concepts of this field of mathematics, we present here a very brief and condensed overview of some topics that might be critical for the developtment of the subject as proposed by this work. In accordance with the rest of the text, the material presented in this appendix is also exposed in a way that should not make any use of unecessery assumptions, for instance, we should not make reference to a local coordinate system, unless if to examplify some definition. To make things easier, we will always be considering a smoth, or infinitely differentiable, manifolds - all the definitions, however, apply to a manifold that is differentiable only $k \in \mathbb{Z}^+$ times, just change C^∞ by C^{k1} where appropriate and you should have the definitions working for nonsmoth, differentiable manifolds. The presentation here is similar and inspired by that of Wald [12] and Carroll [15], and is not intended as a pedagogical exposition, but rather as a dictionary of differential geometry's jargon.

A.1 Manifolds, Vectors and Tensors

Simply put, a differentiable manifold is the generalization of our well-known Euclidean geometry where we allow it to have a complicated global topology, but nonetheless, require that it remains similar enough to a Euclidean geometry in a way that we can make sense of functions and coordinate systems at each point - even though comparisons of properties

[1] C^k is the class of functions that can be differentiated k times, with all derivatives being continuous.

from separated points might not always be possible. So in essence, we build a manifold of dimension n by sewing together, in a continuous way, a bunch of $\mathbb{R}^n$ structures until we achieve the aforementioned topology when we look at it globally. The precise definition as given by Wald is the following.

Definition A.1.1. An n-dimensional, C^∞, real manifold M is a set together with a collection of subsets $\{O_\alpha\}$ satisfying

(i) Each $p \in M$ lies in at least one O_α.

(ii) For each index α there is a one-to-one, onto, map $\phi_\alpha : O_\alpha \longrightarrow U_\alpha$, where $U_\alpha \in \mathbb{R}^n$ is an open subset.

(iii) If O_α and O_β are such that $O_\alpha \cap O_\beta \neq \emptyset$, then we require the map $\phi_\beta \circ \phi_\alpha^{-1}$, which maps $\phi_\alpha[O_\alpha \cap O_\beta] \in U_\alpha$ to $\phi_\beta[O_\alpha \cap O_\beta] \in U_\beta$, to be of class C^∞. Besides, $\phi_\alpha[O_\alpha \cap O_\beta] \in U_\alpha$ and $\phi_\beta[O_\alpha \cap O_\beta] \in U_\beta$ must be open subsets of $\mathbb{R}^n$.

The last condition makes the "sewing in a continuous way" bit mathematically precise. The pair (O_α, ϕ_α) is usually called a chart, or a coordinate system, and the union of charts that covers M is referred to as an atlas. It is worth mentioning that the above definition considers that all possible atlas are taken into account, so that the same manifold with different choices of covering are not counted twice.

In ordinary, flat geometry we have an intuitive notion of vectors as stretching from one point to another, and collectively forming a set, called vector space: elements of this set can, among other properties, be added to form yet another element of the same set. While this notion can be helpful in the absence of a more complex global structure, in a general differentiable manifold the global topology will spoil this vector space structure, but, since we have defined manifolds to be locally $\mathbb{R}^n$, we can work with "infinitesimal displacements" as so to make sense of vectors at a point $p \in M$, i.e, "infinitesimal displacements" on a manifold will continue to possess a vector structure. This discussion motivates the definition of the tangent vector space, defined for each point $p \in M$.

To this end, let $\mathcal{F}$ denote the collection of all C^∞ functions from M into $\mathbb{R}$. We define a tangent vector v at the point p to be a map from $\mathcal{F}$ to $\mathbb{R}$ satisfying

(i) $v(af + bg) = av(f) + bv(g)$, for all $f, g \in \mathcal{F}$; $a, b \in \mathbb{R}$;

(ii) $v(fg) = f(p)v(g) + g(p)v(f)$,

i.e, a vector is a linear map satisfying the Leibnitz rule. The collection of all vectors at $p \in M$ is denoted by V_p and named tangent vector space, with $\dim(V_p) = \dim(M)$. In fact, one can show that for each element $f \in \mathcal{F}$, the directional derivative provides a map $f \longrightarrow df/d\lambda(p)$ that satisfies A.1. Thus, the tangent vector space at $p \in M$ is identified with the set of all directional derivative operators along curves passing through p.

To see this somewhat obscure looking definition more clearly, let $\phi : O \in M \longrightarrow U \subset \mathbb{R}^n$ be a chart, with $p \in O$, and consider a curve $\gamma : \mathbb{R} \longrightarrow M$ parameterized by λ. If $f \in \mathcal{F}$ then

$$\begin{aligned}\frac{d}{d\lambda}f =& \frac{d}{d\lambda}(f \circ \gamma) \\ & \frac{d}{d\lambda}(f \circ \phi^{-1}) \circ (\phi \circ \gamma) \\ & \frac{d}{d\lambda}(\phi \circ \gamma)^\mu \frac{\partial}{\partial x^\mu}(f \circ \phi^{-1}) \\ & \frac{dx^\mu}{d\lambda}\partial_\mu f,\end{aligned} \tag{A.1}$$

where $(x^0, x^1, ..., x^{n-1})$ are the Cartesian coordinates of $\mathbb{R}^n$. Note that we can decompose any directional derivative operator $d/d\lambda$ as a linear combination of the vectors $dx^\mu/d\lambda$, showing that, indeed, they form a basis of V_p. In this special case, when x^μ are the coordinates of $\mathbb{R}^n$, the basis $\{\partial_\mu\}$ is called coordinate basis of V_p.

Now that we have a well defined notion of tangent vector space, consider the collection, V_p^*, of linear maps $f : V_p \longrightarrow \mathbb{R}$. Of course, to define the action of those maps on arbitrary elements of V_p, it suffices to define their actions on its basis. Hence, given a basis $v_1, v_2, ..., v_n$ of V_p, we define elements $\omega^j \in V_p^*$ by imposing

$$\omega^i(v_j) = \delta^i{}_j, \qquad i, j \in \{1, 2, ..., n\}. \tag{A.2}$$

One can check that by the above equation, the set $\{\omega^1, \omega^2, ..., \omega^n\}$ automatically defines a basis for the space V_p^*. An arbitrary element of V_p^* is then written in this basis, and acted upon by means of (A.2). The correspondence $v \leftrightarrow \omega$ defines an isomorphism between the sets V_p and V_p^*, which is in general coordinate-dependent unless more structure is given for V_p.

To finish this section we present yet another type of map. Up to know we have defined maps $\mathcal{F} \longrightarrow \mathbb{R}$ and $v_p \longrightarrow \mathbb{R}$. Now we introduce the idea of a tensor. A tensor of type (k, l) is a multilinear map from the Cartesian product of k copies of V_p^*, and l copies of V_p into the real numbers. More precisely

$$T : \underbrace{V^* \times ... \times V^*}_{k} \times \underbrace{V \times ... \times V}_{l} \longrightarrow \mathbb{R} \tag{A.3}$$

where we have dropped the subscripts p labeling the vector spaces; but note that the above definition is to be understood as defined for every point $p \in M$.

A tensor that stands out among all the other, at least in the context of the General Theory of Relativity, is the metric tensor, usually denoted g. This is a tensor of type $(0, 2)$, that is, it maps two vectors from the cotangent space to the real numbers, and gives to the manifold M the structure of a metric space.

$$g : V^* \times V^* \longrightarrow \mathbb{R}. \tag{A.4}$$

It defines a generalized notion of distance in the manifold M, which can be positive-definite, for a Riemannian metric, or even negative numbers in the case of a Lorentzian metric. To differentiate between those two cases, we look at the signature of the metric, that is, the sequence of signs of the eigenvalues of the metric tensor. If all eigenvalues are nonzero and positive, $(+, +, ..., +)$, the metric is said to be Riemannian, if all eigenvalues are nonzero and only one of them is negative, $(-, +, ..., +)$, then the metric is said to be a Lorentzian metric, finally, if any eigenvalue vanish the tensor is said to be degenerate. This can be further used to classify two events in a manifold. Indeed, given a manifold (M, g) with Lorentzian metric,

two vectors $v, w \in V_p$, where V_p is the tangent vector space at $p \in M$, are said to be timelike separated if $g(w, v) < 0$, null separated if $g(w, v) = 0$, and spacelike separated if $g(w, v) > 0$[2]. This nomenclature will also be used when talking about events and curves in the same fashion.

Even though nothing in this appendix has demanded a metric, such a definition can, among other things, establish a coordinate-independent isomorphism between tangent and cotangent vector spaces, as well as fix the otherwise arbitrary choice of a covariant derivative operator -which we will not be discussing here- by requiring it to be compatible with the metric in the sense that when applied to g we have 0 as a result.

A.2 Maps between manifolds

A simple way to relate quantities defined on one manifold, M, to quantities defined on another one, N, can be achieved by means of a map $\phi : M \longrightarrow N$. Such relations can be of very great use; note that, in particular, when M and N are the same manifold, this transformation may encode symmetries of the system being modeled. Suppose then that ϕ is a C^∞ map, and consider a function $f : N \longrightarrow \mathbb{R}$. We can obviously use ϕ to map a point from M to N, and then use f to map the result into $\mathbb{R}$, thus the composition

$$(f \circ \phi) : M \longrightarrow \mathbb{R}, \tag{A.5}$$

is a function defined on M and is called the "pull back" of f, as if we have pulled f from N back to M - *back* because it moves a quantities in the "direction of the inverse of" ϕ. Similarly, using the above composition, we can go in the other direction and carry tangent vectors from M at a point p, to N at point $\phi(p)$. For every $v \in V_p$, let us define the map $\phi^* v : V_p \longrightarrow V_{\phi(p)}$ acting according to

$$(\phi^* v)(f) = v(f \circ \phi). \tag{A.6}$$

[2]For a metric with signature $(+, +, ..., -)$ the greater than, and less than symbols are interchanged.

From (A.1), we see that v is just mapping a function on M as it should be, on the left-hand side, however, we have the map $(\phi^* v)$ acting on a function defined on N, and as such, belongs to a tangent space of the manifold N. Because of this, we call ϕ^* the "push forward" map, which carries vectors from $V_p,\ p \in M$ to $V_q,\ q \in N$.

Since we are not assuming ϕ to have an inverse, we are unable, for example, to "push forward" a function $g : M \longrightarrow \mathbb{R}$ to N, since the composition $g \circ \phi^{-1} : N \longrightarrow \mathbb{R}$ is not defined because, in general, we lack ϕ^{-1}. Likewise, we cannot "pull back" a tangent vector from N to M, we can, however, do it with dual vectors, the elements of the cotangent space. Define the map $\phi_* \omega : V_p^* \longrightarrow V_{\phi(p)}^*$, $\omega \in V_q,\ q \in N$, by

$$(\phi_* \omega)(v) = \omega(\phi^* v), \tag{A.7}$$

$v \in V_p,\ p \in M$. Again we see that the above definition works fine: on the left-hand side we have our desired map, while on the right-hand side, we carries v from V_p to $V_{\phi(p)}$ and then uses ω to map the result to $\mathbb{R}$. One can extend these operations accordingly to multilinear maps, noting the restrictions due to the absence of ϕ^{-1}. With the maps ϕ^*, and ϕ_* in hands, we can "pull back" tensors of type $(0, l)$ from N to M - from point $\phi(p)$ to the point p, as well as "push forward" tensor of type $(k, 0)$ from M to N - from point p to the point $\phi(p)$. For a tensor $(0, l)$, for example, we obtain

$$(\phi_* T)(v_1, ..., v_l) = T(\phi^* v_1, ..., \phi^* v_1), \tag{A.8}$$

$v_1, ..., v_l \in V_p$ with $p \in M$. And for $(k, 0)$ tensors

$$(\phi^* K)(\omega^1, ..., \omega^k) = K(\phi^* \omega^1, ..., \phi_* \omega^k), \tag{A.9}$$

with obvious notation.

If besides the map $\phi : M \longrightarrow N$, we also have a well-defined $\phi^{-1} : N \longrightarrow M$ that is one-to-one, onto, and C^∞, then the manifolds M and N are said to be diffeomorphic -

implying $\dim(M) = \dim(N)$, which basically means that they are "equal", as far as the manifold properties are concerned; in particular, two physical systems described by N and M are necessarily identical. The existence ϕ and ϕ^{-1} allows one to "pull back" and "push forward" all types of tensor, i.e, $(k, 0)$, $(0, l)$, and even (k, l). As pointed out in the beginning of the section, these maps may encode symmetries. Indeed, if T is a tensor, and $\phi^* T = T$, we say that ϕ^* is a symmetry transformation of the tensor T, a notion we discuss more thoroughly on the next section.

A.3 Symmetries and Lie Derivative

In this section we just employ the idea of maps between manifolds, more specifically, diffeomorphisms, with a bit of real analysis to have a better understanding of what a tensor symmetry transformation means. We thus work with a group of diffeomorphisms parameterized by a continuous parameter.

Definition A.3.1. A one-parameter group of diffeomorphism ϕ_t is a C^∞ map from $\mathbb{R} \times M \longrightarrow M$ such that for fixed $t \in \mathbb{R}$, $\phi_t : M \longrightarrow M$ is a diffeomorphism and for all $s, t \in \mathbb{R}$, we have $\phi_t \circ \phi_s = \phi_{t+s}$.

We can associate a vector field, say v with the map ϕ_t. Indeed, for each $p \in M$ we have that $\phi_t(p) : \mathbb{R} \longrightarrow M$ is a curve on the manifold M. Let $\phi_{t=0}(p) = p$, then we define $v|_p$ to be the tangent vector to the curve $\phi_t(p)$ at $t = 0$; do this for all $p \in M$ and we have a vector field v that is said to be the generator of the group. Now, suppose we have a tensor T and we push it forward by means of ϕ_t^*. A question that naturally arises, since the manifolds are diffeomorphic, is how $\phi_t^* T$ relates to $\phi_s^* T$ for $s \neq t$. In the limit $s \to t$, since ϕ_0 is the identity transformation, the answer o that question is called Lie derivative, $\mathcal{L}_v$ along the vector field

v:

$$\begin{aligned}\mathcal{L}_v T &= \lim_{t\to 0}\left\{\frac{\phi_t^* T - \phi_0^* T}{t}\right\} \\ &= \lim_{t\to 0}\left\{\frac{\phi_t^* T - T}{t}\right\}.\end{aligned} \tag{A.10}$$

One can check that with this definition $\mathcal{L}_v$ is linear and does obey the Leibnitz rule on outer product of tensors. Also, if we apply this derivative to an ordinary function $f : \mathcal{M} \longrightarrow \mathbb{R}$, we get the ordinary directional derivative

$$\mathcal{L}_v f = v(f), \tag{A.11}$$

by immediate application of Equation (A.6).

It is clear now why the idea of maps and Lie derivative are good tools to work with symmetries. Let say that ϕ_t is a symmetry transformation of a given tensor T, then

$$\mathcal{L}_v T = 0, \tag{A.12}$$

that is, the tensor T remains unchanged as one moves it along the integral curves of ϕ_t.

In the context of General Relativity, the most important example are the symmetries of the metric tensor. Say (M, g) is a manifold with metric g, then if

$$\mathcal{L}_v g = 0, \tag{A.13}$$

we say that this transformation is an isometry; further, we call the vector field v a killing vector field. Thus, a killing vector field is but a vector field that generates a group of symmetry transformations of the metric g.

Appendix B

Master Equation Derivation

In this appendix we just present a more detailed calculation of the master equation used in Chapter 3, as well as the Fourier transform of the correlation functions shown in Equation 3.22.

B.1 Master Equation

Ours starting point is the Hamiltonian shown in Equation 3.5

$$H = H_I + H_D + H_F, \tag{B.1}$$

where each term is as explained in the text. We will work in the interaction picture with $H = H_0 + H_I$. In the interaction picture, the density matrix evolves according to eh von Neumann equation

$$\frac{d}{dt}\rho_I(t) = -i[H_I(t), \rho_I(t)] \tag{B.2}$$

where the subscript I denotes we are working in the interaction picture, and $H_I(t)$ is the interaction picture version of H_I. By direct integration the above equation yields

$$\rho_I(t) = \rho_I(t_0) - i\int_{t_0}^{t}[H_I(s), \rho_I(s)]ds \tag{B.3}$$

Plugging Equation (B.3) back into (B.2) we obtain

$$\frac{d}{dt}\rho_I(t) = -i[H_I(t), \rho_I(t_0)] - \int_{t_0}^{t}[H_I(t), [H_I(s), \rho_I(s)]]ds, \tag{B.4}$$

and after tracing out the reservoir degrees of freedom, one is left with

$$\frac{d}{dt}\rho_{DI}(t) = -\int_{t_0}^{t} tr_F[H_I(t), [H_I(s), \rho_I(s)]]ds, \tag{B.5}$$

with $\rho_{DI}(t)$ representing the interaction picture of the density matrix of our two-qubit system. The term $[H_I(t), \rho_I(t_0)]$ was set to zero because we have a stationary reservoir.

Now, as discussed in the main text, we perform the Born approximation by considering that the state of the system + environment is at all times separable, i.e., the weak-coupling approximation,

$$\rho_I(t) \approx \rho_{DI}(t) \otimes \rho_F(0) \tag{B.6}$$

here $\rho_F(0)$ is the unchanging state of the reservoir. After that, Equation (B.5) reads

$$\frac{d}{dt}\rho_{DI}(t) = -\int_{t_0}^{t} tr_F[H_I(t), [H_I(s), \rho_{DI}(s) \otimes \rho_F(0)]]ds, \tag{B.7}$$

Next, to obtain an equation that is local in time, we perform the Markov approximation, that is, we assume that the state of the system at time t will not depend on the state of the system at times s prior to t, $s < t$. With this, we obtain a Markovian dynamics, a dynamics without memory; meaning the state of the system will not depend on its history. We also make the change of variables $s \to s - t$. The result after these changes are

$$\frac{d}{dt}\rho_{DI}(t) = -\int_{0}^{t-t_0} tr_F[H_I(t), [H_I(t-s), \rho_{DI}(t) \otimes \rho_F(0)]]ds, \tag{B.8}$$

With this change of variable we can see that the parameter s indicates how far we go back in time to account for memory effects concerning the bath, with characteristic time τ_F, the reservoir relaxation time; i.e., any change in the reservoir due to the interaction with the two-qubit system will relax in time intervals of order τ_F, which in turn is considered to be much smaller then the system relaxation time τ_D, i.e., $\tau_F \gg \tau_D$. For that matter, under the Markov approximation the integrand decays very quickly for $s \gg \tau_F$, for we are considering that the state of the system should not be affected by the reservoir's state at times before

$t - \tau_F$. Since $\tau_F \gg \tau_D$ we can extend the integration interval to infinity.

$$\frac{d}{dt}\rho_{DI}(t) = -\int_0^{+\infty} tr_F[H_I(t), [H_I(t-s), \rho_{DI}(t) \otimes \rho_F(0)]]ds. \tag{B.9}$$

Now, we plug the Hamiltonian (3.4) into the above equation to obtain

$$\begin{aligned}\frac{d}{dt}\rho_{DI}(t) = \varepsilon_0^2 \sum_{m,\ell} \Big\{ &\int_0^{+\infty} e^{-i\Omega s} tr_F[\tilde{\phi}_m(t)\tilde{\phi}_l(t-s)]ds\big(A^{\dagger\ell}\rho_{DI}A^m - A^m A^{\dagger\ell}\rho_{DI}\big) \\ &+ \int_0^{+\infty} e^{+i\Omega s} tr_F[\tilde{\phi}_m(t)\tilde{\phi}_\ell(t-s)]ds\big(A^{\ell}\rho_{DI}A^{\dagger m} - A^{\dagger m} A^{l}\rho_{DI}\big)\Big\}.\end{aligned} \tag{B.10}$$

where A and $A^\dagger$ are the fermionic ladder operators.

Some comments are in order here. First of all we have introduced the ϕ_m to indicate $\phi(t, \vec{x}_m)$, that is, we use the index to label the trajectory of each atom; in our case, the parameter t is the same regardless of the indeces m, ℓ. Further, we introduced the $\tilde{\phi}$ field. This is the so called smeared field [23][5] and is just the operator-value distribution ϕ integrated against the test function Ψ in (3.4).

$$\tilde{\phi}(t) = \int_{\Sigma_t} \phi(t, \vec{x})\Psi(\vec{x})dV_t, \tag{B.11}$$

where Σ_t is a Cauchy surface and dV_t its volume element. It was shown in Ref. [23] that this procedure naturally takes care of the regularization of the Fourier transform of the field correlation function. It is nothing but a physically motivated way to introduce the $i\epsilon$-prescription. Finally, the ε_0 is just the value assumed by $\varepsilon(t)$ between the switching on and off of the detector. This can be done, because at the times for which $\varepsilon(t)$ differs from ε_0, the integrals vanish due to the correlation functions g of the field

$$g(t, \vec{x}_m; t', \vec{x}_\ell) = tr_F[\phi_m(t)\phi_l(t')], \tag{B.12}$$

for simplicity we are ignoring the tilde here, since all this will be fixed in the end of the calculation. An interesting characteristic of our problem is the fact that our trajectories (2.10) are orbits of a timelike Killing vector field, that is, along these trajectories we have time-

translation symmetry, and the above correlation function is time-independent. Indeed, by shifting the fields in the above equation by $t - s$ we get

$$\Gamma_{m\ell}(\Omega) = \int_0^{+\infty} e^{i\Omega s} g(s, \vec{x}_m; 0, \vec{x}_\ell) ds \tag{B.13}$$

which dos not depend upon the time t. Given this new notation, our master equation can be written as

$$\begin{aligned} \frac{d}{dt}\rho_{DI}(t) = \varepsilon_0^2 \sum_{m,\ell} \Big\{ & \Gamma_{m\ell}(-\Omega)\big(A^{\dagger\ell}\rho_{DI}A^m - A^m A^{\dagger\ell}\rho_{DI}\big) \\ & + \Gamma_{m\ell}(\Omega)\big(A^{\ell}\rho_{DI}A^{\dagger m} - A^{\dagger m}A^{l}\rho_{DI}\big)\Big\}. \end{aligned} \tag{B.14}$$

To separate the unitary and dissipative contribuitions, we decompose Γ as [24]

$$\Gamma_{m\ell}(\Omega) = \frac{1}{2}\gamma_{m\ell}(\Omega) + iS_{m\ell}(\Omega), \tag{B.15}$$

where

$$\gamma_{m\ell}(\Omega) = \int_{-\infty}^{+\infty} e^{i\Omega s} g(s, \vec{x}_m; 0, \vec{x}_\ell) ds, \tag{B.16}$$

is the positive matrix appearing in (3.22). The form of $S(\Omega)$ is not of particular interest to us. The final form of the master equation is then

$$\frac{d}{dt}\rho_{DI}(t) = \mathcal{L}(\rho_{DI}) = -i[H_{eff}, \rho_{DI}] + \mathcal{D}(\rho_{DI}), \tag{B.17}$$

where we set $H_{eff} = 0$[1], since we are not interested in this unitary contribution. Going back to the Schrödinder picture yield precisely the same equation as above, but with $\rho_{DI}(t) \to \rho_I(t)$. The form of $\mathcal{D}$ is

$$\begin{aligned} \mathcal{D}(\rho) = \varepsilon_0^2 \sum_{m,\ell=A,B} \Big\{ & \gamma_{m\ell}(\Omega)\Big(A_{(m)}\rho A^\dagger_{(\ell)} - \frac{1}{2}\big\{A^\dagger_{(\ell)}A_{(m)}, \rho\big\}\Big) \\ & + \gamma_{m\ell}(-\Omega)\Big(A^\dagger_{(m)}\rho A_{(\ell)} - \frac{1}{2}\big\{A_{(\ell)}A^\dagger_{(m)}, \rho\big\}\Big)\Big\}, \end{aligned} \tag{B.18}$$

which is the desired result as in (3.21).

[1]The quantities $S(\pm\Omega)$ are part of this effective Hamiltonian that we disconsider.

B.2 Correlation Functions

In this section we evaluate

$$\gamma_{m\ell}(\Omega) = \int_{-\infty}^{+\infty} e^{i\Omega s} g(s, \vec{x}_m; 0, \vec{x}_\ell) ds, \tag{B.19}$$

by the method of residue. To remember, the trajectory of our two-atoms, A and B, are given by

$$t_A = a^{-1} e^{a\xi} \sinh(a\tau) \qquad x_A = a^{-1} e^{a\xi} \cosh(a\tau) \qquad y = 0 \quad \text{and} \quad z = 0, \tag{B.20}$$

for the atom A, and

$$t_B = a^{-1} e^{a\xi} \sinh(a\tau) \qquad x_B = a^{-1} e^{a\xi} \cosh(a\tau) \qquad y = 0 \quad \text{and} \quad z = L, \tag{B.21}$$

for the atom B.

For the simples case of a massless scalar field in the Minkowski vacuum state -which is true in our case- the correlation functions, also known as positive-frequency Wightman functions, are well known to be [20]

$$g(t, \vec{x}_m; t', \vec{x}_\ell) = tr_F[\phi_m(t)\phi_l(t')] = \frac{1}{4\pi^2} \frac{-1}{|t - t' - i\epsilon|^2 - |\vec{x}_m - \vec{x}_\ell|^2}. \tag{B.22}$$

with $\epsilon > 0$.

Since there is a sum in the master equation, we will have all the terms $g(t, \vec{x}_A; t', \vec{x}_A)$, $g(t, \vec{x}_A; t', \vec{x}_B)$, $g(t, \vec{x}_B; t', \vec{x}_A)$ and $g(t, \vec{x}_B; t', \vec{x}_B)$ appearing there. In fact, the diagonal terms are equal to each other. Also, the cross terms are also equal to each other, so that we only have to calculate two of the above four functions. Since both calculations are essentially the same, we will do only $g(t, \vec{x}_A; t', \vec{x}_A)$ explicitly. If one substitutes the trajectory (B.20) with $\xi = 0$ into the above expression, the result is

$$g(t, \vec{x}_m; t', \vec{x}_\ell) = -\frac{a^2}{16\pi^2} \text{csch}^2 \left(\frac{1}{2} a(\tau - \tau') - ia\epsilon \right), \tag{B.23}$$

where csch is the hyperbolic cosecant function. To proceed, we make use of the Laurent expansion of the cosecant function [20]

$$\operatorname{csch}(\pi x) = \frac{1}{\pi^2} \sum_{k=-\infty}^{+\infty} (x-k)^{-2}, \tag{B.24}$$

that, together with the identity $\operatorname{csch}(x) = i\operatorname{csc}(ix)$, allow us to write (B.23) in the form

$$g(s, \vec{x}_A; 0, \vec{x}_A) = -\frac{1}{4\pi^2} \sum_{k=-\infty}^{+\infty} (s - 2i\epsilon + 2k\pi i/a)^{-2}. \tag{B.25}$$

By plugging this result into (B.19), we are left with the task of evaluating the integral

$$\gamma_{AA}(\Omega) = -\frac{1}{4\pi^2} \sum_{k=-\infty}^{+\infty} \int_{-\infty}^{+\infty} ds \frac{e^{i\Omega s}}{s - 2i\epsilon + 2k\pi i/a}, \tag{B.26}$$

with poles at $s = -2i(k\pi/a - \epsilon)$. This integral can be solved by the methods of residue. Indeed, consider the contour that cover all the real line and is closed by a semicircle in the upper-half part of the complex plane. By the residue theorem [35] the integral over this contour is given by

$$\int_{-\infty}^{+\infty} ds \frac{e^{i\Omega s}}{s - 2i\epsilon + 2k\pi i/a} = 2\pi i \left. \frac{d}{ds} e^{i\Omega s} \right|_{s=-2i(k\pi/a-\epsilon)} = -2\pi\Omega e^{2\Omega(k\pi/a-\epsilon)}, \tag{B.27}$$

Finally

$$\begin{aligned} \gamma_{AA}(\Omega) = \int_{-\infty}^{+\infty} ds e^{i\Omega s} g(s, \vec{x}_A; 0, \vec{x}_A) ds &= \frac{\Omega}{2\pi} \sum_{k=-\infty}^{0} \left(e^{2\pi\Omega/a}\right)^k \\ &= \frac{\Omega}{2\pi} \sum_{k=0}^{+\infty} \left(e^{-2\pi\Omega/a}\right)^k \\ &= \frac{\Omega}{2\pi} \frac{1}{1 - e^{-2\pi\Omega/a}}. \end{aligned} \tag{B.28}$$

the final, desired result. To calculate $\gamma_{AB}(\Omega) = \gamma_{BA}(\Omega)$ one follows the same procedure, the only difference being the position of the poles that will be shifted by an amount proportional the separation distance between the two atoms.

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