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The dynamical Casimir effect and the generation of thermodynamic entropy

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"*We should always push the mathematical formalism of physics and its internal logic to their ultimate conclusions.*"

Bryce DeWitt

"*I dedicate this dissertation to my late grandfathers Anibal Antunes de Oliveira, Vitória Donadel, Senira Fuehr Siebel and to my still living grandfather Marílho Siebel.*"

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List of Publications

The original results presented in this dissertation have been previously reported in collaboration with Lucas C. Céleri in the following paper,

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Resumo

Nesta dissertação investigamos a dinâmica da produção de entropia termodinâmica no efeito dinâmico Casimir. Isto é feito considerando um campo escalar quântico confinado por uma cavidade unidimensional composta por um par de espelhos ideais, um fixo e outro que pode se mover em uma trajetória prescrita. O objetivo central deste trabalho é compreender como a entropia termodinâmica do campo evolui ao longo do tempo devido ao processo de criação de partículas induzido pelas condições de contorno não triviais impostas pelo espelho móvel. Ao empregar uma abordagem hamiltoniana eficaz, mostra-se que a produção de entropia do sistema aumenta com o número de partículas criadas dentro do limite de curto prazo. Além disso, pode-se também demonstrar que esta abordagem está diretamente relacionada com a geração de coerência quântica na base energética do campo. Utilizando um método distinto, fundamentado na teoria dos estados gaussianos, conseguimos analisar o limite de longo prazo da produção de entropia para um único modo do campo. Os resultados obtidos estabelecem uma relação entre o aumento da entropia termodinâmica no modo de campo e o emaranhamento entre o modo considerado e o resto da estrutura do modo de campo. Desta forma, vinculamos a produção de entropia no campo devido ao efeito Casimir dinâmico com duas características fundamentais da mecânica quântica: coerência quântica e emaranhamento.

Palavras-chaves: Efeito Casimir dinâmico; Produção de entropia; Termodinâmica quântica; Entropia diagonal; Irreversibilidade.

Abstract

In this dissertation, we investigate the dynamics of the thermodynamic entropy production in the dynamical Casimir effect. This is done by considering a quantum scalar field confined by a one-dimensional cavity composed of a pair of ideal mirrors, one fixed and the other allowed to move in a prescribed trajectory. The central goal of this work is to understand how the thermodynamic entropy of the field evolves over time due to the particle creation process induced by the non-trivial boundary conditions imposed by the moving mirror. By employing an effective Hamiltonian approach, the system's entropy production is shown to increase with the number of particles created within the short-time limit. Moreover, one can also demonstrate that this approach is directly related to the generation of quantum coherence in the energy basis of the field. Utilizing a distinct method, grounded in the theory of Gaussian states, we were able to analyze the long-time limit of the entropy production for a single mode of the field. The obtained results establish a relationship between the increase in thermodynamic entropy in the field mode and the entanglement between the considered mode and the rest of the field mode structure. In this way, we link the entropy production in the field due to the dynamical Casimir effect with two fundamental features of quantum mechanics: quantum coherence and entanglement.

Keywords: Dynamical Casimir effect; Entropy production; Quantum thermodynamics; Diagonal entropy; Irreversibility.

Contents

Chapter 1

Introduction

The year was 1947, and Hendrik Casimir, a Dutch physicist based at Philips Research Labs, had already obtained, together with Dirk Polder, the expression for the dispersive forces between atoms in the long-distance regime, today known as Casimir-Polder forces [1]. In the summer of that same year, while walking with Niels Bohr and talking about the unexpected simplicity of the obtained expression, Casimir heard from him some murmurs about a loose connection between his results and the "zero-point energy" [2]. Inspired by Bohr's words, Casimir was led to consider the role played by the vacuum fluctuations of the electromagnetic field in those interactions. In 1948, after considering how the presence of a pair of parallel ideal conductor plates could change the zero-point energy of free space, Casimir predicted the emergence of attractive forces between the plates even in vacuum, the surprising physical phenomenon now known as the *Casimir effect* [3].

Particularly, Casimir's result offers us a novel perspective on the nature of the vacuum and invites us to realize that the latter cannot be seen just as an inert actor, but rather as a physical entity full of activity that reacts against distortions caused by the presence of other objects [4]. Indeed, modern arguments, grounded on the fluctuation-dissipation theorem [5], anticipate that because of this fluctuating character, a moving ideal plate (a non-uniform accelerated mirror, more specifically) should perceive the vacuum as a viscous medium, inducing dissipative reactive forces [6]. The immediate question that emerges under those considerations is how such a moving mirror could dissipate energy if the system is in a vacuum. Looking in retrospect, the correct explanation to this puzzle was already contained in the surprising prediction made in 1970, by the American physicist Gerald Moore [7]. There, he showed how non-stationary boundaries, such as a set of moving mirrors, can lead to the creation of pairs of particles from the vacuum state of a quantum field.

In his seminal paper, Moore considered the quantization of a linear polarized electromagnetic field confined by a one-dimensional cavity composed of two ideal mirrors, one at rest and the other describing an arbitrary trajectory in time. Moore could bypass the initial difficulty associated with time-dependent boundary conditions by mapping the problem into the static mirror model, with the help of the conformal symmetry present in his (1+1)-dimensional problem. Using perturbation theory in terms of slow mirror velocities (with respect to c), he predicted that such a time-dependent

cavity should produce only a negligible amount of photons from the vacuum. In 1975, Bryce DeWitt [8] made, independently, the equivalent prediction in terms of a single moving mirror setup. For this reason, such a phenomenon is recurrently called *Moore-DeWitt effect*, although nowadays, it is mostly known as the *dynamical Casimir effect* (DCE): the physical phenomenon of *particle creation from the vacuum due to non-adiabatic changes on quantum field properties, such as non-stationary boundary conditions (like moving mirrors) or time-dependent cavity properties of the filling medium (like changing dielectric constant).*

The notion that the vacuum can lead to the generation of particles dates back to the pioneering work of Schrödinger in 1939, who explored the possibility for particle production in an expanding universe [9]. Similar suggestions (in other curved space-time contexts) were also made by DeWitt [10] in 1953 and Imamura [11] in 1960. But it was Parker [12, 13], in 1969, and Zel'dovich [14], in 1970, who were the first to introduce a comprehensive treatment of cosmological particle creation. Attention to moving mirrors configurations was resumed with the outstanding result obtained by Hawking in 1974 [15], who predicted the thermal emission of particles from black holes, in the phenomenon of *Hawking radiation* [16]. This led, Fulling and Davies in 1976 [17], to a more detailed study on DeWitt's simple model of a moving mirror. The philosophy adopted by the authors was that in order to better grasp Hawking results, one would first need to understand the effects of acceleration (of mirrors) on quantum field theory (QFT). The combined efforts of DeWitt, Davies, Fulling and Unruh on the subject [8, 18–20], culminated, in the same year with the profound realization that the notion of particle in QFT is observer-dependent. This was due to the discovery of the *Unruh effect* [21], which predicts the vacuum to be seen by a uniformly accelerated observer as a thermal bath of particles. In 1977, Davies and Fulling [22] extended previous results on the radiation of moving mirrors with emphasis on an energy-momentum tensor analysis. Such an analysis was also carefully carried out by Candelas and Deutsch [23] in the same year.

Subsequently, research in the DCE gained traction, and an increasing number of physicists began to consider different theoretical configurations with the common theme of the amplification of vacuum fluctuations into real particles by rapidly changing quantum field properties. The name "dynamical Casimir effect" itself was originally coined by Yablonovitch in 1989, in a seminal paper [24] where he drew attention to the possibility of observing the Unruh effect through the sudden change in the index of refraction of the filling medium of a cavity. A competing term for the research program was also put forward by Dodonov et al. [25] as *nonstationary Casimir effect*. Nonetheless, the tag *dynamical* end up gaining overwhelming popularity due to the authority of Schwinger [26] in his fascination with Sonoluminescence —the intriguing phenomena of intense light flashing by a bubble of air in an extreme acoustic field [27]— cultivated in his last years of life, as he conjectured (wrongly, unfortunately) that dynamical consideration of the Casimir effect could explain the phenomenon. In more than fifty years of existence the research program has received a great deal of attention with numerous developments, encompassing the considerations of the particle production from semi-transparent

mirrors [28–33], to quantized motion of the walls [34–40], effective Hamiltonian approaches [34, 41–45], distinct geometrical configurations [46–50], curved spacetime settings [51–53], nonlinear interactions [54–56] and entanglement dynamics [57–60]. For a comprehensive overview, interested readers are directed to a recent review [61].

As widely acknowledged in the literature, the DCE can only occur when a quantum field is subjected to *non-adiabatic* changes of one of its external parameters [61]. For instance, a moving cavity can amplify vacuum fluctuations into particles, but only if the mirror positions change rapidly enough to prevent the field from re-adapting to its instantaneous configuration. Since in thermodynamics, irreversible processes typically unfold similarly, the last behavior strongly suggests that the effect must exhibit some sort of intrinsic irreversibility. However, in spite of this intuition, the literature in the DCE lacks a robust study about this matter, raising the question of how one can confirm such a hypothesis and, if confirmed, what sources might be playing a role in the irreversible dynamics of the phenomenon.

For this reason, by focreated particles.cusing on a physical configuration where a quantum field is constrained by classical moving mirrors, we are then immediately led to consider the study of such dynamics in terms of a familiar measure of irreversibility in physics: the change in the system's thermodynamic entropy. Consequently, the central question of the work can be summarized as: *how much thermodynamic entropy is produced in a quantum field due to the DCE if one takes into account the correspondent time-dependent nature of boundary conditions*?

To do so, we first delimit the object of study, by choosing the most simple (but still non-trivial) system that reproduces the DCE: that of a quantum scalar field confined in a one-dimensional cavity by a pair of ideal mirrors, with one fixed in a given position and the other allowed to move in a prescribed trajectory. Since the system under consideration is isolated from the environment, a satisfactory discussion on the topic can only be achieved if we introduce the necessary ingredients of thermodynamics in isolated quantum systems, which is the subject of Chapter 2. We begin the correspondent chapter with a brief discussion on the irreversibility present in physics and the increasing necessity of understanding thermodynamics from a quantum mechanical point of view. In the second part of the chapter, we then explore a much more deep question associated with the correct definition of microscopic entropy, capable of extending the laws of thermodynamics to the quantum realm. For such, we introduce the most popular candidates for microscopic entropy, in order: von Neumann, Boltzmann, observational and diagonal entropy, discussing their definitions, properties and pros and cons. We complete the corresponding chapter by motivating why we end up choosing the diagonal entropy as our thermodynamics entropy of interest.

Subsequently, in Chapter 3, we properly introduce the mathematical machinery used in the literature to characterize and compute interesting quantities in the DCE. In the first part, we delineate the system in terms of a classical field theory, followed by a formal introduction of the quantization procedure. Finally, the general paradigm of the DCE is introduced, by showing how the imposed

time-dependent boundary conditions translates into the particle creation process. In the second part of the chapter we then introduce the so called, *instantaneous basis approach* to be utilized through the work and whom we derive an effective Hamiltonian in Schrödinger picture and the widely used techniques of instantaneous decomposition in the Heisenberg picture. Finally, in the third part, we give an overview on how to compute the so-called *Bogoliubov coefficients* —a set of coefficients that connect the particle notion before and after the cavity motion— from both techniques outlined in the latter approach.

In Chapter 4, we get hands-on and use the knowledge gathered in the previous chapters to derive the original results presented in this dissertation. Specifically, we compute the explicit expressions for the diagonal entropy, which is the adequate thermodynamic entropy for our context. To implement the calculations, we use two distinct methods: one based on the effective Hamiltonian and the other grounded on the evolution of Gaussian states. In the first part of the chapter, we use the effective Hamiltonian to evolve the density operator through a perturbation theory on the small mirror amplitude. Due to the inherent complexity of the Hamiltonian, the naive expansion leads to secular terms in the special (but important) case of the oscillation cavity, constraining the results to the short-time limit. A general expression for the diagonal entropy is then computed in terms of the density operator, as well as its version for the particular oscillating case. Consequently, we obtain that the entropy production in the short-time regime scales with the number of particles, and it is directly linked with the developments of quantum coherence in the energy eigenbasis. For the second part of the chapter, we briefly revise the literature in the Gaussian states, i.e., a class of quantum states in which its Wigner function has a Gaussian character. In terms of the knowledge of the Wigner function for a single mode of the field, we can then obtain expressions for the reduced density operator in a given field mode. With the help of the Bogoliubov coefficients associated with the parametric oscillations of the second mirror, we can evolve the reduced expression for the diagonal entropy (single-mode). Here we are able to obtain expression in the short-time limit, where the scaling behavior with the number is recovered, and in the long-time regime, where it is found that the diagonal entropy corresponds to the Rényi-2 entropy, and therefore, quantifies the amount of entanglement between the considered mode and the rest of the field.

Finally, in chapter 5 —not surprisingly named conclusions and perspectives—, we conclude the dissertation by summarizing all the important results obtained throughout this work, interpreting them and providing some perspectives for further investigations.

Chapter 2

Thermodynamics of isolated quantum systems

As delineated in Chapter 1, the main goal of this work is to investigate the intrinsic irreversibility associated with the phenomenon of the DCE by analyzing the thermodynamic entropy production. The system we choose to study is composed by a quantum scalar field confined by an idealized time-dependent cavity. Since the cavity properties are described by external prescribed parameters, our system can be said to be isolated in the thermodynamic sense (although not closed), i.e., the quantum system can exchange energy in a controllable manner with the mirrors, but its remain in a pure state since the cavity dynamics in still classical. In this sense it will be important to explore the thermodynamics of isolated quantum systems in order to justify the choice of the thermodynamic entropy to be utilized throughout this description.

2.1 Irreversibility in Physics

Except for gravity, all the known fundamental forces of nature are described by the standard model of particles in terms of quantum field theories [62]. There, each interaction and fundamental particle can be shown to satisfy **CPT** symmetry, with the laws of physics remaining the same over the combined conjugation of **C**harge (interchanging particles by antiparticles), inversion of **P**arity (replacing left hand by right hand) and reversal of **T**ime (reversing the direction of motion of all particles). In most processes, one can neglect the effects of C and P, such that all interactions must be invariant over T. Consequently, the most basic laws of physics can be said to be time-symmetric [63] and devoid of a preferred direction in time for the occurrence of physical phenomena.

Despite this observation about fundamental physical theories, almost all the phenomena we encounter in our daily lives seem to follow very specific temporal orders: milk always dilutes in coffee but never separates again; hot soup always cools to room temperature but never heats up at the expense of cooling the environment; we are born, we grow old, and we die, never the other way around. All of these cases are instances of *irreversible processes*, *i.e.*, processes that seems to follow an "arrow in time" by manifesting spontaneously in one direction but never in the opposite.

Classical thermodynamics was perhaps the first truly physical theory to incorporate into its basic assumptions the distinction between the order of occurrence of such processes. In it, irreversibility is quantified through a state function of the system called entropy. Under the statement of the second law of thermodynamics, the entropy of a closed system, either increases or remains constant over time, but never decreases [64]. This law, which was initially introduced on purely phenomenological grounds, establishes restrictions on the occurrence of phenomena, which despite respecting the conservation of energy, never manifest themselves spontaneously in the macroscopic regime.

It was only with the advent of statistical physics, that the nature of the second law could be understood not as a fundamental law in the strict sense, but a statistical statement that hold true only on average. More specifically, its validity is recovered when dealing with sufficiently complex systems comprised of a vast number of microscopic constituents –as in the case of a typical macroscopic system– where one expects statistical deviations from the second law to become vanishing small and entropy to be likely to increase (or stay constant) in the overwhelming number of cases. When statistical deviations do in fact play a role, stronger principles known as fluctuation theorems emerge [65, 66], and irreversibility is then defined in terms of those processes in which entropy tends to increase on average.

With the rapidly technological advancements in experimental techniques over the last few decades, experimentalists now have an increasing control over the states of quantum systems, enabling the detail study of their behavior even out of equilibrium. In such scenarios physicist not only have access to open systems like small quantum systems in contact with weak thermal baths, which are instances of *open quantum systems* [67], but can also isolate many-body systems from the environment with high degree of tunability, as in the case of ultra cold atoms experiments [68, 69]. While the status of thermodynamics as a phenomenological theory is well-established, challenges arise when attempting to apply it to non-equilibrium situations as captured by those small quantum systems. Hence, the task of understanding how thermodynamics emerges from quantum mechanical and statistical considerations —commonly known as *quantum thermodynamics*– is of increasing importance. For this context, various approaches have emerged in the quest to grasp thermodynamics from a microscopic level. Notable developments include those based on statistical physics [65, 70], resource theory [71], density functional theory [72], axiomatic formulation [73] and information theory [74, 75]. For a comprehensive study of entropy production in both classical and quantum systems, we recommend Ref. [76] and its associated sources.

2.2 In the pursuit of a microscopic definition for entropy

An ongoing conundrum in the foundations of physics refers to the problem of how to establish the correct definition of microscopic entropy that naturally extends the laws of thermodynamics to the context of quantum mechanics for both open and isolated systems [77–80]. In this section, we will

explore some candidates for microscopic entropy and motivate the choice of thermodynamic entropy to be utilized in our work.

2.2.1 The von Neumann Entropy

Perhaps the most natural candidate for microscopic entropy —due to its exceptional success for applications in information and communication theories— is the von Neumann entropy S_{vN}, which is defined for a system's state $\hat{\rho}$ to be simply

$$S_{vN}(\hat{\rho}) = -\operatorname{Tr}\{\hat{\rho}\ln\hat{\rho}\}\,. \tag{2.1}$$

Importantly, the von Neumann entropy is consistent with thermodynamic considerations in two important situations:

1. **When the system is in equilibrium** and is represented by statistical ensembles, such as the Gibbs thermal state
$$\hat{\rho}_{\mathrm{G}}(\beta) := \frac{e^{-\beta\hat{H}}}{Z(\beta)}, \qquad Z(\beta) := \operatorname{Tr}\left\{e^{-\beta\hat{H}}\right\}, \tag{2.2}$$
or its generalizations, as in the case of the grand canonical ensemble (β is inverse of the thermodynamic temperature).

2. **For ensembles, even out of equilibrium, that describes open systems coupled to an ideal heat bath**. Such consideration is justified in light of the extensive use of S_{vN} in quantum thermodynamics to study small quantum systems in weak contact with a large thermal bath.

On the other hand, there are strong arguments against the idea that the von Neumann entropy correspond to the thermodynamic entropy *in general*. The main reason is related with its property of being invariant under a unitary evolution, such as in

$$S_{\mathrm{vN}}(\hat{\rho}) = S_{\mathrm{vN}}(\hat{U}\hat{\rho}\hat{U}^{\dagger}), \tag{2.3}$$

where $\hat{U}$ is the unitary operator associated with the time evolution of $\hat{\rho}$. As a result, if we were to interpret S_{vN} as providing an accurate thermodynamic description, it would logically entail the prediction that the thermodynamic entropy of isolated quantum systems should remain constant for any process. The implication stands in direct contradiction to the second law of thermodynamics that postulate a strict increase in thermodynamic entropy for such systems, as it is observed empirically during the free expansion of a gas or in the mixing of liquids. Other important postulate of thermodynamics that S_{vN} violates is the uniqueness of entropy as a function of energy. This violation occur specifically in isolated systems, where the change in S_{vN} must vanish in any cyclic process despite some possible non-zero change for the system's energy [78].

In principle, one could stand for the position that S_{vN} is in fact our desired microscopic entropy and restricting its use only for situations where an adequate ensemble of states is possibly defined. However, this cannot possibly explain how the laws of thermodynamics can emerge from microscopic considerations, where the time evolution of the system $\hat{\rho}(t)$ is given from its initial state [77]. In this respect, although the von Neumann entropy stands as an essential informational tool and coincides with the thermodynamic entropy for important situations, we can not elevate S_{vN} to the status of microscopic entropy to be used throughout this work (specially because we must consider an isolated quantum system).

2.2.2 Boltzmann Entropy

Another contestant for microscopic definition of entropy is the Boltzmann entropy S_B. To describe it in the context of quantum mechanics, let us first consider any closed system governed by a Hamiltonian operator $\hat{H}$ respecting the following stationary Schrödinger equation

$$\hat{H}\left|E_i\right\rangle = E_i\left|E_i\right\rangle,$$

where $\{|E_i\rangle\}$ corresponds to the set of energy eigenstates with energy eigenvalues E_i.

If the system under consideration is sufficiently simple such that an observer has complete experimental control over it, one can understand an energy measurement in terms of orthogonal projectors $\Pi_i = |E_i\rangle\langle E_i|$, which, as the name itself reveals, projects an arbitrary state into the correspondent energy eigenvalue. However, in a more realistic situation, the inherent complexity of the system implies that measurement instruments should have an intrinsic resolution (or uncertainty) δ_E associated with energy readings E. This means that instead of sharp detection of individual spectrum lines, now, any energy value E measured by the observer should be contained in an interval $\Delta_E := [E, E+\delta_E]$ with a bunch of possible exact energy eigenvalues E_i in it. We call the set of possible macroscopically distinguishable energy outcome $\{E\}$, the *macrostate* of the system.

In this context of ignorance over the complete description of the system, more realistic measurements are characterized by the following projectors

$$\Pi_E = \sum_{E_i\in[E,E+\delta_E]} |E_i\rangle\langle E_i|,$$

whose set $\{\Pi_E\}$ is complete ($\sum_E \Pi_E = 1$) and orthonormal ($\Pi_E\Pi_{E'} = \delta_{E,E'}\Pi_E$). We call such measurements as *coarse-grained* since it capture the loss of perfect knowledge of the system due to the coarseness of our macroscopic apparatuses.

If an observer performing a coarse-grained measurement obtains an energy outcome E, then the Boltzmann entropy of the system can be defined as

$$S_B(E) := \ln V_E, \tag{2.4}$$

where $V_E = \mathrm{Tr}\{\Pi_E\}$, referred as the system's *volume element*, is simply the rank of the projector Π_E. Since V_E represent the amount of exact energy eigenvalues contained in the coarse-grained interval Δ_E, one can then interpret the Boltzmann entropy as counting the number of microstates (energy eigenvalues) which are compatible with the system's macrostate with energy outcome E.

In the situation in which we have information about further macroscopic variables, such as the particle number N, the Boltzmann entropy can be generalized as

$$S_B(E, N, \dots) = \ln V_{E,N,\dots}, \tag{2.5}$$

with $V_{E,N,\dots}$ counting all the microstate compatible with the macroscopic constraints E, N, etc.

An advantage of Eq. (2.4) over S_{vN}, is that Boltzmann entropy in non-zero even for a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$, and therefore, is compatible with the description of isolated quantum systems. Another positive side in favor of S_B is related on how intuitively the Boltzmann entropy explain the second law of thermodynamics without needing the notion of statistical ensembles. If one consider an isolated system initially prepared in a state with small volume term, one expects the same to evolve towards a region with larger volume and to spend most of the time within the largest volume term (correspondent with thermal equilibrium).

But just like S_{vN}, the Boltzmann entropy faces some conceptual obstacles that prevent its identification as our sought microscopic entropy. The first reason is related with its inadequacy in the description of small quantum systems, where an experimenter usually have precise control over the quantum degrees of freedom. Since the volume term is equal to unity for a single microstate, one would expect S_B to vanish, and therefore, to not explain how S_{vN} can possibly describe the thermodynamic entropy for small open systems. The second reason is associated with the unsatisfactory situation in which one needs to introduce a non-uniquely defined coarse-graining on top of the system's dynamics. From those reasons, we should refrain from utilizing the S_B in our description.

2.2.3 Observational entropy

A third possible candidate for microscopic entropic quantity which interpolates the ideas of von Neumann and Boltzmann is the so-called *observational entropy*. To properly define it, we begin considering a set of coarse-grained measurements $\mathcal{C} = \{\Pi_c\}$ characterized by a complete set of orthogonal projectors Π_c respecting the identities $\sum_c \Pi_c = 1$ and $\Pi_c\Pi_{c'} = \delta_{c,c'}\Pi_c$. In this respect, if the system is in a given state $\hat{\rho}$, the observational entropy associated with a coarse-graining $\mathcal{C}$ takes

the following expression

$$S^{\mathcal{C}}_{\text{obs}}(\hat{\rho}) := -\sum_c p_c \ln \frac{p_c}{V_c}, \tag{2.6}$$

where $p_c := \text{Tr}\{\Pi_c \hat{\rho}\}$ is the probability of observing the outcome c, while $V_c := \text{Tr}\{\Pi_c\}$ is the volume term for Π_c.

As we will see next, one can generally interpret the expression (2.6) as quantifying the amount of information an observer can deduce from the system's initial state if he/she perform measurements within a basis defined by the process of coarse-graining. To see this we can rewrite Eq. (2.6) as in

$$S^{\mathcal{C}}_{\text{obs}}(\hat{\rho}) = \sum_c p_c \left(-\ln p_c + \ln V_c\right) = S_{Sh}(p_c) + \sum_c p_c S_B(c), \tag{2.7}$$

where $S_{\text{Sh}}(p_c) = -\sum_c p_c \ln p_c$ is the Shannon entropy associated with the probabilities p_c and the second term corresponds to the average value of the Boltzmann entropy for such outcome. In terms of the last expression (2.7) it easy to identify that

1. In the fine-grained situation where we have access to all physical information about the system, we can choose a coarse-graining $\mathcal{C}_\psi = \{|\psi\rangle \langle\psi|\}$ correspondent to the eigenstate of the system's density operator $\hat{\rho} = \sum_\psi \rho_\psi |\psi\rangle \langle\psi|$ (which is the most informative measurement possible). Since in this case the volume term is $V_\psi = 1$, one obtains

$$S^{\mathcal{C}_\psi}_{\text{obs}}(\hat{\rho}) = S_{\text{Sh}}(\rho_\psi) = S_{vN}(\hat{\rho}); \tag{2.8}$$

2. In the maximum coarse-grained situation where we only have access to imprecise measurements, we are restricted to the coarse-graining $\mathcal{C}_O = \{|o\rangle \langle o|\}$ associated to the measurements of a given macroscopic observable $\hat{O}$. In this context, after measuring the system we will have full confidence that the system's state $\hat{\rho}$ is found in a given macrostate $\Pi_{O'}$. Since the probabilities fall to be $p_O = \delta_{oo'}$ with volume term $V_O = \text{Tr}\{\Pi_{o'}\}$, one obtains,

$$S^{\mathcal{C}_O}_{obs}(\hat{\rho}) = S_B(o'). \tag{2.9}$$

   If the chosen macroscopic observable correspond to the system's energy (or the set of known macroscopic variables), Eq. (2.9) will coincide with the Boltzmann definition of entropy.

Therefore, the observational definition of entropy is sufficient flexible to reproduce the two previous notions of entropic quantities (von Neumann and Boltzmann).

Finally, it is important to notice that although Eq. (2.6) is well-defined for an arbitrary state $\hat{\rho}$ and coarse-graining $\mathcal{C}$, one still need to make the right choice of coarse-graining to connect $S^{\mathcal{C}}_{\text{obs}}(\hat{\rho})$ with the thermodynamic entropy. In this regard, the observation entropy suffers from a difficulty similar to that faced by S_B, relying on the introduction of a coarse-graining, which is non-unique and independent of the Hamiltonian structure.

On the other hand, the observational entropy can still be employed to analyze a third special situation: where we can only perform fine-grained measurements on a fixed basis. Take for example a sufficiently complex isolated quantum system that can exchange energy in a controllable manner due to an external time-dependent parameter, such as a changing magnetic field or a non-stationary cavity property. Although the system's energy eigenstates are now time-dependent $|E_n(t)\rangle$, due to the system's complexity, the experimentalist in general only have control over energy measurements on a fixed basis (such as the initial eigenstate $|E_n(0)\rangle$). In this circumstance, we identify the observational coarse-graining measurement $\mathcal{C}_d$ to be given by the set of fixed initial energy eigenstate projectors $\{|E_n(0)\rangle\langle E_n(0)|\}$, with volume term $V_n = 1$ and probabilities $p_{E_n}(t) = \langle E_n(0)|\hat{\rho}(t)|E_n(0)\rangle$. Consequently, the observational entropy in this context takes the form

$$S_{obs}^{\mathcal{C}_d}(\hat{\rho}(t)) = S_{vN}(\hat{\rho}_d(t)) = -\sum_n p_{E_n}(t)\ln p_{E_n}(t), \tag{2.10}$$

where $\hat{\rho}_d(t) = \sum_i p_{E_n}(t)|E_n(0)\rangle\langle E_n(0)|$ are the diagonal components of the density operator in the fixed initial energy eigenbasis. An interesting feature of expression (2.10) is that since $S_{vN}(\hat{\rho}_d(t))$ is defined instantaneously, there is no requirement for the introduction of any further coarse-graining or to assume the Hamiltonian to be stationary. In the next section, we shall investigate with more attention the properties of the last expression.

2.2.4 Diagonal entropy

From the problems outlined in the previous sections, we can start making sense about the conditions we should expect from a microscopic entropy to be consistent with our system: (i) it should be defined for isolated quantum systems; (ii) it should be independent of coarse-graining procedures; and (iii) it should be defined for time-dependent Hamiltonian. An immediate candidate that satisfies such conditions is given by expression (2.10) and it was first introduced in Ref. [78] as the thermodynamic entropy for closed systems: the so-called *diagonal entropy* or *d-entropy* S_d, which is given in a more clean notation by

$$S_d := -\sum_n \rho_{nn}\ln\rho_{nn}, \tag{2.11}$$

where ρ_{nn} are the diagonal components of the system's density operator in the eigenbasis of the Hamiltonian (energy eigenstates). An ambiguity inherent in the last expression corresponds to the circumstance in which energy eigenstates are degenerate. As such situations occur accidentally or as a result of system symmetries, they are generally absent and will be ignored throughout the text. It is also important to emphasize that S_d expression differs from that of the von Neumann entropy, written in the basis that diagonalize the density operator. As expressed in Eq. (2.11), we are really discarding all the non-diagonal contributions of S_{vN} when expressed in the Hamiltonian eigenbasis.

More specifically, the diagonal entropy can be interpreted as quantifying the amount of randomness observed in the energy eigenbasis of the system. To see this more explicitly, one can consider the commonly encountered situation where the system has a sufficiently large dimensionality and quantum state tomography becomes impractical [81]. For such physical configurations, it is reasonable to assume a physicist to have access to only a restricted number of possible measurements, such as the energy values of the system. Therefore, for a general process (unitary or not) it becomes inevitable the emergence of transitions between instantaneous energy levels and the consequent development of quantum coherence and entanglement among different parts of the system. The diagonal entropy can then be identified as measuring the loss of information due to the limited set of measurements available.

Another interesting result to highlight here is that an emerging notion of entropy exactly equal to the S_d expression can be derived from a recent framework of quantum thermodynamics introduce in Ref. [82]. Indeed, in this formalism, which is grounded in physical quantities that remain invariant under an emergent gauge group, the corresponding gauge-invariant definition of heat can be attributed to the (almost inevitable) transitions between energy levels [82].

In Ref. [78] the author has putted forward some arguments to motivate (on physical grounds) the central importance of the diagonal contributions of $\hat{\rho}$ (in the energy basis) for the thermodynamic characterization of quantum systems.

1. **In general, all information about time-average observables —which appear in any thermodynamic measurements— is contained only at the diagonal elements of $\hat{\rho}$.**

   The reason for such consideration is associated with the description of sufficiently complex systems that have achieved a steady state after some process that occur in the distant past, such as in

   $$\hat{\rho}(t) = \sum_{mn} \rho_{nm} e^{-i(E_m - E_n)t} \left|m\right\rangle \left\langle n\right|, \tag{2.12}$$

   where $\hat{H}\left|n\right\rangle = E_n \left|n\right\rangle$ is the stationary Schrödinger equation for the final time-independent Hamiltonian. If one anticipate that quantum systems, even when isolated, are likely to exhibit some form of ergodic behavior [78, 83], then the time averages for any thermodynamic observable $\hat{O}$ should be comparable to its equilibrium ensemble average, as in

   $$\overline{\left\langle \hat{O}(t) \right\rangle} \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \mathrm{d}t \left\langle \hat{O}(t) \right\rangle, \tag{2.13}$$

   where $\left\langle \hat{O}(t) \right\rangle = \mathrm{Tr}\left\{ \hat{\rho}(t) \hat{O} \right\}$. The last expression means that for the steady state (2.12), one expects such observable time averages to respect

   $$\overline{\left\langle \hat{O}(t) \right\rangle} = \sum_{nm} \rho_{nm} O_{nm} \overline{\langle e^{-i(E_m - E_n)t} \rangle}, \tag{2.14}$$

where $O_{nm} \equiv \langle n| \hat{O} |m\rangle$ are the observable matrix elements in the energy eigenstate basis. For the case in which the eigenstates of the Hamiltonian are not degenerate, the time average over the exponential terms reduces to [83]

$$\overline{\langle e^{-i(E_m-E_n)t}\rangle} = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau \mathrm{d}t e^{-i(E_m-E_n)t} = \delta_{nm}. \tag{2.15}$$

As a result, one obtains from Eq. (2.14) the following expression

$$\overline{\left\langle \hat{O}(t)\right\rangle} = \sum_{nm} \rho_{nn} O_{nn}, \tag{2.16}$$

which is dependent only on the time-independent diagonal elements of the density operator in the eigenbasis of the Hamiltonian. The conclusion we draw from this expression, is that relevant information for thermodynamic measurements is only present in the diagonal terms ρ_{nn}.

2. **The diagonal components of $\hat{\rho}$ are conserved for quasi-static process**.

A second argument for the importance of the diagonal terms ρ_{nn} in Eq. (2.11) is contained in the *adiabatic theorem for quantum mechanics* [84] which asserts that a slowly varying Hamiltonian do not induce transitions between instantaneous eigenstates. More specifically, given a time-dependent Hamiltonian $\hat{H}(t)$ satisfying $\hat{H}(t)\,|n(t)\rangle = E_n(t)\,|n(t)\rangle$ and an arbitrary state vector

$$|\Psi(t)\rangle = \sum_n c_n(t)\,|n(t)\rangle\,, \tag{2.17}$$

which is a solution of the time-dependent Schrödinger equation, if $H(t)$ changes sufficiently slowly on time, *i.e.*, quasi-static process, one can show that the time-dependent amplitudes $c_n(t)$ must evolve in time as [85]

$$c_n(t) = c_n(0)\exp\left\{-i\int_0^t \left[E_m(t') - i\,\langle m(t')|\,\frac{\partial}{\partial t'}\,|m(t')\rangle\right]\mathrm{d}t'\right\}. \tag{2.18}$$

An immediate consequence of Eq. (2.18) is the conservation of the eigenstate probabilities: $|c_n(t)|^2 = |c_n(0)|^2$, meaning that if the system begins as an eigenstate of the initial Hamiltonian $\hat{H}(0)$ then it will remain as an eigenstate of $\hat{H}(t)$ during all time (differing only by a time-dependent phase factor). A trivial corollary of the last result is that the diagonal terms $\rho_{nn}(t) = |c_n(t)|^2$ of the density operator in the instantaneous energy eigenbasis should remain constant over time for any quasi-static processes. If we identify heat in the quantum realm as energy increase due to transitions between different energy levels —as described in Refs. [78, 86, 87]— one can associate thermodynamic adiabaticity with quasi-static processes. Then, any entropic quantity that dependent only on ρ_{nn}, such as S_d, should be conserved for adiabatic processes,

as it is expected from a thermodynamic entropy.

In addition to the last physical motivation, the diagonal entropy can be show to exhibit all the key properties expected from a thermodynamic entropy. For equilibrium states, where the density operator is stationary (and therefore, diagonal), the d-entropy can be shown to be identical to S_{vN}. As a result, S_d must respect important properties such as extensivity, positivity and vanishing under the zero-temperature limit. In more general scenarios, such as when the d-entropy can change in time, one can prove for any time-dependent process in isolated systems that [78]

$$S_d(t) \geq S_{vN}(0), \tag{2.19}$$

as long as the initial state is stationary. It is also important to remark that the last expression is not saying that $S_d(t)$ is always a monotonic function of time. Furthermore, this entropy increase can be better understood in terms of a general unitary evolution of the density operator in time

$$\hat{\rho}(t) = \hat{U}(t)\hat{\rho}(0)\hat{U}^{\dagger}(t), \tag{2.20}$$

$\hat{U}(t)$ being a unitary operator. For an initially diagonal state $\hat{\rho}(0) = \sum_m \rho_{mm}(0) \left|m\right\rangle \left\langle m\right|$, one can easily show that

$$\rho_{nn}(t) = \left\langle n\right| \hat{\rho}(t) \left|n\right\rangle = \sum_m P_{nm}\rho_{mm}(0), \tag{2.21}$$

where the matrix elements $P_{nm} = |U_{mn}|^2$ determines the transition rates between the instantaneous energy eigenvalues. Given that the matrix P is doubly stochastic ($\sum_m P_{mn} = \sum_n P_{nm} = 1$), there is a tendency for the system to promote a uniform spreading of probability density ρ_{nn} among the energy spectrum. As a result, since S_d is a measure of the spread of ρ_{nn}, for any dynamical process the diagonal entropy must necessarily either increase or remain constant. The validity of Eq. (2.19) could also be extended in Ref. [79] for a particular class of non-diagonal initial states called by the authors as "generalized equilibrium pure states".

Another important property one expects a thermodynamic entropy to respect is the fundamental thermodynamic relation

$$\Delta U = T\Delta S - \sum_j \left.\frac{\partial U}{\partial \lambda_i}\right|_S \Delta\lambda_i, \tag{2.22}$$

which is a direct consequence of the thermodynamic postulate that require the entropy to be a unique function of energy E and the external parameters λ_1, λ_2, etc. By expressing the system's energy $U(t) = \sum_n \rho_{nn}(t)E_n(t)$ in terms of a linear order expansion on $\Delta\rho$ and $\Delta\lambda$, one can easily obtain the change in energy ΔU to be

$$\Delta U \approx \sum_n \Delta E_n(t)\rho_{nn}(0) + \sum_n E_n(0)\Delta\rho_{nn}(t), \tag{2.23}$$

where $\Delta E_n(t) = E_n(t) - E_n(0)$ is the change in the instantaneous energy eigenstates due to the time evolution while $\Delta\rho_{nn}(t) = \rho_{nn}(t) - \rho_{nn}(0)$ is the change in the diagonal components of $\hat{\rho}$. Since we do not expect transitions between instantaneous energy levels in the adiabatic limit ($\Delta\rho_{nn}(t) = 0$), one can identify the first term in Eq. (2.23) given by $\Delta E_{\text{ad}} = \sum_n \Delta E_n(t)\rho_{nn}(0)$, as the adiabatic change of the system's energy, or simply the work done by the system due to the changes of the external parameters λ_i. As a result, the second term $Q(t) = \sum_n E_n(0)\Delta\rho_{nn}(t)$ must correspond to the non-adiabatic change of the energy, i.e., the heat generated by the system when the process is non-quasi-static.

Similarly, a change in the diagonal entropy to the leading order in $\Delta\hat{\rho}_{nn}(t)$ can be obtained as

$$\Delta S_d \approx -\sum_n \left[\Delta\rho_{nn}(t)\ln\rho_{nn}(0) + \rho_{nn}(0)\Delta\ln\rho_{nn}(t)\right] = -\sum_n \Delta\rho_{nn}(t)\ln\rho_{nn}(0), \tag{2.24}$$

since $\sum_n \Delta\rho_{nn}(t) = \text{Tr}\{\rho(t)\} - \text{Tr}\{\rho(0)\} = 0$. If we consider an initially thermal state in Eq. (2.2) in the energy eigenbasis

$$\hat{\rho}(0) = \sum_n \frac{1}{Z} e^{-E_n(0)/T} \left|n\right\rangle \left\langle n\right| \qquad \text{with} \quad Z = \sum_m e^{-E_m|(0)/T}, \tag{2.25}$$

is immediately from Eq. (2.24) that $\Delta S_d = \sum_n \Delta\rho_{nn}(t)E_n(0)/T = Q(t)/T$. Using this last identity one can then show by that Eq. (2.23) can be written as

$$\Delta U \approx \Delta E_{\text{ad}} + T\Delta S_d. \tag{2.26}$$

Since ΔE_{ad} is understood as a function of the state of the system, one can write it in terms of the partial derivatives of the system energy over the external parameters as in

$$\Delta E_{\text{ad}} = \sum_j \left.\frac{\partial U}{\partial\lambda_i}\right|_{S_d} \Delta\lambda_i, \tag{2.27}$$

making Eq. (2.26) equivalent to the fundamental thermodynamic relation (2.22). In fact, the last demonstration can be extended for more general scenarios. More specifically, the diagonal entropy can be shown to be a unique function of energy (and therefore, to satisfy the fundamental relation) as long as the system's Hamiltonian is local and non-integrable [78].

Finally, due to the correspondence between S_d and S_{vN} for equilibrium states, is immediately that the diagonal entropy must be additive in thermal equilibrium. However, in non-equilibrium conditions, the situation becomes more subtle. While the sum of thermodynamic entropies for subsystems aligns with the sum of their diagonal entropies, they are not equal to the total S_d. Additionally, it can also be demonstrated that when two initially uncoupled systems each in local equilibrium, are allowed to interact, the sum of their diagonal entropy should satisfy the inequality $S_d^{(1)}(t) + S_d^{(2)}(t) \geq S_d^{(1)}(0) + S_d^{(2)}(0)$. This last result is aligned with thermodynamic expectation, since

the second law of thermodynamics demands the sum of entropies to increase or remain constant.

Chapter 3

The dynamical Casimir effect

As explained in the introduction, the DCE can be understood as the particle creation phenomenon that occurs when the structure of the vacuum state of a quantum field is distorted in a non-adiabatically manner. Although this description is qualitatively informative, to better grasp the origin of such effect is essential to define it in a mathematically precise manner. In this chapter we must, therefore, derive the DCE for the particular system under consideration, i.e., a quantum field constrained by a moving cavity. After this we shall complement the discussion by introducing the mathematical formalism needed to compute, in the next chapter, the system's entropy production.

3.1 QFT through the DCE

The goal of this section is twofold: in one hand we introduce the DCE and the notation to be used in the rest of this work, but on the other hand, we also make the discussion sufficiently rigorous from the mathematical point of view, such that will serve as an introduction to quantum field theory in time-dependent backgrounds.

3.1.1 The system

In order to reproduce the DCE, we begin considering a real massless Klein-Gordon field $\Phi(x^\mu)$ embedded in a $(1+1)$-dimensional Minkowski spacetime with a Lagrangian density

$$\mathcal{L}(x^\mu) = \frac{1}{2}\eta^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi, \tag{3.1}$$

where $x^\mu \equiv (x^0, x^1)$ is the two-vector and $\eta^{\mu\nu} = \text{diag}\{1, -1\}$ with $\mu, \nu = 1, 2$. If we introduce a global inertial coordinate system (t, x), one can account for the dynamical nature of the phenomenon by confining the field into a time-dependent space interval $\Sigma(t) = \{x \in [x_1, x_2(t)]\}$ with two boundaries at its endpoints: one fixed at $x_1 = x_0$, and another allowed to move in a prescribe trajectory $x_2(t) = x_0 + l(t)$, with $l(t)$ being the time-dependent interval distance. For simplification purposes, we account for the interaction between the field and the moving boundaries by applying the following

Dirichlet conditions on the field

$$\Phi(x_1, t) = \Phi(x_2(t), t) = 0. \tag{3.2}$$

Therefore, if one consider Φ as describing a convenient component of the electromagnetic vector potential when the polarization effects are negligible, then the imposed boundary conditions describes a time-dependent cavity composed by two ideal mirrors (perfect reflectors). From now on we must adopt this last interpretation for $\Sigma(t)$.

From the previous considerations, the system can then be described (in the chosen inertial reference frame) in terms of the following action

$$A = \frac{1}{2}\int \mathrm{d}t \int_{\Sigma(t)} \mathrm{d}x \left[(\partial_t\Phi)^2 - (\partial_x\Phi)^2\right]. \tag{3.3}$$

By demanding Eq. (3.3) to be stationary ($\delta A = 0$), one obtain the equation of motion for the field to be simply the wave equation with dynamical boundaries conditions

$$\left(\partial_t^2 - \partial_x^2\right)\Phi(x,t) = 0, \quad \text{with} \quad \Phi(x_0, t) = \Phi(x_0 + l(t), t) = 0. \tag{3.4}$$

A more rigorous account of the system can be implemented by introducing a phase space formulation for the theory. Since we are describing a quadratic lagrangian with linear dynamical equations, this last procedure can be accomplished by specifying the symplectic linear space $(\mathscr{M}, \Omega(.,.))$ of the theory, where $\mathscr{M}$ is a vector space and $\Omega(.,.)$, is the canonical symplectic form, i.e., a non-degenerate, antisymmetric, bilinear map. From the Lagrangian density (3.1), one can begin this task by introducing the canonically conjugate field

$$\Pi(x,t) = \frac{\partial\mathcal{L}}{\partial(\partial_t\Phi)} = \partial_t\Phi(x,t), \tag{3.5}$$

which is defined on the spatial interval $\Sigma(t)$ at a given global inertial time t. Here, we restrict attention to the class of smooth functions which live in $C_0^\infty(\Sigma(t))$: the space of infinite differentiable functions which are compact support at this last space interval. As a result, one can define the system's phase space $\mathscr{M}$, at a given instant of time t_0, by specifying the values of the pair of function Φ and Π, on $\Sigma(t_0)$, forming

$$\mathscr{M} \equiv \{[\Phi, \Pi] | \Phi : \Sigma(t_0) \to \mathbb{R}, \Pi : \Sigma(t_0); \Phi, \Pi \in C_0^\infty(\Sigma(t_0))\}. \tag{3.6}$$

Since Eq. (3.4) is well-posed (there exist a unique solution for a given initial-value specification), one can define $\mathcal{S}$ to be the space of solutions of the Klein-Gordon equation (with dynamical boundary conditions (3.2)) which arise from an initial data in $\mathscr{M}$.

Lastly, to define the system's symplectic structure, we search for bilinear maps of the form $\mathcal{S} \times \mathcal{S} \to \mathbb{R}$, which are conserved over time. A hint to this task can be found by looking to the

structure of the Klein-Gordon equation $\partial_\mu \partial\mu\Phi = 0$. If we consider two solutions Φ_1 and Φ_2 on $\mathscr{S}$, one must expect them to respect,

$$0 = \Phi_1 \partial_\mu \partial^\mu \Phi_2 - \Phi_2 \partial_\mu \partial^\mu \Phi_1 = \partial^\mu J_\mu(\Phi_1, \Phi_2), \tag{3.7}$$

where we have defined $J_\mu(\Phi_1, \Phi_2) =: \Phi_1 \partial_\mu \Phi_2 - \Phi_2 \partial_\mu \Phi_1$. Using $\partial^\mu J_\mu = \eta^{\mu\sigma} \partial_\sigma J_\mu$, follow immediately from the last expression that

$$\partial_t \int_{\Sigma(t)} \mathrm{d}x J_0 = \int_{\Sigma(t)} \mathrm{d}x \partial_x J_1 = \left[\Phi_1 \partial_x \Phi_2 - \Phi_2 \partial_x \Phi_1\right] \Big|_{x_1}^{x_2(t)} = 0, \tag{3.8}$$

meaning that the spatial integral of $J_0 = \Phi_1 \partial_t \Phi_2 - \Phi_2 \partial_t \Phi_1$ over the interval $\Sigma(t)$, remain the same for all instants of time. As an immediate consequence, by giving two points of $\mathscr{M}$, say, $[\Phi_1, \Pi_1]$ and $[\Phi_2, \Pi_2]$, both at an arbitrary instant of time t_0, we are compelled to introduce as the symplectic structure $\Omega : \mathscr{M} \times \mathscr{M} \to \mathbb{R}$, the following invariant bilinear map,

$$\Omega\left([\Phi_1, \Pi_1], [\Phi_2, \Pi_2]\right) := \int_{\Sigma(t_0)} \mathrm{d}x \left[\Pi_1 \Phi_2 - \Phi_1 \Pi_2\right]. \tag{3.9}$$

With it, the Poisson brackets $\{\cdot, \cdot\}_{\mathrm{PB}}$ on $\mathscr{M}$ is defined to be simply

$$\left\{\Omega\left([\Phi_1, \Pi_1], \cdot\right), \Omega\left([\Phi_2, \Pi_2], \cdot\right)\right\}_{PB} = -\Omega\left([\Phi_1, \Pi_1], [\Phi_2, \Pi_2]\right), \tag{3.10}$$

where $\Omega\left([f, g], \cdot\right) \equiv \int_{\Sigma(t_0)} \mathrm{d}x[g\Phi - \Pi f]$ can be viewed as a linear map on $\mathscr{M}$. As an example, by choosing $[\Phi_1, \Pi_1] = [0, f_1]$ and $[\Phi_2, \Pi_2] = [f_2, 0]$, one obtain from Eq. (3.10) the following expression

$$\left\{\int_{\Sigma(t_0)} \mathrm{d}x f_1 \Phi, \int_{\Sigma(t_0)} \mathrm{d}x f_2 \Pi\right\}_{PB} = \int_{\Sigma(t_0)} \mathrm{d}x f_1 f_2, \tag{3.11}$$

which is the smeared version on space of the familiar expression (but mathematically ill-defined)

$$\{\Phi(x_1, t_0), \Pi(x_2, t_0)\} = \delta(x_1 - x_2).$$

As a matter of fact, any attempt to define the functions Φ and Π in a particular space-time point (x, t), is doomed from the start to run into serious convergence problems. For this reason we generally "smear" Φ in spacetime averages over test functions f, such as in

$$\Phi(f) := \int_{\mathbb{R}^2} \mathrm{d}t\mathrm{d}x \; f(x, t)\Phi(x, t) \tag{3.12}$$

$$\Pi(g) := \int_{\mathbb{R}^2} \mathrm{d}t\mathrm{d}x \; g(x, t)\Pi(x, t) \tag{3.13}$$

where f and g are smooth and of compact support on Minkowski spacetime, i.e., are elements of $\mathscr{T} = C_0^\infty(\mathbb{R}^2)$. By identifying a linear map $\mathcal{E} : \mathscr{T} \to \mathscr{S}$ between the space of test functions and the

space of classical solutions, one can enunciate the following properties which are gonna be relevant latter (see Ref. [88] for proofs and exact definitions):

(i) Every $\Phi_a \in \mathscr{S}$ can be expressed as $\Phi_a = \mathcal{E}f$ for some $f \in \mathscr{T}$;

(ii) $\mathcal{E}f = 0$ if and only if $f = \partial_\mu \partial^\mu g$ for some $g \in \mathscr{T}$;

(iii) For all $\Phi_a \in \mathscr{S}$ and all $f \in \mathscr{T}$, one have

$$\int_{\mathbb{R}^2} \mathrm{d}t\mathrm{d}x \; f\Phi_a = \Omega(\mathcal{E}f, \Phi_a). \tag{3.14}$$

System's Hamiltonian

Despite the introduction of the phase formalism for the classical theory, we have not yet touched on the explicit expression for the system's Hamiltonian H. With the conjugate field Π, as defined in Eq. (3.5), the standard way to define H would be by following the Legendre transformation

$$H(t) = \int_{\Sigma(t)} \mathrm{d}x \mathcal{H}(x,t) \qquad \text{with} \quad \mathcal{H} = \Pi \partial_t \Phi - \mathcal{L}(\Phi, \Pi, t). \tag{3.15}$$

Then, the time evolution of any given functional $F[\Phi, \Pi]$ is provided by the Poisson equation

$$\dot{F}(t) = \{F(t), H(t)\}_{\mathrm{PB}}, \qquad \text{where} \quad \{F, G\}_{\mathrm{PB}} = \int \mathrm{d}x \left(\frac{\delta F}{\delta \Phi} \frac{\delta G}{\delta \Pi} - \frac{\delta F}{\delta \Pi} \frac{\delta G}{\delta \Phi} \right) \tag{3.16}$$

is the field version of the classical Poisson bracket and $\frac{\delta}{\delta \psi} = \frac{\partial}{\partial \psi} + \frac{\partial}{\partial x} \frac{\partial}{\partial (\partial_x \psi)}$ is the functional derivative.

But contrary to naive expectations, the explicit expression for the Hamiltonian function cannot be straightforwardly defined in our system. As pointed out by Moore in his seminal work [7], given the oversimplified description of the mirror (in terms of the idealized boundary conditions (3.2)), our system cannot have a fundamental Hamiltonian H to generate the exact time evolution. To show this, let us suppose, by contradiction, that H indeed exists. From Eq. (3.16), one should be able to evolve the field configuration from $(x, t_0) \to (x, t_0 + \mathrm{d}t)$ in terms of the following infinitesimal expression

$$\Phi(x, t_0 + \mathrm{d}t) = \Phi(x, t_0) + \{\Phi(x, t_0), H\}_{\mathrm{PB}} \mathrm{d}t. \tag{3.17}$$

Now, if by chance we require the point (x, t_0) to lie on the mirror trajectory, since $\Phi(x, t_0) = 0$ (by the boundary conditions), one cannot expect an initially vanishing function to evolve into a non-zero value. Showing that H cannot exist.

For this reason, the overwhelming majority of papers on the subject tend to address the problem in the Heisenberg picture, where the relevant quantities are calculated without making reference to the explicit state vector of the system. However, contrary to Moore conclusion, the description of the DCE in Schrödinger picture is indeed attainable through the introduction of effective Hamiltonians.

The first description along these lines was obtained by Razavy [41] and Law [34, 42] in terms of an instantaneous decomposition of the field. Other important contributions to the effective description of field dynamics can be found in the works of Johnston *et al.* [89], Schützhold *et al.* [43] and Haro *et al.* [44, 45]. Throughout this work, we shall explore such effective Hamiltonian techniques to compute relevant properties.

3.1.2 Quantization procedure

In contrast to the paradigm of classical mechanics, —where a state is represented by a point in finite dimensional manifold $\mathscr{M}$, an observable is a real-valued function defined on $\mathscr{M}$ and the dynamical evolution is given by a canonical transformation generated by a Hamiltonian function— the quantum realm presents a remarkable different scenario. There, a quantum state is represented by a ray in an infinite dimensional projective Hilbert space $\mathscr{F}$, observables are self-adjoint operators acting on $\mathscr{F}$ and its dynamical evolution is given by unitary transformations generated by a Hamiltonian operator [88].

In this respect, to construct a quantum version of our classical theory we are going to need to account for:

1. An *algebraic structure*: How to establish the correct algebra from classical Poisson-brackets $\{\cdot,\cdot\}_{\rm PB}$ to quantum commutators $[\cdot,\cdot]$, wherein classical observables O_i are mapped to self-adjoint operators $\hat{O}_i$;

2. The *choice of a Hilbert Space*: How to appropriately choose the Hilbert space $\mathscr{F}$ in which the quantum states of the theory reside and the system observables $\hat{O}:\mathscr{F}\to\mathscr{F}$ operate.

Building the Hilbert space of our theory

The first challenge we are faced with, is how to extract a Hilbert space $\mathscr{F}$ out of the ingredients already present in our theory. If we remember, such as space $\mathscr{F}$ needs to be: *(i)* A linear vector space over the complex numbers $\mathbb{C}$; *(ii)* Equipped with a positive definite inner product $\langle\cdot,\cdot\rangle$; *(iii)* Complete with respect to the norm defined by $\langle\cdot,\cdot\rangle$.

We already have a linear vector space, the space of smooth solutions of the Klein-Gordon equation, denoted by $\mathscr{S}$. But unfortunately, the same is defined over the real numbers $\mathbb{R}$. We can overcome this issue with the complexification of $\mathscr{S}$ into $\mathscr{S}^{\mathbb{C}}=\mathscr{S}\otimes\mathbb{C}$. To define the notion of "orthonormality" on $\mathscr{S}^{\mathbb{C}}$ we can introduce the Klein-Gordon inner product, by

$$(\Phi_1,\Phi_2)_{\rm KG}=-i\Omega(\Phi_1^*,\Phi_2)=i\int_{\Sigma(t)}\mathrm{d}x\ \left[\Phi_1^*\partial_t\Phi_2-\Phi_2\partial_t\Phi_1^*\right], \tag{3.18}$$

which is invariant in the cavity interval $\Sigma(t)$. An unpleasant feature of Eq. (3.18) is that is not positive definite on $\mathscr{S}^{\mathbb{C}}$, as it assigns negative values for complex solutions $(\Phi_i^*,\Phi_j^*)=-(\Phi_i,\Phi_j)$. To

remedy this last difficult one can always choose any subspace $\mathcal{S}^{\mathbb{C}+} \subset \mathcal{S}^{\mathbb{C}}$ (space of positive solutions) which satisfies the following properties:

(i) The inner product (3.18) is positive definite on $\mathcal{S}^{\mathbb{C}+}$;

(ii) $\mathcal{S}^{\mathbb{C}} = \mathcal{S}^{\mathbb{C}+} \oplus \overline{\mathcal{S}^{\mathbb{C}+}}$, with the bar designating the complex conjugated of the space;

(iii) For all $\psi^+ \in \mathcal{S}^{\mathbb{C}+}$ and $\psi^- \in \overline{\mathcal{S}^{\mathbb{C}+}}$, we have $(\psi^+, \psi^-)_{\mathrm{KG}} = 0$.

With the last considerations we now have an inner product space $(\mathcal{S}^{\mathbb{C}+}, (\cdot,\cdot)_{\mathrm{KG}})$. If $\mathcal{S}^{\mathbb{C}+}$ is not complete in the norm defined by its inner product, we can extend it into a Hilbert space $\mathscr{H}$ with the well-defined mathematical procedure called *Cauchy completion*, whereby taking equivalence classes of Cauchy sequences[1] on $(\mathcal{S}^{\mathbb{C}+}, (\cdot,\cdot)_{\mathrm{KG}})$, one can endow it with a Hilbert space structure.

Although we have successfully obtained $\mathscr{H}$ from $\mathcal{S}$, this is not yet the Hilbert space for the theory that we are looking for. In the meantime, we shall denote by $|\psi_a\rangle$ and $\langle\psi_a|$, the vectors and dual vectors belonging, respectively, to $\mathscr{H}$ and $\overline{\mathscr{H}}$. In this manner $\langle\psi_a|\psi_a\rangle \equiv (\psi,\psi)_{\mathrm{KG}}$ designate its inner product. Furthermore, we can also introduce the symmetric n-th tensor product of Hilbert spaces as

$$\mathscr{H}_S^{(n)} = \bigotimes_S^n \mathscr{H} = \overbrace{\mathscr{H} \otimes_s \cdots \otimes_s \mathscr{H}}^{n \text{ times}},$$

with elements denoted by $|\psi_{a_1}, \psi_{a_2}, \ldots, \psi_{a_n}\rangle_S = |\psi_{a_1}\rangle \otimes_s |\psi_{a_2}\rangle \otimes_s \cdots \otimes_s |\psi_{a_n}\rangle$, where $\otimes_s$ is the symmetric tensor product defined by

$$|\psi_{a_1}\rangle \otimes_s |\psi_{a_2}\rangle = \frac{1}{2}\left(|\psi_{a_1}\rangle \otimes |\psi_{a_2}\rangle + |\psi_{a_2}\rangle \otimes |\psi_{a_1}\rangle\right).$$

Here, $\mathscr{H}^{(0)}$ with elements $|\psi\rangle$, is simply the set of complex scalars $\mathbb{C}$ (which is also a Hilbert space).

With all the ingredients at hands, we are now in a position to define the Hilbert space of our theory, the so-called symmetric Fock space $\mathscr{F}_S(\mathscr{H})$ in terms of the following direct sum

$$\mathscr{F}_S(\mathscr{H}) = \bigoplus_{n=0}^{\infty} \mathscr{H}_S^{(n)} = \mathbb{C} \oplus \mathscr{H} \oplus (\mathscr{H} \otimes_s \mathscr{H}) \oplus \ldots, \tag{3.19}$$

where a general vector $|\Psi\rangle \in \mathscr{F}_S(\mathscr{H})$ is represented by

$$|\Psi\rangle = (|\psi\rangle, |\psi_{a_1}\rangle, |\psi_{a_1}, \psi_{a_2}\rangle_S, \ldots, |\psi_{a_1}, \psi_{a_2}, \ldots, \psi_{a_n}\rangle_S, \ldots). \tag{3.20}$$

Before continuing, we must introduce two important operators for the theory: the annihilation operator $\hat{a}(\langle\xi_a|) : \mathscr{F}_S(\mathscr{H}) \to \mathscr{F}_S(\mathscr{H})$ acting on a general state of system as

$$\hat{a}(\langle\xi_a|)\,|\Psi\rangle = (\langle\xi_a|\psi_a\rangle, \sqrt{2}\,\langle\xi_a|\psi_a\rangle\,|\psi_{a_1}\rangle, \sqrt{3}\,\langle\xi_a|\psi_a\rangle\,|\psi_{a_1}, \psi_{a_2}\rangle_S, \ldots), \tag{3.21}$$

[1] If V is a normed vector space, a sequence $\{v_n\}$ of elements of V is said to be a *Cauchy sequence* in V if given a $\epsilon > 0$ there exists a $N \in \mathbb{Z}$ such that $||v_n - v_m|| < \epsilon$ for all $n, m > N$.

and the creation operator $\hat{a}^\dagger(|\xi_a\rangle) : \mathscr{F}_S(\mathscr{H}) \to \mathscr{F}_S(\mathscr{H})$, defined as

$$\hat{a}^\dagger(|\xi_a\rangle)\,|\Psi\rangle = (0, \psi\,|\xi_{a_1}\rangle\,, \sqrt{2}\,|\xi_{a_1}, \psi_{a_2}\rangle_S\,, \sqrt{3}\,|\xi_{a_1}, \psi_{k_2}, \psi_{k_3}\rangle_S\,, \dots). \tag{3.22}$$

Defining the algebraic structure of the quantum theory

In order to obtain the quantum version of our theory, we need to construct a bridge where functions $\Omega([\Phi, \Pi], \cdot)$ defined on $\mathscr{M}$ are represented by operators $\hat{\Omega}([\Phi, \Pi], \cdot)$ on $\mathscr{F}$, with the following commutation relations

$$\left[\hat{\Omega}\left([\Phi_1, \Pi_1], \cdot\right), \hat{\Omega}\left([\Phi_2, \Pi_2], \cdot\right)\right] = -i\Omega\left([\Phi_1, \Pi_1], [\Phi_2, \Pi_2]\right)\hat{I}, \tag{3.23}$$

where $\hat{I}$ is the identity operator. Inspecting the definition for the creation and annihilation operators in Eqs. (3.21) and (3.22) with more attention, one can have a hint of how to define $\hat{\Omega}([\Phi, \Pi], \cdot)$. By looking at their commutation relations applied to a general state $|\Psi\rangle$, one have

$$\hat{a}(\langle\xi_a|)\hat{a}^\dagger(|\eta_a\rangle)\,|\Psi\rangle - \hat{a}^\dagger(|\eta_a\rangle)\hat{a}(\langle\xi_a|)\,|\Psi\rangle = \langle\xi_a|\eta_a\rangle\,|\Psi\rangle\,, \tag{3.24}$$

or more explicitly

$$\left[\hat{a}(\langle\xi_a|), \hat{a}^\dagger(|\eta_a\rangle)\right] = \langle\xi_a|\eta_a\rangle\,\hat{I}. \tag{3.25}$$

Using the fact that $\langle\xi_a|\eta_a\rangle = -i\Omega(\xi_a^*, \eta_a)$ and comparing Eq. (3.25) with Eq. (3.23), it is possible to see that the correct expression for $\hat{\Omega}([\Phi, \Pi], \cdot)$ must be an adequate combination of creation and annihilation operators.

From the previous discussion we have seen that for any $\Phi_a \in \mathcal{S}$, we can always proceed with a decomposition in positive and negative solutions $\Phi_a = \Phi_a^+ + \Phi_a^-$ with $\Phi_a^+ \in \mathcal{S}^{\mathbb{C}+}$ and $\Phi_a^- \in \overline{\mathcal{S}^{\mathbb{C}+}}$. This correspondence $\Phi_a \to \Phi_a^+$ of selecting positive solutions must then guarantee the existence of a linear one-to-one map $\mathcal{K} : \mathcal{S} \to \mathscr{H}$ with the following properties

$$\mathcal{K}\Phi_a \to \left|\Phi_a^+\right\rangle \in \mathscr{H}, \qquad \overline{\mathcal{K}\Phi_a} \to \left\langle\Phi_a^+\right| \in \overline{\mathscr{H}}. \tag{3.26}$$

By considering any $\Phi_a \in \mathcal{S}$, the quantization procedure can then be achieved by defining the operator $\hat{\Omega}\left(\Phi_a, \cdot\right)$ on $\mathscr{F}_S(\mathscr{H})$, by

$$\hat{\Omega}\left(\Phi_a, \cdot\right) = i\hat{a}(\overline{\mathcal{K}\Phi_a}) - i\hat{a}^\dagger(\mathcal{K}\Phi_a). \tag{3.27}$$

In particular, the quantum field operator $\hat{\Phi}(f)$ weighted by a test function f —which is now interpreted as an *operator-value distribution*— is defined by

$$\hat{\Phi}(f) = i\hat{a}(\overline{\mathcal{K}\left(\mathcal{E}f\right)}) - i\hat{a}^\dagger(\mathcal{K}\left(\mathcal{E}f\right)). \tag{3.28}$$

A more familiar form for the last expression can be introduced if one chooses a particular orthonormal basis $\{u_k(x,t)\}$ for the space of positive solutions $\mathcal{S}^{\mathbb{C}+}$. In this case, $\mathcal{K}(\mathcal{E}f) \equiv |\mathcal{E}f\rangle = \sum_k \langle u_k|\mathcal{E}f\rangle\,|u_k\rangle$ and by writing $\langle u_k|\mathcal{E}f\rangle = i\Omega(\mathcal{E}f, u_k^*) = i\int \mathrm{d}x\mathrm{d}t f u_k^*$ with the help of the identity (3.14), we can explore the linearity of the creation and annihilation operators to obtain

$$\begin{aligned}\hat{\Phi}(f) &= i\hat{a}\left(-i\sum_k \int_{\mathbb{R}} \mathrm{d}t\mathrm{d}x f(x,t)u_k(x,t)\,\langle u_k|\right) - i\hat{a}^\dagger\left(i\sum_k \int_{\mathbb{R}} \mathrm{d}t\mathrm{d}x f(x,t)u_k^*(x,t)\,|u_k\rangle\right)\\ &= \sum_k \int_{\mathbb{R}} \mathrm{d}t\mathrm{d}x f(x,t)u_k(x,t)\hat{a}\,(\langle u_k|) + \sum_k \int_{\mathbb{R}} \mathrm{d}t\mathrm{d}x f(x,t)u_k^*(x,t)\hat{a}^\dagger\,(|u_k\rangle)\\ &= \int_{\mathbb{R}} \mathrm{d}t\mathrm{d}x f(x,t)\hat{\Phi}(x,t),\end{aligned}$$

where

$$\hat{\Phi}(x,t) = \sum_k \left[u_k(x,t)\hat{a}_k + u_k^*(x,t)\hat{a}_k^\dagger\right], \tag{3.29}$$

is the spacetime expression for the quantum field with $\hat{a}_k := \hat{a}\,(\langle u_k|)$ and $\hat{a}_k^\dagger := \hat{a}^\dagger\,(|u_k\rangle)$ being the creation and annihilation operators in the chosen basis. Since the basis $\{u_k\}$ is orthonormal, one can show from relation (3.23) that $\hat{a}_k$ and $\hat{a}_k^\dagger$ respects the standard commutation relations

$$\left[\hat{a}_k, \hat{a}_j\right] = \left[\hat{a}_k^\dagger, \hat{a}_j^\dagger\right] = 0 \quad \text{and} \quad \left[\hat{a}_k, \hat{a}_j^\dagger\right] = \delta_{kj}. \tag{3.30}$$

The above construction show to us in a very explicit manner that the field expansion in terms of creation and annihilation operators is completely dependent on the choice of the space of positive solutions $\mathcal{S}^{\mathbb{C}+}$ and its correspondent orthonormal basis $\{u_k\}$. This means that, in general, we will have infinitely many unitarily nonequivalent choices to decompose our field. In particular, we call u_k to be positive-frequency if it satisfies the relation

$$i\partial_t u_k = +\omega_k u_k, \qquad \omega_k > 0, \tag{3.31}$$

while its complex conjugated counterpart u_k^* its called negative-frequency as it respects

$$i\partial_t u_k^* = -\omega_k u_k^*, \qquad \omega_k > 0. \tag{3.32}$$

The significance of this nomenclature becomes apparent when expressing the system's Hamiltonian in terms of the operators $\hat{a}_k$ and $\hat{a}_k^\dagger$. By expressing the naive Hamiltonian choice given by Eq. (3.15), together with the Lagrangian density (3.1), one obtains: $H = \frac{1}{2}\left[(\partial_t\Phi)^2 + (\partial_x\Phi)^2\right]$. Introducing the field expansion from Eq. (3.29) into the quantum version for the last Hamiltonian and using the orthonormality conditions given by the inner product (3.18), one obtains the expression

$$\hat{H} = \sum_{kj}\left\{\Theta[u_k^*, u_j]\hat{a}_k^\dagger\hat{a}_j + \Theta[u_k, u_j^*]\hat{a}_k\hat{a}_j^\dagger + \Theta[u_k, u_j]\hat{a}_k\hat{a}_j + \Theta[u_k^*, u_j^*]\hat{a}_k^\dagger\hat{a}_j^\dagger\right\}, \tag{3.33}$$

where we have defined the functional $\Theta[u,v] := \frac{1}{2}\int_{\Sigma(t)} \mathrm{d}x\,[\partial_t u \partial_t v + \partial_x u \partial_x v]$ for any complex functions u and v. In the special case in which u_k is positive-frequency (and u_k^* negative-frequency), we automatically obtain[2] $\Theta[u_k^*, u_j] = \Theta[u_k, u_j^*] = \omega_j \delta_{kj}$ and $\Theta[u_k, u_k] = \Theta[u_k^*, u_k^*] = 0$, so the Hamiltonian (3.33) becomes

$$\hat{H} = \sum_k \omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}\right) \quad \text{for} \quad \partial_t u_k = -i\omega_k u_k, \tag{3.34}$$

which is diagonal with respect to the operators $\hat{a}_k$ and $\hat{a}_k^\dagger$. In this exceptional circumstance, a part from the infinite number of terms proportional to $\omega_k/2$ (the zero-point energy contributions of the theory), one can understand the system's Hamiltonian as made up of a collection of mode terms $\hat{N}_k = \hat{a}_k^\dagger \hat{a}_k$, each one accompanied by a discrete quantum of energy $\hbar\omega_k$ (in SI units). We then interpret the average value of the operator $\hat{N}_k$ as counting the number of energy excitation with a definite field mode k, which in the context of quantum field theory we simply refer as particles.

As we have seen in more formally in definitions (3.21) and (3.22), another important use for the operators $\hat{a}_k$ and $\hat{a}_k^\dagger$ is to construct a particular basis for $\mathscr{F}_s(\mathscr{H})$. One can reformulate such a construction by defining the ground state of the theory $|0\rangle$, as the state annihilated by all $\hat{a}_k$, that is,

$$\hat{a}_k |0\rangle = 0, \qquad \text{for all } k. \tag{3.35}$$

As the ground state is the lowest energetic state of the field, we call it the vacuum state of the theory. A general state with n_{k_i} particles populated in all the k_i-th modes can be constructed by the repeated application of the creation operator $\hat{a}_k^\dagger$ on this vacuum state,

$$|\boldsymbol{n}\rangle = |n_{k_1}, n_{k_2}, \ldots\rangle = \prod_i \frac{1}{\sqrt{n_{k_i}!}} \left(\hat{a}_{k_i}^\dagger\right)^{n_{k_i}} |0\rangle\,. \tag{3.36}$$

In this language, Eqs. (3.21) and (3.22) can be rewritten as

$$\hat{a}_k |n_1, \ldots, n_k, \ldots\rangle = \sqrt{n_k}\, |n_1, \ldots, n_k - 1, \ldots\rangle\,, \tag{3.37}$$

$$\hat{a}_k^\dagger |n_1, \ldots, n_k, \ldots\rangle = \sqrt{n_k + 1}\, |n_1, \ldots, n_k + 1, \ldots\rangle\,. \tag{3.38}$$

3.1.3 Field expansion in different time intervals

Let us consider at the initial time interval *in*: $\{t \in (-\infty, 0]\}$ that the system is in a static configuration with a constant cavity length $l(t \leq 0) = l_0$. Given the static nature of the imposed boundary conditions, a natural set of orthonormal solutions for the wave equation (3.4) can be

[2] When u_k is positive-frequency, follow immediately from the Klein-Gordon inner product properties that: $\omega_j \int_{\Sigma(t)} \mathrm{d}x u_k^* u_j = \delta_{kj} - \omega_k \int_{\Sigma(t)} \mathrm{d}x u_j u_k^*$ whereas $\omega_j \int \mathrm{d}x u_k u_j = -\omega_k \int_{\Sigma(t)} \mathrm{d}x u_k^* u_j^*$. Using the wave equation together with the boundary conditions on $\Sigma(t)$ we can also obtain the expression $\int_{\Sigma(t)} \mathrm{d}x \partial_x f_k^* \partial_x u_j = \omega_j^2 \int_{\Sigma(t)} \mathrm{d}x u_k^* u_j$.

introduced simply as

$$u_k^{\text{in}}(x,t) = \frac{1}{\sqrt{\pi k}} \sin\left[\omega_k^{in}(x-x_0)\right] e^{-i\omega_k^{in}t} \qquad \text{for } t \leq 0, \tag{3.39}$$

where $\omega_k^{in} = k\pi/l_0$ with $k = 1, 2, \ldots$ are the initial time-independent frequencies and, again, the set $\{u_k^{in}(x,t)\}$ form a basis that spams the subspace $\mathcal{S}^{\mathbb{C}+}$ for this time interval. In this circumstance, the field expansion takes the form

$$\hat{\Phi}(x, t \leq 0) = \sum_k \left[\hat{a}_k^{in} u_k^{in}(x,t) + \hat{a}_k^{in\dagger} u_k^{in*}(x,t)\right], \tag{3.40}$$

where $\hat{a}_k^{in}$ and $\hat{a}_k^{in\dagger}$ are the initial annihilation and creations operators which defines an initial vacuum state $|0; in\rangle$ by the same arguments presented in Eq. (3.35).

In particular, the modes u_k^{in} are said to be positive-frequency from the criterion introduced in Eq. (3.31). As a consequence, not only the system's Hamiltonian is automatically diagonalized and we have a well-defined notion of particles for all $t \leq 0$, but one can also show that this is the preferred and most natural choice for a mode decomposition. To display this, we begin considering an inertial observer with coordinates $(\tilde{x}, \tilde{t})$ which is moving with velocity v in respect to the fixed mirrors x_0 and $x_0 + l_0$ of our cavity. One can relate the two descriptions by means of the following Lorentz transformation

$$t = \gamma\left(\tilde{t} + v\tilde{x}\right), \qquad \begin{aligned} x &= \gamma(\tilde{x} + v\tilde{t}), \\ x_0 &= \gamma(\tilde{x}_0 + v\tilde{t}), \end{aligned} \tag{3.41}$$

where $\gamma = 1/\sqrt{1-v^2}$ is the Lorentz factor. The time derivative of our positive-frequency modes in the new frame is then

$$\begin{aligned} i\partial_{\tilde{t}} u_k^{in} &= i\frac{\partial u_k^{in}}{\partial x}\frac{\partial x}{\partial \tilde{t}} + i\frac{\partial u_k^{in}}{\partial x_0}\frac{\partial x_0}{\partial \tilde{t}} + i\frac{\partial u_k^{in}}{\partial t}\frac{\partial t}{\partial \tilde{t}} \\ &= \frac{\gamma\omega_k^{in}}{\sqrt{k\pi}} \sin[\omega_k^{in}(x-x_0)] e^{-i\omega_k^{in}t} = \tilde{\omega}_k^{in} u_k^{in}, \end{aligned} \tag{3.42}$$

where $\tilde{\omega}_k^{in} = \gamma\omega_k^{in}$ is the frequency in the boosted frame. This is telling us that although the field frequency is re-scaled by the Lorentz factor, different observers will agree on how to distinguish between positive and negative frequencies solutions and, therefore, on how to define the creation and annihilation operators and the correspondent vacuum state for the system.

Now let us discuss the situation in which the field is subjected to dynamical boundary conditions (3.4). For the time interval $t > 0$, when the cavity returns to motion, the natural criterion of selecting positive-frequency solutions is no longer available, meaning we do not expect to find an unambiguous choice for $\mathcal{S}^{\mathbb{C}+}$ and, therefore, to the notion of particles. To see this in a more explicit manner, let's consider the approach pioneered by Gerald Moore in 1970 [7]. At this time interval, an expansion for the quantum field can still be decomposed in terms of the initial operators $\hat{a}_k^{in}$ and

$\hat{a}_k^{in\dagger}$ in the form,

$$\hat{\Phi}(x,t>0)=\sum_k\left[\hat{a}_k^{in}u_k(x,t)+\hat{a}_k^{in\dagger}u_k^*(x,t)\right], \tag{3.43}$$

as long as the new set of mode functions $\{u_k(x,t)\}$ satisfies: (i) the wave equation (3.4), (ii) the time-dependent boundary condition (3.4), and (iii) the initial condition $u_k(x,0)=u_k^{\rm in}(x,0)$. To attack such problem one can exploit the conformal invariance of the wave equation (3.4) in $(1+1)$-dimension, which under the coordinate transformation $(x,t)\to(\xi,\eta)$ with equations

$$\eta-\xi=R\,(t-x)\qquad\text{and}\qquad\eta+\xi=R\,(t+x)\,, \tag{3.44}$$

preserve its structure under the form of

$$\frac{\partial^2\hat{\Phi}}{\partial\eta^2}-\frac{\partial^2\hat{\Phi}}{\partial\xi^2}=0\rightleftarrows\frac{\partial^2\hat{\Phi}}{\partial t^2}-\frac{\partial^2\hat{\Phi}}{\partial x^2}=0,$$

where $R(t\pm x)$ is an arbitrary function, and we have used new set of abstract coordinates ξ and η restricted to the finite interval $0\leq\xi,\eta\leq1$.

To map the time-dependent problem into a more amenable where the boundary conditions are static, we choose the function R in such a way that the ξ-coordinate (analogous to our original spatial coordinate) is defined to satisfy

$$\xi=0\quad\text{when}\quad x=x_0, \tag{3.45}$$

$$\xi=1\quad\text{when}\quad x=x_0+l(t). \tag{3.46}$$

Solving the conformal transformed scalar field equation for $\hat{\Phi}(\xi,\eta)$, under the boundary conditions $\hat{\Phi}(0,\tau)=\hat{\Phi}(1,\eta)=0$, we can analogously find solutions in the form

$$\begin{aligned}u_k=\frac{1}{\sqrt{k\pi}}\sin(k\pi\xi)e^{-ik\pi\eta}&=\frac{i}{\sqrt{4k\pi}}\left\{e^{-ik\pi(\eta+\xi)}-e^{-ik\pi(\eta-\xi)}\right\}\\&\equiv\frac{i}{\sqrt{4k\pi}}\left\{e^{-ik\pi R(t+x)}-e^{-ik\pi R(t-x)}\right\},\end{aligned} \tag{3.47}$$

whereby using the inverses of the conformal transformations (3.44) we have returned to the (x,t) representation in the second line. Using the identity

$$R(t+x)-R(t-x)=(\tau+\xi)-(\tau-\xi)=2\xi, \tag{3.48}$$

we can then force the function R to satisfy $R(t-x)=F(t-x)$ and $R(t+x)=G(t+x)$ (where F and G are another arbitrary functions) and search for solutions in the form

$$u_k\,(x,t)=\frac{i}{\sqrt{4k\pi}}\left[e^{-ik\pi G(t+x)}-e^{-ik\pi F(t-x)}\right], \tag{3.49}$$

provided that we can solve the following functional relations derived from Eq. (3.48)

$$G\Big(t+x_0+l(t)\Big)-F\Big(t-x_0-l(t)\Big)=2 \quad \text{and} \quad G\Big(t+x_0\Big)-F\Big(t-x_0\Big)=0. \tag{3.50}$$

The above relations are known as generalized Moore's equation [7]. If one consider the special case in which $x_0=0$, we recover the original function relation due to Moore in which $G(z)=F(z)=R(z)$ such that $R(t+l(t))-R(t-l(t))=2$.

To exemplify the last discussion, let us consider the special case in which $x_0=0$ and the second mirror perform small oscillations with $l(t)=l_0\left[1+\epsilon\sin\left(q\omega_1^{in}t\right)\right]$, where $q=1,2,\ldots$ and $\varepsilon\ll 1$. In the long-time limit $\varepsilon t/l_0\gg 1$, solutions to the Moore's equations (3.50) can be found exactly as [90, 91]

$$R(z)=\frac{z}{l_0}-\frac{2}{\pi q}\,\mathrm{Im}\left\{\ln\left[\frac{1+\xi(t)}{1-\xi(t)}+e^{i\frac{\pi q z}{l_0}}\right]\right\} \quad \text{with} \quad \xi(t)=\exp\left\{(-1)^{q+1}\frac{\pi q\varepsilon t}{l_0}\right\}. \tag{3.51}$$

Introducing Eq. (3.51) into expression (3.49), one finds that in the time-dependent case, the resulting mode decomposition is not separable on the time and space coordinates. Since different observers will not agree on how to expand the field in terms of modes with positive and negative frequencies, in general there will not exist a preferred and unambiguous choice for the vacuum state of the theory. Thus, unless we can specify a measurement process, the usual notion of particle loses its well-defined meaning, and only when the cavity comes to rest we can associate a definite particle interpretation to the quanta described by these operators [42].

As a result, in order to define a meaningful particle state for our theory, we must consider that, after a finite period of time T, the cavity returns to a static configuration with a constant cavity length $l(t\geq T)=l_T$. At this final time interval *out*: $\{t\in[T,\infty)\}$ one can then reintroduce a preferred choice for the mode function

$$u_k^{\mathrm{out}}(x,t)=\frac{1}{\sqrt{\pi k}}\sin\left[\omega_k^{\mathrm{out}}\left(x-x_0\right)\right]e^{-i\omega_k^{\mathrm{out}}t} \quad \text{for } t\geq T \tag{3.52}$$

and described in terms of the positive frequencies $\omega_k^{\mathrm{out}}=k\pi/l_T$. Due to the continued distortion of the mode function structure, the initial operators $\hat{a}_k^{in}$ and $\hat{a}_k^{in\dagger}$ cease to have a physical significance at the end of the cavity motion, and we then decompose the field as

$$\hat{\Phi}(x,t\geq T)=\sum_k\left[\hat{a}_k^{out}u_k^{\mathrm{out}}(x,t)+\hat{a}_k^{out\dagger}u_k^{\mathrm{out}*}(x,t)\right], \tag{3.53}$$

with the new set of final operators $\hat{a}_k^{out}$ and $\hat{a}_k^{out\dagger}$ satisfying analogous commutation relations as in Eq. (3.30), as well as defining a new vacuum state $|0;\mathrm{out}\rangle$ as the state annihilated by all $\hat{a}_k$.

3.1.4 Connecting the two field representations

As pointed out in Ref. [59], although both sets $\mathcal{G}^{in} = \{u_k^{\text{in}}, u_k^{\text{in}*}\}$ and $\mathcal{G}^{out} = \{u_k^{\text{out}}, u_k^{\text{out}*}\}$ form a basis for the space of solutions $\mathcal{S}$, they represent different decompositions into the subspaces $\mathcal{S}^{\mathbb{C}+}$ and $\overline{\mathcal{S}^{\mathbb{C}+}}$. The two sets of mode functions (3.39) and (3.52) should then be related by a linear transformation

$$u_k^{\text{in}} = \sum_j \left[\alpha_{jk} u_j^{\text{out}} + \beta_{jk} u_j^{\text{out}*}\right], \tag{3.54}$$

where α_{jk} and β_{jk} are complex numbers called Bogoliubov coefficients satisfying the following unitary conditions

$$\sum_{m=1}^{\infty} (\alpha_{nm}^* \alpha_{km} - \beta_{nm}^* \beta_{km}) = \delta_{nk}, \quad \sum_{n=1}^{\infty} (\alpha_{nm}^* \alpha_{nj} - \beta_{nm}^* \beta_{nj}) = \delta_{mj}, \quad \sum_{n=1}^{\infty} (\beta_{nm}^* \alpha_{nk} - \beta_{nk}^* \alpha_{nm}) = 0.$$

Inserting Eq. (3.54) into the field decomposition (3.40), and comparing with Eq. (3.53), we obtain the set of Bogoliubov transformations

$$\hat{a}_j^{out} = \sum_k \left[\alpha_{kj} \hat{a}_k^{in} + \beta_{kj}^* \hat{a}_k^{in\dagger}\right]. \tag{3.55}$$

For the special case in which the final position of the moving boundary coincides with initial position l_0 and $\omega_k^{in} = \omega_k^{out} = \omega_k$, the Bogoliubov coefficients can be found using the definition of inner product given in Eq. (3.18) in the expression (3.55) as in [92]

$$\left.\begin{aligned} \alpha_{jk} &= \left(u_k^{in}, u_j\right) \\ \beta_{jk} &= -\left(u_k^{in}, u_j^*\right) \end{aligned}\right\} = \frac{1}{2l_0}\sqrt{\frac{j}{k}} \int_{t-l_0}^{t+l_0} dx' \exp\left\{-i\left[\omega_k^{in} l_0 R\left(x'\right) \mp \omega_j^{in} x'\right]\right\}, \tag{3.56}$$

where a transformation of variable $t \pm x \to x'$ were used throughout the derivation.

Observe that the vacuum defined by $\hat{a}_k^{out}$ and $\hat{a}_k^{in}$ are not equivalent in general and as a consequence, when computing the number of particles defined by the final operators $\hat{a}_k^{out}$ and $\hat{a}_k^{out\dagger}$ with respect to the initial vacuum $|0; \text{in}\rangle$, it is obtained

$$N(T) = \sum_j \langle 0; \text{in} | \hat{a}_j^{out\dagger} \hat{a}_j^{out} | 0; \text{in} \rangle = \sum_{jk} |\beta_{jk}(T)|^2. \tag{3.57}$$

In general, β_{jk} is non-zero when time-dependent boundary conditions are imposed on the field. This last equation characterizes the DCE as the quantum field phenomenon of particle creation from the vacuum due to the time-dependent nature of the imposed boundary conditions.

3.2 The instantaneous basis approach

Although we have presented Moore's approach to handle with the problem of particle creation due to the DCE, for the purposes of this work it will more suitable to utilize a different method known as the *instantaneous basis approach*. The main feature of this approach revolves around the introduction of the auxiliary function

$$\varphi_k(x,t) = \sqrt{\frac{2}{l(t)}}\sin\left[\omega_k(t)\,(x-x_0)\right], \quad \text{with} \quad \omega_k(t) = \frac{k\pi}{l(t)} \quad (k=1,2,\dots) \tag{3.58}$$

which is an eigenfunction of the one-dimensional Laplacian operator $-\partial_x^2$ with the dynamical boundary condition (3.4). It is easy to check that the set $\{\varphi_k(x,t)\}$ satisfies the orthonormality and completeness conditions

$$\int_{x_0}^{x_0+l(t)} \mathrm{d}x\ \varphi_k(x,t)\varphi_j(x,t) = \delta_{kj}, \tag{3.59}$$

$$\sum_k \varphi_k(x,t)\varphi_k(x',t) = \delta(x-x'), \tag{3.60}$$

and therefore, constitute an orthonormal basis. With this at hands, the classical field $\Phi(x,t)$ can then be expanded as

$$\Phi(x,t) = \sum_k q_k(t)\varphi_k(x,t), \tag{3.61a}$$

with $q_k(t)$ being the position quadrature for the field. At this point, as we have already discussed the system's quantization, it may seem a little odd to reintroduce classical expressions for the field, and we agree. Nevertheless, in this section we will try to reformulate the problem by re-expressing the system's Lagrangian $L = \int_{\Sigma(t)} dx\mathcal{L}$ in terms of the new generalized coordinate function $q_k(t)$ and its derivatives $\dot{q}_k(t) := \mathrm{d}q_k(t)/\mathrm{d}t$[3]. By inserting Eqs. (3.61a) into expression (3.1), one obtains,

$$\begin{aligned} L\left(q_k,\dot{q}_k,t\right) &= \int_{x_0}^{x_0+l(t)} \mathrm{d}x\mathcal{L}\left(q_k,\dot{q}_k,t\right) \\ &= \frac{1}{2}\sum_k\left\{\dot{q}_k^2 - \omega_k^2(t)q_k^2 + \sum_j\left[2G_{kj}(t)q_k\dot{q}_j + H_{kj}(t)q_kq_j\right]\right\}, \end{aligned} \tag{3.62}$$

[3]The presentation was heavily inspired by Ref. [93].

where we have used the relations (3.1) as well as the identity $\int_0^{l(t)} \partial_x\varphi_k(x,t)\partial_x\varphi_j(x,t) = -\omega_k^2(t)\delta_{kj}$. The time-dependent coefficients in Eq. (3.62) are defined as

$$G_{jk}(t) := \int_{x_0}^{x_0+l(t)} \mathrm{d}x\ \dot{\varphi}_j(x,t)\varphi_k(x,t) = \lambda(t)g_{jk}, \tag{3.63a}$$

$$H_{jk}(t) := \int_{x_0}^{x_0+l(t)} \mathrm{d}x\ \dot{\varphi}_j(x,t)\dot{\varphi}_k(x,t) = \lambda^2(t)\sum_l g_{jl}g_{kl}, \tag{3.63b}$$

with $\lambda(t) = \dot{l}(t)/l(t)$ and

$$g_{jk} = \begin{cases} (-1)^{j-k}\frac{2kj}{k^2-j^2}, & k \neq j \\ 0, & k = j. \end{cases} \tag{3.64}$$

The dynamical equation for the quadrature position $q_k(t)$ can be obtained from the correspondent version of the Euler-Legendre equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \ddot{q}_k + \omega_k^2(t)q_k - 2\lambda(t)\sum_j g_{kj}\dot{q}_j + \dot{\lambda}(t)\sum_j g_{kj}q_j - \lambda^2(t)\sum_{jl} g_{kl}g_{jl}q_j = 0. \tag{3.65}$$

Finally, the Hamiltonian description of the system can be introduced by expanding the momentum density field in terms of the basis function (3.58) as in

$$\Pi(x,t) = \sum_k p_k(t)\varphi_k(x,t), \tag{3.66}$$

where the generalized conjugated momenta $p_k(t)$ is defined from the Lagrangian Eq. (3.62) its terms of

$$p_k(t) = \frac{\partial L}{\partial \dot{q}_k} = \dot{q}_k(t) - \sum_j G_{kj}(t)q_j(t). \tag{3.67}$$

With it, the system's Hamiltonian can be obtained with the following Legendre transformation

$$\begin{aligned} H(t) &= \sum_k \dot{q}_k(t)p_k(t) - L\left[q_k(t), p_k(t), t\right] \\ &= \frac{1}{2}\sum_k \left[p_k^2(t) + \omega_k^2(t)q_k^2(t)\right] + \sum_{kj} G_{jk}(t)q_j(t)p_k(t). \end{aligned} \tag{3.68}$$

The last expression makes clear that the dynamical description of the system is equivalent to that of a system of coupled harmonic oscillators with time dependent frequencies. Using the terminology due to Ref. [43], after the Hamiltonian is quantized we will be able to identify two different effects responsible for the particle creation process: a *squeezing effect* due to the time-dependence in the mode frequencies $\omega_k(t)$ present in the first part of the Hamiltonian and a *acceleration effect* due to the time-dependence of coupling coefficient $G_{kj}(t)$ in the second part.

Comparatively speaking, the energy of the field E is defined in terms of the 00-component of the energy-momentum density and the Lagrangian density (3.1), as

$$E(t) = \int_{\Sigma(t)} \mathrm{d}x\ T_{00} = \frac{1}{2}\int_{\Sigma(t)} \mathrm{d}x\ \left[\Pi^2 + (\partial_x \Phi)^2\right] \tag{3.69}$$

$$= \frac{1}{2}\sum_k \left[p_k^2(t) + \omega_k^2(t) q_k^2(t)\right] \tag{3.70}$$

where $T_{\mu\nu} = \partial_\nu \Phi \frac{\partial \Phi}{\partial(\partial_\nu \Phi)} - \eta_{\mu\nu}\mathcal{L}$.

Therefore, the system's effective Hamiltonian differs from the field energy (3.69) with the following form

$$H(t) - E(t) = \sum_{kj} G_{kj}(t) q_j(t) p_k(t). \tag{3.71}$$

3.2.1 Decomposition with static operators

As we have obtained the necessary ingredients for further discussion, we can return to the quantum description. This is done straightforwardly by promoting the functions $q_k(t)$ and $p_k(t)$ to the status of operators $\hat{q}_k(t)$ and $\hat{p}_k(t)$ acting on $\mathscr{F}_S(\mathscr{H})$ and imposing the following set of equal-time commutation relations

$$\left[\hat{q}_k(t), \hat{p}_j(t)\right] = i\delta_{kj} \quad \text{and} \quad \left[\hat{q}_k(t), \hat{q}_j(t)\right] = \left[\hat{p}_k(t), \hat{p}_j(t)\right] = 0. \tag{3.72}$$

By comparing the initial field decomposition (3.40) in terms of mode function (3.39) (and its time derivative), with the quantum version of the Eqs. (3.61), one can straightforwardly obtain the following expressions for the quadrature operators at the initial interval of time $t \leq 0$:

$$\hat{q}_k(t \leq 0) = \frac{1}{\sqrt{2\omega_k^{in}}}\left[\hat{a}_k^{in}(t) + \hat{a}_k^{in\dagger}(t)\right], \tag{3.73a}$$

$$\hat{p}_k(t \leq 0) = i\sqrt{\frac{\omega_k^{in}}{2}}\left[\hat{a}_k^{in\dagger}(t) - \hat{a}_k^{in}(t)\right], \tag{3.73b}$$

where $\hat{a}_k^{in}(t) = \hat{a}_k^{in} e^{-i\omega_k^{in} t}$ and $\hat{a}_k^{in\dagger}(t) = \hat{a}_k^{in\dagger} e^{i\omega_k^{in} t}$ are the correspondent initial annihilation and creation operators in the Heisenberg picture.

For the intermediate interval of time $0 < t < T$, when the cavity is in motion, we expect the quadrature operators to follow a more complicated dependence on time due to the dynamical equations (3.65) and (3.67). A popular approach widely used in Refs. [94–98] to obtain expressions for $\hat{q}_k(t)$ and $\hat{p}_k(t)$ at $t > 0$, is based in a decomposition in terms of the static initial operators $\hat{a}_k^{in\dagger}$

and $\hat{a}_k^{in}$, but with a mode structure distorted on time by the following Fourier expansion

$$\hat{q}_k(t>0) = \sum_j \frac{1}{\sqrt{2\omega_j^{in}}} \hat{a}_j^{in} Q_j^{(k)}(t) + \text{h.c}, \tag{3.74a}$$

$$\hat{p}_k(t>0) = -i \sum_j \sqrt{\frac{\omega_j^{in}}{2}} \hat{a}_j^{in} P_j^{(k)}(t) + \text{h.c}, \tag{3.74b}$$

where the Fourier coefficients $Q_j^{(k)}(t)$ and $P_j^{(k)}(t)$ are complex functions satisfying the differential equations

$$\dot{Q}_j^{(k)}(t) = P_j^{(k)}(t) - \sum_l G_{jl}(t) Q_l^{(k)}(t), \tag{3.75}$$

$$\dot{P}_j^{(k)}(t) = -\omega_j^2(t) Q_j^{(k)}(t) + \sum_l G_{lj}(t) Q_l^{(k)}(t), \tag{3.76}$$

together with the initial conditions

$$Q_j^{(k)}(0) = P_j^{(k)}(0) = \delta_{kj} \quad \text{and} \quad \dot{Q}_j^{(k)}(0) = \dot{P}_j^{(k)}(0) = -i\omega_k^{in}\delta_{kj}. \tag{3.77a}$$

Finally, for completeness, at the final interval of time $t \geq T$ (when the cavity return to its static configuration), as in the initial case, the quadrature operators take the following expression in the Heisenberg picture

$$\hat{q}_k(t \geq 0) = \frac{1}{\sqrt{2\omega_k^{out}}} \left[\hat{a}_k^{out}(t) + \hat{a}_k^{out\dagger}(t)\right], \tag{3.78a}$$

$$\hat{p}_k(t \geq 0) = i\sqrt{\frac{\omega_k^{out}}{2}} \left[\hat{a}_k^{out\dagger}(t) - \hat{a}_k^{out}(t)\right], \tag{3.78b}$$

where, again, $\hat{a}_k^{out}(t) = \hat{a}_k^{out} e^{-i\omega_k^{out} t}$ and $\hat{a}_k^{out\dagger}(t) = \hat{a}_k^{out\dagger} e^{i\omega_k^{out} t}$.

3.2.2 Decomposition with instantaneous operators

Another possible decomposition for $\hat{q}_k(t)$ and $\hat{p}_k(t)$ involves the introduction of a time-dependent version for the operators $\hat{a}_k^{in}$ and $\hat{a}_k^{in\dagger}$ in close analogy with Eq. (3.73a) and (3.73b). In this spirit, we define for the interval of time $0 \leq t \leq T$, the following quadrature operators [42]

$$\hat{q}_k(t) = \frac{1}{\sqrt{2\omega_k(t)}} \left[\hat{a}_k(t) + \hat{a}_k^{\dagger}(t)\right], \tag{3.79a}$$

$$\hat{p}_k(t) = i\sqrt{\frac{\omega_k(t)}{2}} \left[\hat{a}_k^{\dagger}(t) - \hat{a}_k(t)\right], \tag{3.79b}$$

where the *instantaneous* creation and annihilation operators $\hat{a}_k(t)$ and $\hat{a}_k^\dagger(t)$ respect, as expected, the usual commutation relations

$$\left[\hat{a}_k(t), \hat{a}_j(t)\right] = \left[\hat{a}_k^\dagger(t), \hat{a}_j^\dagger(t)\right] = 0 \quad \text{and} \quad \left[\hat{a}_k(t), \hat{a}_j^\dagger(t)\right] = \delta_{kj}. \tag{3.80}$$

The name instantaneous refers to the physical interpretation that if we "freeze" the system at some instant t_0, then $\hat{a}_k(t_0)$ and $\hat{a}_k^\dagger(t_0)$ must describe the operators as if the cavity has been stopped at a fixed size $l(t_0)$ (where the notion of particle is well-defined) [42].

Here we will consider an effective description in which the annihilation and creation operators $\hat{a}_k(t)$ and $\hat{a}_k^\dagger(t)$ are supposed to have an explicit time-dependence in the Heisenberg picture (exclusively due to the system's dynamics), while the frequency $\omega_k(t)$ and the coefficients $G_{kj}(t)$ and $H_{kj}(t)$ are dependent on time only implicitly through the cavity size $l(t)$[4]. Therefore, we are compelled to express the total time derivative of $\hat{q}_k(t)$ in terms of its explicit and implicit time dependence:

$$\dot{\hat{q}}_k(t) = \partial_t \hat{q}_k(t) - \frac{1}{2}\frac{\dot{\omega}_k(t)}{\omega_k(t)}\hat{q}_k(t), \tag{3.81}$$

where $\dot{\omega}_k(t) \equiv \frac{\partial \omega_k(t)}{\partial l(t)}\dot{l}(t)$. Our strategy to construct an effective Hamiltonian out of this last description can be obtained by rewriting the instantaneous Lagrangian (3.62) in terms of q_k and $\partial_t q_k$ (instead of $\dot{q}_k$) with help of the classical version of relation (3.81). Following the same steps performed in the beginning of this section, one can update the classical Hamiltonian (3.68) by simply changing $G_{kj} \to G_{kj} - \frac{1}{2}\frac{\dot{\omega}_j}{\omega_j}\delta_{kj}$.

An effective Hamiltonian operator can then be obtained from the structure of the classical expression (3.68) (with the updated coefficients), by promoting its quadrature functions $q_k(t)$ and $p_k(t)$ to quantum operators $\hat{q}_k(t)$ and $\hat{p}_k(t)$ in such a way that

$$\hat{H}_{\text{eff}}(t) = \frac{1}{2}\sum_k \left[\hat{p}_k^2(t) + \omega_k^2(t)\hat{q}_k^2(t)\right] - \frac{1}{2}\sum_{kj}\left[\sqrt{\frac{\omega_j(t)}{\omega_k(t)}}\mu_{jk}(t)\hat{q}_j(t)\hat{p}_k(t) + \sqrt{\frac{\omega_k(t)}{\omega_j(t)}}\mu_{kj}(t)\hat{p}_j(t)\hat{q}_k(t)\right], \tag{3.82}$$

where we have symmetrized the term $\hat{q}_j(t)\hat{p}_k(t) \to \frac{1}{2}\left[\hat{q}_j(t)\hat{p}_k(t) + \hat{p}_k(t)\hat{q}_j(t)\right]$ and exchanged the indexes in $\sqrt{\frac{\omega_j(t)}{\omega_k(t)}}\mu_{kj}(t)\hat{p}_k(t)\hat{q}_j(t) \to \sqrt{\frac{\omega_k(t)}{\omega_j(t)}}\mu_{jk}(t)\hat{p}_j(t)\hat{q}_k(t)$, since k and j are muted in the double sum. In this context, the new time-dependent coefficient is defined to be

$$\mu_{jk}(t) := \sqrt{\frac{\omega_k(t)}{\omega_j(t)}}\left[G_{kj}(t) + \frac{1}{2}\frac{\dot{\omega}_j(t)}{\omega_j(t)}\delta_{kj}\right]. \tag{3.83}$$

[4]Although we will continue to display the same notation $*(t)$, a more rigorous reading of its time-dependence should take into account the following form: $\omega_k(t) \equiv \omega_k[l(t)]$, $G_{kj}(t) \equiv G_{kj}[l(t)]$ and $H_{kj}(t) \equiv H_{kj}[l(t)]$.

In terms of the instantaneous creation and annihilation operators $\hat{a}_k^\dagger(t)$ and $\hat{a}_k(t)$ from definitions (3.79), the effective Hamiltonian (3.82) takes the form (after normal ordering)

$$\hat{H}_{\text{eff}}(t) = \sum_k \omega_k(t)\hat{a}_k^\dagger(t)\hat{a}_k(t) \tag{3.84}$$

$$+ \frac{i}{2}\sum_{kj}\left\{\mu_{(k,j)}(t)\left[\hat{a}_j(t)\hat{a}_k(t) - \hat{a}_k^\dagger(t)\hat{a}_j^\dagger(t)\right] + \mu_{[k,j]}(t)\left[\hat{a}_k^\dagger(t)\hat{a}_j(t) - \hat{a}_j^\dagger(t)\hat{a}_k(t)\right]\right\}, \tag{3.85}$$

where the correspondent coefficients are defined to be

$$\mu_{[k,j]}(t) = \frac{1}{2}\left[\mu_{kj}(t) - \mu_{jk}(t)\right] \quad \text{and} \quad \mu_{(k,j)}(t) = \frac{1}{2}\left[\mu_{kj}(t) + \mu_{jk}(t)\right]. \tag{3.86}$$

Here, we can clearly see the existence of two different contributions: the terms containing the coefficients $\mu_{(k,j)}(t)$ govern the process of creation and annihilation of pairs of particles, while the ones proportional to $\mu_{[k,j]}(t)$ are responsible for scattering of particles between distinct modes. Indeed, our effective Hamiltonian (3.84) has the exact expression of the Hamiltonian obtained in Ref [99] and is completely equivalent to the ones presented in Ref. [41, 42] even though having slightly different presentation for its time-dependent coefficients. For consistency, we derive the effective Hamiltonian (3.84) with a different approach and compare it with the Hamiltonian presented in the referred literature in Appendix A.

Until now the description of the system has been restricted to the Heisenberg picture of quantum mechanics, where operators have a dependence on time, but state vectors are time-independent. In particular, in this picture, an observable $\hat{O}(t)$ will evolve on time through the Heisenberg equation of motion

$$\dot{\hat{O}}^H(t) = i\left[\hat{H}^H(t), \hat{O}^H(t)\right] + \left(\partial_t \hat{O}^S(t)\right)^H, \tag{3.87}$$

where the superscript H and S designate operator's representation in the Heisenberg and Schrödinger pictures. We can also express our effective Hamiltonian, in the Schrödinger picture, with the following decomposition

$$\hat{H}^S_{\text{eff}}(t) = \hat{H}^S_{\text{free}}(t) + \hat{H}^S_{\text{int}}(t) \tag{3.88}$$

where the contributions

$$\hat{H}^S_{\text{free}}(t) := \sum_k \omega_k(t)\hat{a}_k^{in\dagger}\hat{a}_k^{in}, \tag{3.89}$$

$$\hat{H}^S_{\text{int}}(t) := \frac{i}{2}\sum_{kj}\left[\mu_{(k,j)}(t)\left(\hat{a}_k^{in}\hat{a}_j^{in} - \hat{a}_k^{in\dagger}\hat{a}_j^{in\dagger}\right) + \mu_{[k,j]}(t)\left(\hat{a}_k^{in\dagger}\hat{a}_j^{in} - \hat{a}_j^{in\dagger}\hat{a}_k^{in}\right)\right], \tag{3.90}$$

are respectively, the *free* and the *interacting* part of the Hamiltonian. Here the instantaneous annihilation operator (in the Heisenberg picture) can be expressed in terms of $\hat{a}_k(t) = \hat{U}^\dagger(t,0)\hat{a}_k^{in}\hat{U}(t,0)$, with $\hat{U}(t,0)$ being the time evolution operator generated by the Hamiltonian (3.88) from the initial

instant of time $t = 0$. In particular, an arbitrary *instantaneous* vector state $|\psi(t)\rangle_S = \hat{U}(t,0)\,|\psi; in\rangle_H$ evolves in time by the correspondent Schrödinger equation

$$i\partial_t\,|\Psi(t)\rangle_S = \hat{H}^S(t)\,|\Psi(t)\rangle_S\,. \tag{3.91}$$

3.3 Bogoliubov coefficients

As it was possible to learn in section 3.2.4, all the information associated with the temporal evolution of the system can be recovered from the knowledge of the Bogoliubov coefficients connecting the two static representations of the field (*in* and *out*). As we will see in the next chapter, knowing these coefficients will prove fundamental to investigate the behavior of thermodynamic entropy in the DCE. The main goal of this section, therefore, is to obtain explicit expressions for the Bogoliubov coefficients of the system.

3.3.1 Instantaneous decomposition

As we do not expect the annihilation (and creations) operators to depend explicitly on time in the Schrödinger picture (only thorough the mirror trajectory $l(t)$), the correspondent Heisenberg equation for the instantaneous operator $\hat{a}_k(t)$ must take the following form

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{a}_k(t) = i\Big[\hat{H}_{\text{eff}}(t), \hat{a}_k(t)\Big]. \tag{3.92}$$

Inserting effective Hamiltonian (3.84) into the last expression, one can derive the following dynamical equation

$$\dot{\hat{a}}_k(t) = -i\omega_k(t)\hat{a}_k(t) + \sum_j \Big[\mu_{[k,j]}(t)\hat{a}_j(t) + \mu_{(k,j)}(t)\hat{a}_j^\dagger(t)\Big]\,. \tag{3.93}$$

In terms of the initial operators $\hat{a}_k^{in} = \hat{a}_k(0)$ and $(\hat{a}_k^{in\dagger} = \hat{a}_k^\dagger(0))$, one can obtain solutions for the differential equation (3.93) with the instantaneous version of the Bogoliubov transformations

$$\hat{a}_k(t) = \sum_j \Big[\alpha_{jk}(t)\hat{a}_j^{in} + \beta_{jk}^*(t)\hat{a}_j^{in\dagger}\Big]\,, \tag{3.94}$$

where $\alpha_{jk}(t)$ and $\beta_{jk}(t)$ are the "instantaneous" Bogoliubov coefficients with the initial conditions $\alpha_{jk}(0) = \delta_{jk}$ and $\beta_{jk}(0) = 0$. Inserting the expression (3.94) and its hermitian conjugated into Eq. (3.93) one obtains

$$\dot{\hat{a}}_k(t) = \sum_{jj'}\Bigg\{ \Big[-i\omega_j(t)\alpha_{j'j}(t)\delta_{jk} + \mu_{[k,j]}(t)\alpha_{j'j}(t) + \mu_{(k,j)}(t)\beta_{j'j}(t)\Big]\,\hat{a}_{j'}^{in} \tag{3.95}$$

$$+ \Big[-i\omega_j(t)\beta_{j'j}^*(t)\delta_{jk} + \mu_{(k,j)}(t)\alpha_{j'j}^*(t) + \mu_{[k,j]}(t)\beta_{j'j}(t)\Big]\,\hat{a}_{j'}^{in\dagger}\Bigg\}. \tag{3.96}$$

Equating the last expression with the time derivative of Eq. (3.94) given by $\dot{\hat{a}}_k(t) = \sum_j \left[\dot{\alpha}_{jk}(t)\hat{a}_j^{in} + \dot{\beta}^*_{jk}(t)\hat{a}_j^{in\dagger}\right]$, we find the following differential equations for the Bogoliubov coefficients

$$\dot{\alpha}_{jk}(t) = -i\omega_k(t)\alpha_{jk}(t) + \sum_{j'}\left[\mu_{[k,j]}(t)\alpha_{j'j}(t) + \mu_{(k,j)}(t)\beta^*_{j'j}(t)\right], \tag{3.97a}$$

$$\dot{\beta}_{jk}(t) = +i\omega_k(t)\beta_{jk}(t) + \sum_{j'}\left[\mu_{(k,j)}(t)\alpha_{j'j}(t) + \mu_{[k,j]}(t)\beta^*_{j'j}(t)\right]. \tag{3.97b}$$

Since the coefficients $\mu_{[k,j]}(t)$ and $\mu_{(k,j)}(t)$ are of order $\mathcal{O}[\lambda(t)]$ (where $\lambda(t) = \dot{l}(t)/l(t)$), if we suppose the cavity to move much slower than the speed of light ($\dot{l}(t) \ll 1$), one can expand the Bogoliubov coefficients in terms of $\lambda(t)$. The zeroth order solution is

$$\alpha^{(0)}_{jk}(t) = \alpha_{jk}(0)e^{-i\Theta_k(t)} = \delta_{jk}e^{-i\Theta_k(t)}, \tag{3.98a}$$

$$\beta^{(0)}_{jk}(t) = \beta_{jk}(0)e^{+i\Theta_k(t)} = 0. \tag{3.98b}$$

with $\Theta_k(t) = \int_0^t dt\omega_k(t')$. Substituting Eq. (3.98) into the right side of Eq. (3.97) we obtain

$$\dot{\alpha}^{(1)}_{jk}(t) = -i\omega_k(t)\alpha_{jk}(t) + \sum_{j'}\left[\mu_{[k,j]}(t)\alpha^{(0)}_{jj'}(t) + \mu_{(k,j)}(t)\beta^{(0)*}_{jj'}(t)\right], \tag{3.99a}$$

$$\dot{\beta}^{(1)}_{jk}(t) = +i\omega_k(t)\beta_{jk}(t) + \sum_{j'}\left[\mu_{(k,j)}(t)\alpha^{(0)}_{jj'}(t) + \mu_{[k,j]}(t)\beta^{(0)*}_{jj'}(t)\right], \tag{3.99b}$$

whose solutions are given by

$$\alpha^{(1)}_{jk}(t) = e^{-i\Theta_k(t)}\int_0^t dt'\ \mu_{[k,j]}(t')e^{-i[\Theta_k(t')-\Theta_j(t')]}, \tag{3.100a}$$

$$\beta^{(1)}_{jk}(t) = e^{i\Theta_k(t)}\int_0^t dt'\ \mu_{(k,j)}(t')e^{-i[\Theta_k(t')+\Theta_j(t')]}. \tag{3.100b}$$

Continuing with the same iteration scheme, one can obtain solutions at high orders of the perturbation parameter. As we will only be interested the first order solution for the Bogoliubov coefficients, we resume the Bogoliubov expressions to be

$$\alpha_{jk}(t) = e^{-i\Theta_k(t)}\left[\delta_{jk} + \int_0^t \mathrm{d}t'\ \mu_{[k,j]}(t')e^{-i[\Theta_k(t')-\Theta_j(t')]}\right] + \mathcal{O}(\lambda^2), \tag{3.101a}$$

$$\beta_{jk}(t) = e^{i\Theta_k(t)}\int_0^t \mathrm{d}t'\ \mu_{(k,j)}(t')e^{-i[\Theta_k(t')+\Theta_j(t')]} + \mathcal{O}(\lambda^2). \tag{3.101b}$$

The last expression will be of great importance when we utilize an effective Hamiltonian description to compute the system's density operator.

With the help of Eqs. (3.64), (3.83) and (3.86), explicit expressions for the second Bogoliubov coefficient can be found as

$$\alpha_{jk} = e^{i\Theta_k(t)}\left[\delta_{kj} + (-1)^{j-k+1}\frac{\sqrt{jk}}{k-j}\int_0^t \mathrm{d}t'\lambda(t')e^{-i\left(\omega_k^{in}-\omega_j^{in}\right)t'}\right] + \mathcal{O}(\lambda^2) \tag{3.102}$$

$$\beta_{jk} = (-1)^{j-k+1}\frac{\sqrt{jk}}{k+j}e^{i\omega_k^{in}t}\int_0^t \mathrm{d}t'\lambda(t')e^{-i\left(\omega_k^{in}+\omega_j^{in}\right)t'} + \mathcal{O}(\lambda^2). \tag{3.103}$$

3.3.2 Static decomposition

A more robust method to find the Bogoliubov coefficients used in Ref. [95–98] involves the field decomposition (3.74) in terms of the initial operators $\hat{a}_k^{in}$ and $\hat{a}_k^{in\dagger}$. In this description, during the cavity motion, one can expand the quantum field as

$$\hat{\Phi}(x,t) = \sum_k \frac{1}{\sqrt{2\omega_k^{in}}}\left[\sum_j Q_j^{(k)}(t)\varphi_j(x,t)\ \hat{a}_k^{in} + \text{h.c.}\right] \quad \text{for} \quad 0 \le t \le T, \tag{3.104}$$

where the Fourier coefficients $Q_j^{(k)}(t)$ satisfies the initial conditions

$$Q_j^{(k)}(0) = \delta_{jk}, \qquad \dot{Q}_j^{(k)}(0) = -i\omega_k^{in}\delta_{kj}.$$

Substituting the quadrature position function (3.74a) (where the Fourier coefficients are defined) into the equation of motion (3.65), one can obtain the following set of coupled differential equations

$$\ddot{Q}_j^{(k)} + \omega_j^2(t)Q_j^{(k)} = 2\lambda(t)\sum_l g_{kl}\dot{Q}_l^{(k)} + \dot{\lambda}(t)\sum_l g_{kl}Q_l^{(k)} + \mathcal{O}(\lambda^2). \tag{3.105}$$

For $t \ge T$, after the mirror returns to its stationary configuration, the quantum field is decomposed in terms of Eq. (3.52) with mode function (3.52) as

$$\hat{\Phi}(x,t) = \sum_k \frac{1}{\sqrt{2\omega_k^{out}}}\left[\varphi_k(x,T)e^{-i\omega_k^{out}t}\hat{a}_k^{out} + \text{h.c.}\right] \quad \text{for} \quad t \ge T. \tag{3.106}$$

Since at this interval we have $\lambda(T) = 0$, the right-hand side of Eq. (3.105) vanishes and one can write the Fourier coefficients as

$$Q_j^{(k)}(t) = \xi_j^{(k)}e^{-i\omega_j^{out}t} + \eta_j^{(k)}e^{i\omega_j^{out}t}, \tag{3.107}$$

where $\xi_k^{(k)}$ and $\eta_k^{(k)}$ are complex coefficients. Inserting the last expression (3.107) with $\varphi_k(x,T)$ into the field decomposition (3.104), by comparing the resulting expression with the decomposition (3.106) one can relate the Bogoliubov transformations (3.55) in terms of the complex coefficients (3.107) as

$$\alpha_{jk} = \sqrt{\frac{\omega_k^{out}}{\omega_j^{in}}}\xi_k^{(j)} \qquad \text{and} \qquad \beta_{jk} = \sqrt{\frac{\omega_k^{out}}{\omega_j^{in}}}\eta_k^{(j)}. \tag{3.108}$$

This means that to calculate the Bogoliubov coefficients one only need to solve the infinite set of coupled differential equations (3.105) to obtain the complex numbers $\xi_j^{(k)}$ and $\eta_j^{(k)}$. To complicate things, the equation that we need to solve have an arbitrary time-dependence in an infinite number of terms. So, in order to simplify the problem, we concentrate in this section on the particular case in which the walls perform harmonic oscillations with small amplitude. We then impose the following equation of motion for the moving mirror

$$l(t) = l_0 \left[1 + \epsilon \sin\left(\omega_p t\right)\right], \tag{3.109}$$

where $\omega_p = p\pi/l_0$ is an unperturbed field frequency and $\epsilon \ll 1$ is a small dimensionless parameter.

Since we impose the field to be weakly perturbed by the mirror oscillations (3.109), is natural to search for solutions to $Q_j^{(k)}$ by allowing the coefficients $\xi_j^{(k)}$ and $\eta_j^{(k)}$ in Eq. (3.107) to be functions that vary slowly in time, i.e., $\dot{\xi}_j^{(k)}, \dot{\eta}_j^{(k)} \sim \epsilon$. Then by substituting Eq. (3.107) into (3.105) and employing the usual prescriptions of the method of slowly varying amplitudes [100] (shown in Appendix B), it is possible to obtain a set of coupled first order differential equations with time independent coefficients. For $k \geq p$, their expressions take the form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\xi_k^{(n)} = (-1)^p \left[(k+p)\xi_{k+p}^n - (k-p)\xi_{k-p}^n\right], \tag{3.110a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\eta_k^{(n)} = (-1)^p \left[(k+p)\eta_{k+p}^n - (k-p)\eta_{k-p}^n\right], \tag{3.110b}$$

whereas for $1 \leq k \leq p-1$ we have $p-1$ equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\xi_k^{(n)} = (-1)^p \left[(p+k)\xi_{k+p}^n - (p-k)\eta_{p-k}^n\right], \tag{3.111a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\eta_k^{(n)} = (-1)^p \left[(p+k)\eta_{k+p}^n - (p-k)\xi_{p-k}^n\right], \tag{3.111b}$$

where it was defined the dimensionless time parameter $\tau = \epsilon\omega_1 t/2$ and the initial conditions for the complex coefficients are $\xi_j^{(k)}(0) = \delta_{kj}$ and $\eta_j^{(k)}(0) = 0$. From such conditions, one can show that solutions for the system of equation (3.111) form a set $\xi_{mp+j}^{(np+j)} = \eta_{mp-j}^{(np+j)} \equiv 0$ if $k \neq j$ with $j, k = \{0, 1, \ldots, p-1\}$ and $n, m = \{0, 1, 2, \ldots\}$. Complete solutions for the non-vanishing coefficients

were found in Ref. [96] and are expressed in terms of the hypergeometrical function $F(a,b;c;z)$ as

$$\xi^{(pn+j)}_{pm+j} = \frac{\Gamma(1+n+j/p)(\sigma\kappa)^{n-m}}{\Gamma(1+m+j/p)\Gamma(1+n-m)} F(n+j/p,-m-j/p;1+n-m;\kappa^2) \qquad 0\leq m\leq n \tag{3.112}$$

$$\xi^{(pn+j)}_{pm+j} = \frac{\Gamma(m+j/p)(-\sigma\kappa)^{m-n}}{\Gamma(n+j/p)\Gamma(1+m-n)} F(m+j/p,-n-j/p;1+m-n;\kappa^2), \qquad m\geq n, \tag{3.113}$$

$$\begin{aligned}\eta^{(pn+j)}_{pm-j} = &-\frac{\Gamma\left(m-j/p\right)\Gamma\left(1+j/p\right)}{\pi\Gamma(1+n+m)}\sin\left[\pi\left(m-j/p\right)\right](\sigma\kappa)^{n+m} \\ &\times F\left(n+j/p,m-j/p;1+n+m;\kappa^2\right),\end{aligned} \tag{3.114}$$

where $\Gamma(n)$ is the gamma function, $\sigma=(-1)^p$ and

$$\kappa(\tau) = \frac{\sinh(p\tau)}{\sqrt{1+\sinh^2(p\tau)}}. \tag{3.115}$$

One can also write equivalent expressions for Eqs. (3.110) and (3.111) in terms of coefficients with the same lower indexes [98]. For $n>p$, their expressions take the form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\xi^{(n)}_k = n(-1)^p\left[\xi^{(n-p)}_k - \xi^{(n+p)}_k\right], \tag{3.116a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\eta^{(n)}_k = n(-1)^p\left[\eta^{(n-p)}_k - \eta^{(n+p)}_k\right], \tag{3.116b}$$

whereas for $1<n<p-1$ we have $p-1$ equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\xi^{(n)}_k = n(-1)^{p+1}\left[\eta^{(p-n)*}_k + \xi^{(p+n)}_k\right], \tag{3.117a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\eta^{(n)}_k = n(-1)^{p+1}\left[\xi^{(p-n)*}_k + \eta^{(p+n)}_k\right]. \tag{3.117b}$$

Nonetheless, in the next chapter, we will be interested in computing the diagonal entropy generated in particular modes of the field when the cavity oscillates in resonance with the first unperturbed field frequency. As a result, for reasons that will become clear later, it will be sufficient for us to pay attention only to the asymptotic behavior of the Bogoliubov coefficients in which one of the index is equal to 1. In the special case of parametric resonance, i.e., when the cavity oscillates with twice the fundamental unperturbed frequency ($p=2$), the set of differential equations for the creation and

annihilation operators in Eqs. (3.110) and (3.117) reduces to the following structure:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau}\alpha_{1m} &= -\beta^{*}_{1m} - \sqrt{3}\alpha_{3m}, \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\beta_{1m} &= -\alpha^{*}_{1m} - \sqrt{3}\beta_{3m}, \end{cases} \qquad \text{for} \quad k = 1 \tag{3.118a}$$

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}\tau}\alpha_{km} = \sqrt{k(k-2)}\alpha_{(k-2)m} - \sqrt{k(k+2)}\alpha_{(k+2)m}, \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\beta_{km} = \sqrt{k(k-2)}\beta_{(k-2)k} - \sqrt{k(k+2)}\beta_{(k+2)m}. \end{cases} \qquad \text{for} \quad k > 2, \tag{3.118b}$$

In the particular case when the first index is equal to 1, one can express its solutions in the asymptotic cases: for $\tau \ll 1$, in the short-time regime, we have

$$\alpha_{1(2\mu+1)}(\tau \ll 1) = (-1)^{\mu}\sqrt{2\mu+1}\frac{\Gamma(\mu+1/2)\kappa^{\mu}}{\Gamma(1/2)\Gamma(1+\mu)} = (\mu+1)K_{\mu}J_{\mu}\ \tau^{\mu}, \tag{3.119a}$$

$$\beta_{1(2\mu+1)}(\tau \ll 1) = (-1)^{\mu}\sqrt{2\mu+1}\frac{\Gamma(\mu+1/2)\Gamma(3/2)\kappa^{\mu+1}}{\pi\Gamma(2+\mu)} = -K_{\mu}J_{\mu}\ \tau^{\mu+1}. \tag{3.119b}$$

where $\mu = 0, 1, 2, \ldots$ and we have defined the coefficients J_{μ} and K_{μ} to be

$$J_{\mu} := \frac{(2\mu)!}{2^{\mu}(\mu!)^{2}} \qquad \text{and} \qquad K_{\mu} := (-1)^{\mu}\frac{\sqrt{2\mu+1}}{\mu+1}. \tag{3.120}$$

For $\tau \gg 1$, in the long-time regime, the expression obtained is

$$\alpha_{1(2\mu+1)}(\tau \gg 1) \approx \frac{2}{\pi}\frac{(-1)^{\mu}}{\sqrt{2\mu+1}}, \tag{3.121a}$$

$$\beta_{1(2\mu+1)}(\tau \gg 1) \approx \frac{2}{\pi}\frac{(-1)^{\mu}}{\sqrt{2\mu+1}}. \tag{3.121b}$$

In the next chapter we shall explore the above results to obtain explicit expressions for the thermodynamic entropy for a system reproducing the DCE.

Chapter 4

Thermodynamic Entropy production in the DCE

In this chapter we will be interested in studying the irreversibility associated with the DCE in terms of the production of the system's thermodynamic entropy. As discussed earlier, for this endeavor, we shall consider as the main figure of merit the diagonal entropy [78]

$$S_d(\hat{\rho}) = -\sum_{\boldsymbol{n}} \rho_{\mathbf{nn}} \ln \rho_{\mathbf{nn}}, \tag{4.1}$$

where $\rho_{\mathbf{nn}} = \langle \text{in}; \boldsymbol{n}| \, \hat{\rho} \, |\boldsymbol{n}; \text{in}\rangle$ are the diagonal elements of the system's density operator in the initial energy eigenbasis.

4.1 Effective Hamiltonian approach

To investigate the entropy production within the proposed scheme, one first needs to obtain an explicit expression for the system's density operator $\hat{\rho}$ after the cavity returns to its stationary configuration. A first intuitive approach is to use the effective Hamiltonian (3.88) derived in the previous chapter to evolve the system's density operator by the Liouville–von Neumann equation

$$\dot{\hat{\rho}}(t) = -i \left[\hat{H}^S_{\text{eff}}(t), \hat{\rho}(t)\right]. \tag{4.2}$$

Since the time-dependent boundary condition on the field introduces an interacting contribution to the effective Hamiltonian (3.88), calculations are more naturally performed if we move to the interaction picture of quantum mechanics, as we are going to do next.

4.1.1 The interaction picture

Given a state vector $|\psi(t)\rangle_S$ and an arbitrary observable $\hat{O}^S$, both in the Schrödinger picture, one can write their counterpart expressions in the interaction picture, namely $|\psi(t)\rangle_I$ and $\hat{O}^I(t)$, by the

following definitions

$$\begin{aligned} |\psi(t)\rangle_I &:= \hat{U}^\dagger_{\text{free}}(t)|\psi(t)\rangle_S, \\ \hat{O}^I(t) &:= \hat{U}^\dagger_{\text{free}}(t)\hat{O}^S\hat{U}_{\text{free}}(t), \end{aligned} \tag{4.3}$$

where $\hat{U}_{\text{free}}$ is a unitary operator associated to the time evolution of the free part of the Hamiltonian given by

$$\hat{U}_{\text{free}}(t) := \exp\left\{-i\int_0^t \mathrm{d}t'\hat{H}^S_{\text{free}}(t')\right\} = \exp\left\{-i\Theta_k(t)\hat{a}^{in\dagger}_k\hat{a}^{in}_k\right\}, \tag{4.4}$$

with $\Theta_k(t) = \int_0^t \mathrm{d}t'\omega_k(t')$. As illustrated by Eq. (4.3), state vectors in the interaction picture inherently contain a time-dependent contribution due to the free component of the Hamiltonian. Therefore, their correct time evolution is exclusively governed by the interaction part of the Hamiltonian, as exemplified by the following Schrödinger equation:

$$i\partial_t \left|\Psi(t)\right\rangle_I = \hat{H}^I_{\text{int}}(t)\left|\Psi(t)\right\rangle_I, \tag{4.5}$$

where the Hamiltonian in the interaction picture $\hat{H}^I_{\text{int}}(t)$ is given explicitly by

$$\begin{aligned} \hat{H}^I_{\text{int}}(t) &= \hat{U}^\dagger_{\text{free}}(t)\hat{H}^S_{\text{int}}(t)\hat{U}^\dagger_{\text{free}}(t) \\ &= \frac{i}{2}\sum_{kj}\left[\mathcal{B}^*_{jk}(t)\hat{a}^{in}_j\hat{a}^{in}_k - \mathcal{B}_{jk}(t)\hat{a}^{in\dagger}_k\hat{a}^{in\dagger}_j + \mathcal{A}_{jk}(t)\hat{a}^{in\dagger}_k\hat{a}^{in}_j - \mathcal{A}^*_{jk}(t)\hat{a}^{in\dagger}_j\hat{a}^{in}_k\right]. \end{aligned} \tag{4.6}$$

For simplification purposes the time-dependence of the creation and annihilation operators (in the interaction picture) has been included into the Hamiltonian's coefficients in the form

$$\mathcal{A}_{jk}(t) := \mu_{[k,j]}(t)e^{-i[\Theta_k(t)-\Theta_j(t)]} \qquad \text{and} \qquad \mathcal{B}_{jk}(t) := \mu_{(k,j)}(t)e^{-i[\Theta_j(t)+\Theta_k(t)]}, \tag{4.7}$$

which are connected with the first-order Bogoliubov coefficient defined in Eqs. (3.100) by

$$\tilde{\alpha}^{(1)}_{jk}(t) := \alpha^{(1)}_{jk}(t)e^{+i\Theta_k(t)} = \int_0^t dt'\ \mathcal{A}_{jk}(t'), \tag{4.8a}$$

$$\tilde{\beta}^{(1)}_{jk}(t) := \beta^{(1)}_{jk}(t)e^{-i\Theta_k(t)} = \int_0^t dt'\ \mathcal{B}_{jk}(t'). \tag{4.8b}$$

The above correspondence will be of fundamental importance when expressing solutions for the density operator in a more compact form. From the Eq. (4.5), the interacting version of the system's density operator $\hat{\rho}_I(t) = \left|\Psi(t)\right\rangle_I\left\langle\Psi(t)\right|_I$ now evolves on time through the updated version of the Liouville-von Neumann equation

$$\dot{\hat{\rho}}_I(t) = -i\left[\hat{H}^I_{\text{int}}(t), \hat{\rho}_I(t)\right]. \tag{4.9}$$

4.1.2 Solving the dynamical equation for $\hat{\rho}_I(t)$

Equipped with the dynamical equation (4.9) we can begin searching for explicit solutions for $\hat{\rho}_I$. The first difficulty encountered in this approach is related to the inherent complexity associated with the effective Hamiltonian and its infinite set of coupled creation and annihilation operators with time-dependent coefficients. To circumvent this problem, we opt to narrow our focus of study to the subset of cavity configurations where the second mirror performs the following trajectory

$$l(t) = l_0 \left[1 + \epsilon\xi(t)\right], \tag{4.10}$$

where $\xi(t)$ is an arbitrary continuous function of order unity —as well as its first derivative —, while $\epsilon \ll 1$ is a small dimensionless parameter needed to ensure that the mirror motion amplitude is sufficiently small such that the field is only weakly perturbed. For simplicity, we also choose the first mirror to be located at a fixed position $x_1 = 0$.

A key insight from Eq. (4.10) is to acknowledge that the time-dependent coefficients $\mathcal{A}_{jk}(t)$ and $\mathcal{B}_{jk}(t)$ defined in Eq. (4.7) —which are proportional to $\lambda(t) = \dot{l}(t)/l(t)$ because the coefficients in Eq. (3.83)— are now dependent on the small dimensional parameter ϵ at least in first order. Therefore, by imposing the mirror trajectories (4.10), we expect the corrections to the solvable part of the field to be sufficiently weak, allowing for the application of perturbation theory in terms of ϵ.

A formal solution to Eq. (4.9) can then be obtained, up to second order in ϵ, in terms of the following expansion

$$\hat{\rho}(T) = \hat{\rho}(0) - i\int_0^T dt' \left[\hat{H}^I_{\rm int}(t'), \hat{\rho}(0)\right] - \int_0^T dt' \int_0^{t'} dt'' \left[\hat{H}^I_{\rm int}(t'), \left[\hat{H}^I_{\rm int}(t''), \hat{\rho}(0)\right]\right]. \tag{4.11}$$

As the central goal of this chapter is to investigate the thermodynamic aspects of the particle creation process due to the DCE, is reasonable to assume the system to be initially prepared in the initial vacuum state $\hat{\rho}(0) = |0;\text{in}\rangle\langle\text{in};0|$. From this initial state, the second order term in the last expansion is now separable in terms of the integration time variables t' and t'', such as in

$$\int_0^T dt' \int_0^{t'} dt'' \left[\hat{H}^I_{\rm int}(t'), \left[\hat{H}^I_{\rm int}(t''), \hat{\rho}(0)\right]\right] = \frac{1}{2}\left[\int_0^T dt' \hat{H}^I_{\rm int}(t'), \left[\int_0^T dt'' \hat{H}^I_{\rm int}(t''), \hat{\rho}(0)\right]\right], \tag{4.12}$$

where the right side integral is twice as larger as the left side one because of the parameterization in different domains of integration over the plane (t', t''). Another useful property for the calculation of the pertubative expansion comes from the fact that the time-dependence in the Hamiltonian (3.88) is concentrated only on the coefficients. Consequently, using Eqs. (4.8), one can write

$$\int_0^T dt'' H(t'') = \frac{i}{2}\sum_{nl}\left[\tilde{\beta}^{(1)}_{ln}\hat{a}^{in}_k\hat{a}^{in}_j - \tilde{\beta}^{(1)*}_{kj}\hat{a}^{in\dagger}_k\hat{a}^{in\dagger}_j + \tilde{\alpha}^{(1)}_{ln}\hat{a}^{in\dagger}_j\hat{a}^{in}_k - \tilde{\alpha}^{(1)*}_{ln}\hat{a}^{in\dagger}_k\hat{a}^{in}_j\right]. \tag{4.13}$$

Inserting the last identities (4.12) and (4.13) into the operator expansion (4.11) and after lengthy

algebraic manipulations with the commutators, one can obtain the following expression for the system's density operator (up to second order in ϵ)

$$\hat{\rho}(T) = \hat{\rho}_0 - \tfrac{1}{2}\sum_{kj}\Bigg\{\tilde{\beta}^{(1)*}_{kj}\left(\hat{a}^{in\dagger}_k\hat{a}^{in\dagger}_j\hat{\rho}_0\right) - \tfrac{1}{4}\sum_{nm}\Bigg[\tilde{\beta}^{(1)}_{mn}\tilde{\beta}^{(1)*}_{kj}\left(\hat{a}^{in\dagger}_k\hat{a}^{in\dagger}_j\hat{\rho}_0\hat{a}^{in}_m\hat{a}^{in}_n\right) - \tilde{\beta}^{(1)}_{mn}\tilde{\beta}^{(1)*}_{kj}\left(\hat{a}^{in}_m\hat{a}^{in}_n\hat{a}^{in\dagger}_k\hat{a}^{in\dagger}_j\hat{\rho}_0\right) + \tilde{\beta}^{(1)*}_{mn}\tilde{\beta}^{(1)*}_{kj}\left(\hat{a}^{in\dagger}_m\hat{a}^{in\dagger}_n\hat{a}^{in\dagger}_k\hat{a}^{in\dagger}_j\hat{\rho}_0\right) + 2\tilde{\alpha}^{(1)*}_{mn}\tilde{\beta}^{(1)*}_{kj}\left(\hat{a}^{in\dagger}_m\hat{a}^{in}_n\hat{a}^{in\dagger}_k\hat{a}^{in\dagger}_j\hat{\rho}_0\right)\Bigg] + \text{h.c.}\Bigg\}. \tag{4.14}$$

Despite appearing a mess from a first look, expression (4.14) shows a relative compactness due to the notation for the first order Bogoliubov coefficients. With the last expression we can directly calculate the number of particles created inside the cavity due to the DCE, taking the following form

$$N(T) = \mathrm{Tr}\left\{\sum_k \hat{\rho}(T)\hat{a}^{in\dagger}_k\hat{a}^{in}_k\right\} = \frac{1}{4}\,\mathrm{Re}\sum_{kk'jmn}\tilde{\beta}^{(1)}_{mn}\tilde{\beta}^{(1)*}_{k'j}\,\mathrm{Tr}\left\{\hat{a}^{in\dagger}_{k'}\hat{a}^{in\dagger}_j\hat{\rho}_0\hat{a}^{in}_m\hat{a}^{in}_n\hat{a}^{in\dagger}_k\hat{a}^{in}_k\right\} = \sum_{kj}|\beta^{(1)}_{kj}|^2, \tag{4.15}$$

in agreement with Eq. (3.57), thus showing the consistency of our calculations. Moreover, one can obtain explicit expressions for the diagonal elements of the density operator in the initial energy basis directly from Eq. (4.14). Those elements come from three contributions

$$\langle 0|\,\hat{\rho}(T)\,|0\rangle = \langle 0|\,\hat{\rho}(0)\,|0\rangle - \frac{1}{4}\,\mathrm{Re}\sum_{kjmn}\tilde{\beta}^{(1)}_{mn}\tilde{\beta}^{(1)*}_{kj}\,\langle 0|\,\hat{a}^{in}_m\hat{a}^{in}_n\hat{a}^{in\dagger}_k\hat{a}^{in\dagger}_j\hat{\rho}(0)\,|0\rangle = 1 - \frac{1}{2}N(T), \tag{4.16}$$

$$\langle 2_k|\,\hat{\rho}(T)\,|2_k\rangle = \frac{1}{4}\,\mathrm{Re}\sum_{k'jmn}\tilde{\beta}^{(1)}_{mn}\tilde{\beta}^{(1)*}_{k'j}\,\langle 2_k|\,\hat{a}^{in\dagger}_{k'}\hat{a}^{in\dagger}_j\hat{\rho}(0)\hat{a}^{in}_m\hat{a}^{in}_n\,|2_k\rangle = \frac{1}{2}|\beta^{(1)}_{kk}|^2, \tag{4.17}$$

$$\langle 1_k,1_j|\,\hat{\rho}(T)\,|1_k,1_j\rangle = \frac{1}{4}\,\mathrm{Re}\sum_{k'j'mn}\tilde{\beta}^{(1)}_{mn}\tilde{\beta}^{(1)*}_{k'j'}\,\langle 1_k,1_j|\,\hat{a}^{in\dagger}_{k'}\hat{a}^{in\dagger}_{j'}\hat{\rho}(0)\hat{a}^{in}_m\hat{a}^{in}_n\,|1_k,1_j\rangle = |\beta^{(1)}_{kj}|^2. \tag{4.18}$$

With the help of the last expressions we are now ready to discuss the entropy production due to the particle creation process.

4.1.3 Entropy production

From the diagonal contributions to the density operator, as shown in Eq. (4.14), the diagonal entropy S_d can be directly computed, resulting in[1]

$$\begin{aligned} S_d(T) &= -\langle 0|\,\hat{\rho}\,|0\rangle\ln\langle 0|\,\hat{\rho}\,|0\rangle - \sum_k\langle 2_k|\,\hat{\rho}\,|2_k\rangle\ln\langle 2_k|\,\hat{\rho}\,|2_k\rangle - \tfrac{1}{2}\sum_{kj}\langle 1_k,1_j|\,\hat{\rho}\,|1_k,1_j\rangle\ln\langle 1_k,1_j|\,\hat{\rho}\,|1_k,1_j\rangle \\ &= -\left[1-\tfrac{1}{2}N(T)\right]\ln\left[1-\tfrac{1}{2}N(T)\right] - \sum_{kj}\tfrac{1}{2}|\beta_{kj}(T)|^2\ln(1-\tfrac{1}{2}\delta_{kj})|\beta_{kj}(T)|^2, \end{aligned} \tag{4.19}$$

[1]The 1/2 factor in the last term is employed to prevent the over-counting of the diagonal contributions $\langle 1_k,1_j|\,\hat{\rho}\,|1_k,1_j\rangle$ when summing over k and j, i.e., $\langle 1_k,1_j|\,\hat{\rho}\,|1_k,1_j\rangle$ is the same as $\langle 1_j,1_k|\,\hat{\rho}\,|1_j,1_k\rangle$.

which is the system's thermodynamic entropy expression for the subclass of mirror motions that weak perturb the field (at second order in ϵ). The very first noticeable aspect of Eq. (4.19) is its scaling behavior with the number of particles created in the cavity as a result of its motion. This is also explicit due to the presence of the terms $|\beta_{kj}|^2$, which can be interpreted as the j-th mode contribution of the in-field to the number of particles created at the k-th mode on the *out*-field.

In general, this link between irreversibility and the particle creation process is consistent with the well established fact that the DCE only occurs if the field is perturbed non-adiabatically, *i.e.*, if the motion of the mirror is fast enough to prevent the field to instantly readjust and thus causing the amplification of quantum vacuum fluctuations. Therefore, the dependence of our thermodynamic entropy with the number of particle created during the process is a feature we already should expect.

Since the initial number of particles in the cavity is zero, *i.e.*, $\sum_{kj} |\beta_{jk}(0)| = N(0) = 0$, it follows immediately that the system's entropy production, represented by ΔS_d, is identical to Eq. (4.19), as illustrated by

$$\Delta S_d(T) := S_d(T) - S_d(0) = S_d(T). \tag{4.20}$$

Another insight given by Eq. (4.19) to the interpretation of the thermodynamic entropy production is related in how the initial vacuum state is transformed in a coherent superposition of excited states as time passes. To connect these two notions, we first need to introduce the concept of quantum coherence and how it is quantified.

Quantum coherence

Let us consider the famous double-slit experiment for a moment. In that experiment, an electron is allowed to pass through a wall with two slits and its subsequent position is measured by a detector placed at a rear screen. In general, one can consider the electron's state $|\Psi\rangle$ after passing through the two slits, to be described by the superposition

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left(|\Psi_1\rangle + e^{i\theta} |\Psi_2\rangle \right), \tag{4.21}$$

where $|\Psi_1\rangle$ and $|\Psi_2\rangle$ designate the states correspondent to the different paths taken and θ is relative phase between them. More specifically, if one calculate the probability of measuring the electron in a given position x, we obtain

$$|\Psi(x)|^2 = \frac{1}{2} \left[|\Psi_1(x)|^2 + |\Psi_2(x)|^2 + 2\,\mathrm{Re}\Big\{ \Psi_1(x)^* \Psi_2(x) e^{i\theta} \Big\} \right], \tag{4.22}$$

which characterizes, in an explicit manner, the existence of interference between the two states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ (here $\Psi_i(x) = \langle x|\Psi_i\rangle$ is the wave function associated to the state $|\Psi_i\rangle$). We then say that the state $|\Psi\rangle$ described by the Eq. (4.21) exhibit *quantum coherence*, since the phase relationship between the superimposed states allows the existence of interference phenomena.

To see the importance of the relative phase between the two states, let us consider the correspondent system's density matrix $\hat{\rho}_\Psi = |\Psi\rangle\langle\Psi|$ in the basis spanned by the vectors $\{|\Psi_1\rangle, |\Psi_2\rangle\}$

$$\hat{\rho}_\Psi = \frac{1}{2}\begin{pmatrix} 1 & e^{-i\theta} \\ e^{i\theta} & 1 \end{pmatrix}. \tag{4.23}$$

For the chosen basis, its diagonal terms $\langle\Psi_1|\hat{\rho}|\Psi_1\rangle$ and $\langle\Psi_2|\hat{\rho}|\Psi_2\rangle$ are the system's *populations* and represent the probabilities of measuring each one of the possible outcomes. The off-diagonal contributions $\langle\Psi_1|\hat{\rho}|\Psi_2\rangle$ and $\langle\Psi_2|\hat{\rho}|\Psi_1\rangle$, on the other hand, are known as the *coherences* of the system, since it carries information about the relative phase between the states.

To see the reason for the last nomenclature, if we consider the experiment to not being performed in a good vacuum, for each interaction between the electron and the environment (collisions with the air molecules, for example), one expects the relative phase θ between the superposed states to change randomly. With the phase relationship between $|\Psi_1\rangle$ and $|\Psi_2\rangle$ becoming uncorrelated, the average value of the exponential term $e^{i\theta}$ tends to zero (net sum of phasors with random direction tends to vanish) and the density operator becomes diagonal

$$\hat{\rho}_\Psi \longrightarrow \hat{\rho}'_\Psi = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.24}$$

If we now calculate the probability to measure a given outcome at position x with the last density matrix, one obtains

$$|\Psi(x)|^2 = \mathrm{Tr}\{\hat{\rho}'_\Psi |x\rangle\langle x|\} = \frac{1}{2}\left[|\Psi_1(x)|^2 + |\Psi_2(x)|^2\right], \tag{4.25}$$

which does not exhibit the interference term present in Eq. (4.22). We then say that due to the absence of a phase relationship, the system has lost coherence through a process called *quantum decoherence*. Showing the important role played by the off-diagonal terms of the density operator in characterizing the quantum coherence.

Another rightful question in this context, is how to quantify the amount of quantum coherence exhibited by a state in a given basis. The first rigorous mathematical account to this issue was introduced in Ref. [101] in terms of the concepts of incoherent states and operations. There, once we fix a particular basis $\{|i\rangle\}$ for $i = 1, \ldots, d$ in a d-dimensional Hilbert space, a quantum state is said to be incoherent if its density operator $\hat{\delta}$ is diagonal in that basis, that is

$$\hat{\delta} = \sum_{i=1}^{d} \delta_i |i\rangle\langle i|. \tag{4.26}$$

Furthermore, if we denote by $\mathcal{I}$ the set of all incoherent states, an operation described by a set of Kraus operators $\{\hat{K}_n\}$ (satisfying the closure condition $\sum_n \hat{K}_n^\dagger \hat{K}_n = \mathbb{I}$) is said to be an *incoherent operation* if it maps any incoherent state into another incoherent operator, *i.e.*, if $\hat{K}_n \mathcal{I} \hat{K}_n^\dagger \subset \mathcal{I}$ for

all n.

Then, in regard to pure states (see Ref. [101] to the general characterization), a *coherence measure* is defined to be any mapping C from quantum states to non-negative number which satisfies the following necessary conditions

1. $C(\hat{\delta}) = 0$, for all $\hat{\delta} \in \mathcal{I}$
2. $C(\hat{\rho}) \geq C(\sum_n \hat{K}_n \hat{\rho} \hat{K}_n^\dagger)$, for all $\{\hat{K}_n\}$ such that $\hat{K}_n \mathcal{I} \hat{K}_n^\dagger \subset \mathcal{I}$.

Considering the decomposition $\hat{\rho} = \sum_{ij} \rho_{ij} |i\rangle \langle j|$ for a given density operator, one very intuitive exemplar of a coherence measure can be introduced in terms of the l_1-*norm of coherence*

$$C_{l_1}(\hat{\rho}) = \sum_{\substack{i,j \\ i \neq j}} |\rho_{ij}|, \tag{4.27}$$

which is given by the sum of the off-diagonal contributions of the density operator. The name l_1-norm comes from the distance measure of order one: $\mathcal{D}_{l_1}(\hat{\rho}, \hat{\delta}) = ||\hat{\rho} - \hat{\delta}||^1 = \sum_{ij} |\rho_{ij} - \delta_{ij}|$ which quantify the distance between a given state $\hat{\rho}$ to the correspondent set of incoherent states. Another intuitive measure of coherence is the *relative entropy of coherence* $C_{\text{re}}(\hat{\rho})$ described by

$$C_{\text{re}}(\hat{\rho}) = S_{vN}(\hat{\rho}_{\text{d}}) - S_{vN}(\hat{\rho}), \tag{4.28}$$

where $\hat{\rho}_{\text{d}} = \sum_i \rho_{ii} |i\rangle \langle i|$ are the diagonal contributions for the density operator in the chosen basis. An insightful feature of both definitions (4.27) and (4.28) is the quantification of the amount of quantum coherence the system's state $\hat{\rho}$ has by adequately counting how much off-diagonal contributions $\hat{\rho}$ exhibit in a given basis.

From definition (4.28) it is also immediate to notice that the entropy production given by our formula (4.19) is exactly equal to the creation of coherence in the energy eigenbasis of the field. To see this, we pick up the initial energy eigenbasis to measure the amount of coherence generated throughout the cavity motion. This is consistent with a definition of a thermodynamic entropy, since the system's energy is the most natural thermodynamic measurement one can expect to recover in the quantum settings. Under the above choice, we directly see that $S_{vN}(\hat{\rho}_d) = S_d(\hat{\rho})$. As the system's evolution is unitary, and the initial state is pure, we have $S_{vN}(\hat{\rho}) = 0$, thus implying that

$$C_{\text{re}}(\hat{\rho}) = S_d(T). \tag{4.29}$$

Note that, differently from Eq. (4.19), such a result is a general one, independent of the perturbation theory used here. This result implies that we will observe irreversibility (positive entropy production) for every process that creates coherence in the system. Therefore, reversible processes must be the ones that are performed slowly enough in order to not induce transitions among the energy

eigenstates. This result is in agreement with the discussions presented in Refs. [78, 82, 86, 87], where both entropy production and heat are associated with processes that generates coherences.

Oscillating mirror

In order to illustrate our results in terms of a specific cavity configuration, let us consider that the second mirror performs harmonic oscillations of the form

$$\xi(t) = \sin(\omega_p t), \tag{4.30}$$

where $\omega_p = p\omega_1^{in}$ is an integral multiple of the first unperturbed field frequency. Substituting the last expression into Eq. (3.102) with $\omega_\nu = \omega_k^{in} + \omega_j^{in}$, we have

$$\begin{aligned}\beta_{jk} &= (-1)^{j-k}\epsilon\omega_p\frac{\sqrt{jk}}{k+j}e^{i\omega_k^{in}T}\int_0^T \mathrm{d}t'\cos(\omega_p t')e^{-i\omega_\nu t'} \\ &= (-1)^{j-k}\epsilon\omega_p\frac{\sqrt{jk}}{k+j}e^{i\omega_k^{in}T}\begin{cases}\frac{\sin\left(\frac{\omega_p-\omega_\nu}{2}T\right)}{\omega_p-\omega_\nu}e^{\frac{i}{2}(\omega_p-\omega_\nu)T} + \frac{\sin\left(\frac{\omega_p+\omega_\nu}{2}T\right)}{\omega_p+\omega_\nu}e^{-\frac{i}{2}(\omega_p+\omega_\nu)T} & \text{for} \quad \omega_p \neq \omega_\nu, \\ \frac{1}{2}\left[T + \frac{\sin(\omega_p T)}{\omega_p}e^{-i\omega_p T}\right] & \text{for} \quad \omega_p = \omega_\nu.\end{cases}\end{aligned}$$

If we assume the case in which the mirror returns to its initial position at time $t = T$ after performing a certain number of complete cycles ($\omega_p T = 2\pi m$ with $m = 1, 2, \dots$), then the last expression can be further simplified as

$$\beta_{jk} = (-1)^{j-k}\epsilon\omega_p\frac{\sqrt{jk}}{k+j}e^{i\omega_k^{in}T}\begin{cases}\frac{2\omega_\nu}{\omega_\nu^2-\omega_p^2}\sin\left(\frac{\omega_\nu T}{2}\right)e^{\frac{i}{2}\omega_\nu T} & \text{for} \quad \omega_p \neq \omega_\nu, \\ \frac{1}{2}T & \text{for} \quad \omega_p = \omega_\nu.\end{cases} \tag{4.31}$$

By applying a rotating-wave approximation on the Eq. (4.31), where we ignore all the rapidly oscillatory terms associated with the modes $\omega_p \neq \omega_k + \omega_j$, one obtains the following expression for the second Bogoliubov coefficient

$$\beta_{jk} = (-1)^{j-k}\frac{\sqrt{jk}}{k+j}\frac{\epsilon\omega_p T}{2}e^{i\omega_k^{in}T}\delta_{j,(p-k)}, \quad \text{for} \quad k = 1, \dots, p-1. \tag{4.32}$$

In terms of the last expression we can calculate the number of particles created on time as

$$\begin{aligned}N(\tau) &= \sum_{jk}|\beta_{jk}(\tau)|^2 = p^2\tau^2\sum_{kj}\frac{jk}{(k+j)^2}\delta_{j,(p-k)} \\ &= \tau^2\sum_{k=1}^{p-1}(p-k)k = \frac{p(p^2-1)}{6}\tau^2,\end{aligned} \tag{4.33}$$

where we have defined the dimensionless time $\tau = \frac{1}{2}\epsilon\omega_1^{in}T$. Note that the above expression in agreement with Ref. [102]. It is also important to observe that in the specific scenario of motion, where the cavity oscillates in resonance with the p-th unperturbed field frequency, the application of

naive perturbation theory outlined in Eq. (4.11) introduces secular terms proportional to orders of ϵT, analogous to those found in Eq. (4.32). To see the inconvenience of those terms, if the timescale is of order $T \sim \frac{1}{\epsilon}$, then perturbation theory breaks down, and subsequent terms become greater than their predecessors. As a consequence, the expression (4.33) serves as a reliable approximation only when $\tau \ll 1$.

In respect to the thermodynamic entropy production, by considering the relation

$$|\beta_{jk}(\tau)|^2 = \frac{6}{p(p^2-1)} \frac{p^2 kj}{(k+j)^2} N(\tau)\delta_{k,(p-j)},$$

one can rewrite expression (4.19) in the oscillatory mirror configuration as

$$S_d(\tau) \;=\; \frac{1}{2}N(\tau)\left[1 - \ln\frac{1}{2}N(\tau) + \ln\frac{p(p^2-1)}{6} - \frac{6\,\mathrm{v}(p)}{p(p^2-1)}\right], \tag{4.34}$$

with

$$\mathrm{v}(p) = \sum_{k=1}^{p-1}(p-k)k\ln(p-k)k.$$

Figure 4.1 shows the diagonal entropy for this particular case. As it is clear from the figure, entropy will be produced in the field for every value of the mirror frequency p, except for $p = 1$, where the number of created particles vanishes.

The present approach, based on the effective Hamiltonian, allowed us to compute the entropy production in the system by means of the time evolution of the density operator. This leads to a direct connection between entropy production and the generation of coherences in the field. In the next section we rely on the Heisenberg picture and compute the entropy production in terms of the time evolution of Gaussian states. This allows us to investigate the contribution to the entropy of the entanglement between a single mode and the rest of the field.

4.2 Gaussian state approach

As shown in the previous section, a downside of the effective Hamiltonian approach is the restriction of the obtained results to the short-time regime, whenever the mirror oscillates in resonance with one of the unperturbed field frequencies. To address this issue, this section will introduce a distinct technique based on the evolution of Gaussian states in order to obtain expressions for the reduced diagonal entropy (in a specific field mode) for all time regimes. As will become evident later, such method will enable us to relate the irreversibility dynamics with that of the entanglement between the considered mode and the rest of the mode structure.

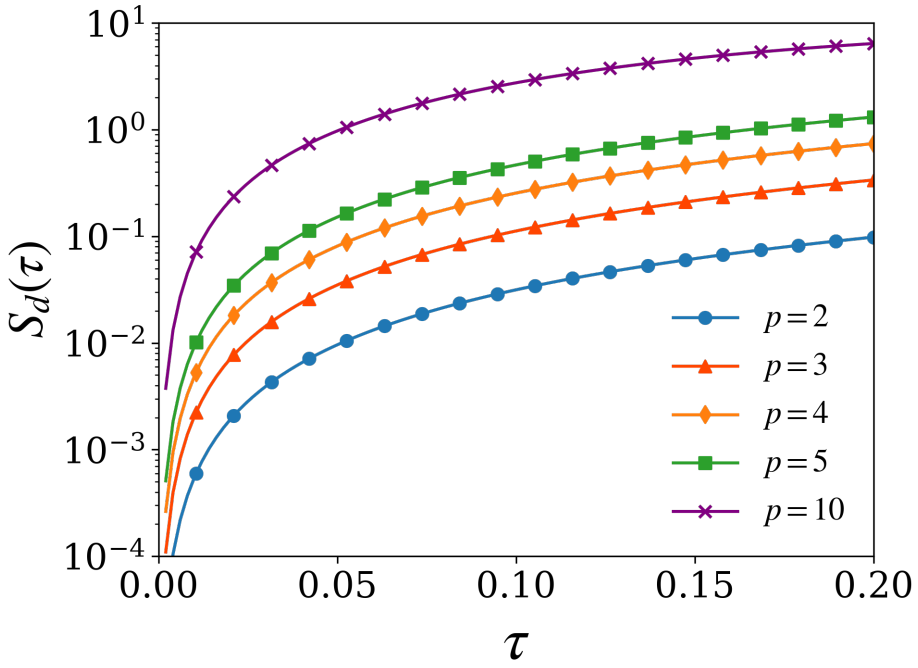

FIGURE 4.1: The diagonal entropy expression given by Eq. (4.34) is plotted in terms of the dimensionless time τ for distinct values of the frequency of oscillation of the mirror. It is important to observed that each point of the graphic represents the thermodynamical entropy for a given instant of time τ as if the cavity has stopped at the instantaneous cavity length $l(\tau)$.

4.2.1 Basic elements in the theory of Gaussian states

Before we delve into the theory of Gaussian states, let us first consider the field quadrature operators $\hat{q}_k$ and $\hat{p}_k$ in the time interval *out*: $t \geq T$, when the cavity has already returned to its static configuration as described by the Eqs. (3.78). Moving to the Schrödinger picture, their expressions take the form

$$\hat{q}_k = \frac{1}{\sqrt{2\omega_k^{out}}}\left(\hat{a}_k^{out} + \hat{a}_k^{out\dagger}\right), \tag{4.35a}$$

$$\hat{p}_k = -i\sqrt{\frac{\omega_k^{out}}{2}}\left(\hat{a}_k^{out} - \hat{a}_k^{out\dagger}\right), \tag{4.35b}$$

and both satisfies the following commutation relations

$$\left[\hat{q}_k, \hat{p}_j\right] = i\delta_{kj} \quad \text{and} \quad \left[\hat{q}_k, \hat{q}_j\right] = \left[\hat{p}_k, \hat{p}_j\right] = 0. \tag{4.36}$$

Although we are dealing with an infinite dimensional system, for presentation purpose, we will

consider the description of only a finite number N of modes. To this end, a more compact mathematical formulation for the system can be obtained with the introduction of the vector

$$\hat{\mathbf{R}} = \bigoplus_{k=1}^{N} \hat{\mathbf{r}}_k \qquad \text{with} \quad \hat{\mathbf{r}}_k = \frac{1}{\sqrt{\omega_k^{out}}}(\omega_k^{out}\hat{q}_k, \hat{p}_k)^T, \tag{4.37}$$

and the commutation relations, now summarized to a single expression

$$\left[\hat{R}_k, \hat{R}_j\right] = i\Omega_{kj}, \tag{4.38}$$

with the orthogonal matrix $\mathbf{\Omega}$ defining the system's simplectic form

$$\mathbf{\Omega} = \bigoplus_{k=1}^{N} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.39}$$

In this notation, the Heisenberg uncertainty relations are generalized to the so-called, *Robertson-Schrödinger uncertainty relations* [103, 104]

$$\mathbf{\Sigma} + \frac{i}{2}\mathbf{\Omega} \geq 0, \tag{4.40}$$

where $\mathbf{\Sigma}$ is the covariance matrix with elements

$$\Sigma_{kj} := \mathrm{Cov}(\hat{R}_k, \hat{R}_j) := \frac{1}{2}\langle \hat{R}_k\hat{R}_j + \hat{R}_j\hat{R}_k\rangle - \langle \hat{R}_k\rangle\langle \hat{R}_j\rangle, \tag{4.41}$$

from the standard definition of covariance $\mathrm{Cov}(\hat{A}, \hat{B}) = \frac{1}{2}\left\{\hat{A} - \langle\hat{A}\rangle, \hat{B} - \langle\hat{B}\rangle\right\}$ where $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$ is the anti-commutator of $\hat{A}$ and $\hat{B}$.

Coherent states

At this stage, it would be instructive for our discussion to introduce an important set of states spanning the single-mode Hilbert space $\mathscr{H}$: the so-called *coherent states*, denoted for a given field mode k as $|\alpha_k\rangle$. Perhaps the most prominent defining feature for the last set of states is captured by the relation

$$\hat{a}_k\,|\alpha_k\rangle = \alpha_k\,|\alpha_k\rangle\,,$$

which means that $\{|\alpha_k\rangle\}$ are eigenstates of the annihilation operator $\hat{a}_k^{out} \equiv \hat{a}_k$, with the eigenvalues $\{\alpha_k\}$ being complex numbers. In terms of the set of number states $\{|n_k\rangle\}$ of a given field mode k, one can represent the coherent state $|\alpha_k\rangle$ as [105]

$$|\alpha_k\rangle = e^{-\frac{1}{2}|\alpha_k|^2}\sum_{n_k}\frac{\alpha_k^{n_k}}{\sqrt{n_k!}}\,|n_k\rangle = \hat{D}(\alpha_k)\,|0_k\rangle\,. \tag{4.42}$$

where $\hat{D}(\alpha_k)$ is the displacement operator with the form

$$\hat{D}(\alpha_k) := e^{\alpha_k \hat{a}_k^\dagger - \alpha_k^* \hat{a}_k}. \tag{4.43}$$

In terms of the action of the last operator in the vacuum state $|0_k\rangle$ as in Eq. (4.42), one can also demonstrate two important relations for a given coherent state, say

$$\frac{1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2\alpha_k \, |\alpha_k\rangle \langle\alpha_k| = \mathbb{I} \qquad \text{(completeness)}, \tag{4.44a}$$

$$\langle\beta_k|\alpha_k\rangle = e^{\frac{1}{2}|\alpha_k - \beta_k|^2} e^{\frac{1}{2}\left(\alpha_k\beta_k^* - \alpha_k^*\beta_k\right)} \qquad \text{(non-orthogonality)}, \tag{4.44b}$$

where $\mathrm{d}^2\alpha_k := \mathrm{d}\,\mathrm{Re}[\alpha_k]\mathrm{d}\,\mathrm{Im}[\alpha_k]$ is the differential element in the complex plane. Indeed, as a direct consequence of the completeness relation (4.44a), the characterization of any bounded operator $\hat{O}$ (on the Hilbert space of one bosonic mode k) can be expressed in the complex plane in terms of the following identities

$$\mathrm{Tr}\left\{\hat{O}\right\} = \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2\alpha_k \, \langle\alpha_k| \hat{O} |\alpha_k\rangle \quad \textit{(trace relation)}, \tag{4.45a}$$

$$\hat{O} = \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2\alpha_k \, \mathrm{Tr}\left\{\hat{O}\hat{D}(\alpha_k)\right\}\hat{D}(-\alpha_k) \quad \textit{(Fourier-Weyl relation)}. \tag{4.45b}$$

Wigner functions

Having gathered all the results thus far, we are now in a position to discuss how the state of a given quantum system can be characterized in the coherent state formulation. In terms of the Fourier-Weyl identity introduced in Eq. (4.45b), the system's density operator $\hat{\rho}$ for a single bosonic mode takes the following form

$$\hat{\rho} = \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2\alpha_k \chi(\alpha_k)\hat{D}(\alpha_k), \tag{4.46}$$

where $\chi(\alpha_k)$ is system's *characteristic function* and is defined as

$$\chi(\alpha_k) := \mathrm{Tr}\left\{\hat{\rho}\hat{D}(-\alpha_k)\right\}. \tag{4.47}$$

This description essentially implies that having knowledge of $\chi(\alpha_k)$ provides complete information about the system's state $\hat{\rho}$. So, an alternative description for a quantum system can be obtained with the introduction of the Wigner quasi-probability function $W(\alpha_k)$, defined as the Fourier transform of the characteristic function in the complex plane, such as

$$W(\alpha_k) = \frac{1}{\pi}\int_{\mathbb{C}} \mathrm{d}^2\beta e^{\alpha_k\beta_k^* - \alpha_k^*\beta_k}\chi(\beta_k). \tag{4.48}$$

In particular, the Wigner function can be demonstrated to be normalized

$$\int_{\mathbb{C}} \mathrm{d}^2\alpha_k W(\alpha_k) = \chi(0) = \mathrm{Tr}\{\hat{\rho}\} = 1, \tag{4.49}$$

while the purity μ for the system's state $\hat{\rho}$ can be calculated through [106]

$$\mu(\hat{\rho}) = \mathrm{Tr}\left\{\hat{\rho}^2\right\} = \int_{\mathbb{C}} \mathrm{d}^2\alpha_k W^2(\alpha_k) = \int_{\mathbb{C}} \mathrm{d}^2\beta_k \chi^2(\beta_k). \tag{4.50}$$

Although we have defined W in the coherent state formulation, we can also rewrite it in terms of eigenvalues of the quadrature operators $\hat{q}_k$ and $\hat{p}_k$. In order to do this we begin considering the inverse relation $\hat{a}_k = \frac{1}{\sqrt{2\omega_k^{out}}}\left(\omega_k^{out}\hat{q}_k + i\hat{p}_k\right)$. With the latter, one can express the displacement operator as

$$\hat{D}(\alpha_k) = e^{\alpha_k \hat{a}_k^\dagger - \alpha_k^* \hat{a}_k} = e^{-i(q_k\hat{p}_k - p_k\hat{q}_k)} = e^{-i\mathbf{r}_k^T \mathbf{\Omega} \hat{\mathbf{r}_k}} = \hat{D}(-\hat{\mathbf{r}}_k), \tag{4.51}$$

where $\mathbf{r}_k = (q_k, p_k)^T \equiv \sqrt{\frac{2}{\omega_k^{out}}}(\mathrm{Re}\{\alpha_k\}, \omega_k^{out}\,\mathrm{Im}\{\alpha_k\})^T$ are the eigenvalues of $\hat{\mathbf{r}}_k$, as defined in Eq. (4.37). By converting the differential element on the complex plane as in $\mathrm{d}^2\alpha_k = \mathrm{d}\,\mathrm{Re}\{\alpha_k\}\mathrm{d}\,\mathrm{Im}\{\alpha_k\} = \frac{1}{2}\mathrm{d}q_k\mathrm{d}p_k$, we can finally represent the Wigner function in the phase space formulation as [107]

$$\begin{aligned}
W(q_k, p_k) &= \frac{1}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{2}\mathrm{d}q_k'\mathrm{d}p_k' e^{i\left(p_k q_k' - q_k p_k'\right)} \chi(q_k', p_k') && (4.52)\\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{d}q_k'\mathrm{d}p_k' e^{i\left(p_k q_k' - q_k p_k'\right)} \int_{\mathbb{R}} \mathrm{d}q_k \left\langle q_k\right| \hat{D}(-\hat{\mathbf{r}}'/2)\hat{\rho}\hat{D}(-\hat{\mathbf{r}}'/2)\left|q_k\right\rangle && (4.53)\\
&= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{\mathbb{R}} \mathrm{d}q_k\mathrm{d}q_k'\mathrm{d}p_k' e^{i\left(p_k q_k'\right)} e^{ip_k'\left(q_k - q_k'\right)} \left\langle q_k - q_k'/2\right| \hat{\rho} \left|q_k - q_k'/2\right\rangle && (4.54)\\
&= \int_{\mathbb{R}} \mathrm{d}q_k' e^{ip_k q_k'} \left\langle q_k - q_k'/2\right| \hat{\rho} \left|q_k - q_k'/2\right\rangle, && (4.55)
\end{aligned}$$

which is its standard presentation. From the last expression we can show the following properties

$$\begin{aligned}
\frac{1}{2}\int_{\infty}^{\infty} \mathrm{d}p_k\; W(q_k, p_k) &= \left\langle q_k\right| \hat{\rho} \left|q_k\right\rangle && (4.56)\\
\frac{1}{2}\int_{\infty}^{\infty} \mathrm{d}q_k\; W(q_k, p_k) &= \left\langle p_k\right| \hat{\rho} \left|p_k\right\rangle, && (4.57)
\end{aligned}$$

meaning that up to a factor $1/2$, the integral of the $W(q,p)$ over the quadrature parameter q_k (p_k), gives the probability of measuring the system in its conjugated quadrature parameter p_k (q_k) [107]. In fact, the Wigner function was introduced firstly by Eugene Wigner in 1932 [108] to map the language of wave functions to a phase space formulation of quantum mechanics. But contrary to initial hopes, $W(p,q)$ can not be truly probability distribution function since it can assign negative values.

Considering a system of N bosonic modes, the Displacement operator acting on the global Hilbert space can be defined as

$$\hat{D}(\boldsymbol{R}) = \bigotimes_{k=1}^{N} \hat{D}(\boldsymbol{r}_k), \tag{4.58}$$

where $\mathbf{R} = \oplus_{k=1}^{N} \mathbf{r}_k$ are the eigenvalues of $\hat{\mathbf{R}}$. In this representation one can rewrite the Wigner function definition as

$$W(\mathbf{R}) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^{2N}} \mathrm{d}^{2N}\boldsymbol{R}\; e^{i\boldsymbol{R}^T\boldsymbol{\Omega}\hat{\boldsymbol{R}}} \chi(\boldsymbol{R}), \tag{4.59}$$

for $\chi(\boldsymbol{R}) = \mathrm{Tr}\Big\{\hat{\rho}\hat{D}(-\mathbf{R})\Big\}$ and $\mathrm{d}^{2N}\mathbf{R} = \mathrm{d}q_1\mathrm{d}p_1 \ldots \mathrm{d}q_N\mathrm{d}p_N$.

4.2.2 Gaussian states

A N-mode state described by a density operator $\hat{\rho}$ is called a *Gaussian state* if its characteristic function is of Gaussian type, namely if

$$\chi_G(\mathbf{R}) = \exp\Big\{-\frac{\mathbf{1}}{\mathbf{2}}\mathbf{R^T\Omega\Sigma\Omega^T R} - \mathbf{i R^T\Omega}\langle\hat{\mathbf{R}}\rangle\Big\}. \tag{4.60}$$

The correspondent Wigner function for a N-mode Gaussian state then takes the form

$$W_G(\mathbf{R}) = \frac{\mathbf{1}}{\sqrt{\det \mathbf{\Sigma}}} \exp\Big\{-\frac{\mathbf{1}}{\mathbf{2}}\Big(\mathbf{R} - \langle\hat{\mathbf{R}}\rangle\Big)^{\mathbf{T}} \mathbf{\Sigma^{-1}} \Big(\mathbf{R} - \langle\hat{\mathbf{R}}\rangle\Big)\Big\}, \tag{4.61}$$

which is therefore, fully characterized by the averages values of $\hat{\mathbf{R}}$ (called statistical first moment) and its covariance matrix $\mathbf{\Sigma}$ (the system's second statistical moment).

Beyond this intrinsic simplicity of describing a system in terms of few parameters, the true importance of Gaussian states in our discussion comes from two results:

1. Unitary transformations that preserve the Gaussian character of a given state $\hat{\rho}_G$ (transforms any Gaussian state into another Gaussian state) are those generated by Hamiltonians which are at most quadratic in the canonical operator $\hat{\mathbf{R}}$, such as in [98, 107]

$$\hat{H} = \frac{1}{2}\hat{\mathbf{R}}^T\mathbf{H}\hat{\mathbf{R}} + \hat{\mathbf{R}}^{\mathbf{T}}\hat{\mathbf{R}}, \tag{4.62}$$

   where $\mathbf{H}$ is symmetric matrix.

2. If $\hat{H}$ is a quadratic Hamiltonian (as in Eq. (4.62)) with positive definite matrix $\mathbf{H}$, then any Gaussian state $\hat{\rho}_G$ can be written as a thermal state [107]

$$\hat{\rho}_G = \frac{e^{-\beta\hat{H}}}{\mathrm{Tr}\Big\{e^{-\beta\hat{H}}\Big\}}, \tag{4.63}$$

   where $\beta > 0$, including the limiting case $\beta \to \infty$ (ground state).

The importance of these properties will become evident in the next section, where we study the evolution of the diagonal entropy in the long-time regime. However, before delving into that, let's finish this discussion by briefly introducing some definitions of entanglement entropies within the Gaussian state formalism.

Given a pure state $|\psi\rangle$, the entanglement between two complementary subsystems A and B is contained in the reduced state $\hat{\rho}_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$. In special, one particularly useful measure of entanglement between the referred subsystems is characterized by the family of Rényi-α entropy

$$R_\alpha^{(A)}(\hat{\rho}_A) = \frac{\ln \mathrm{Tr}\{\hat{\rho}_A^\alpha\}}{1-\alpha}, \tag{4.64}$$

where α is any positive number different than 1. Two important special cases of Eq. (4.64) are the Rényi-2 entropy

$$S_R^{(A)} := R_2^{(A)} = -\ln \mathrm{Tr}\left\{\hat{\rho}_A^2\right\}, \tag{4.65}$$

and the von Neumann entropy

$$S_{vN}(\hat{\rho}_A) := \lim_{\alpha\to+1} R_\alpha^{(A)} = -\mathrm{Tr}\{\hat{\rho}_A \ln \hat{\rho}_A\}. \tag{4.66}$$

In the case of Gaussian states, one can show that the Rényi-2 entropy takes the following expression [107]

$$S_R^{(A)} = \frac{1}{2}\ln\left[\det \boldsymbol{\Sigma}_A\right], \tag{4.67}$$

where $\boldsymbol{\Sigma}_A$ is the covariance matrix for the reduced state.

4.2.3 Reduced density operator

Since the DCE induced by our system is described at all times by a Hamiltonian which is quadratic in terms of the creation and annihilation operators (see the effective Hamiltonian (4.6)), by initially preparing the system in the vacuum state $|0; in\rangle$ (which is a Gaussian state) one expects from the then the evolved state must also be of a Gaussian character and, therefore, be completely characterized by its covariant matrix and quadrature averages. The two results specified previously in Eqs. (4.62) and (4.63), open to us the possibility of studying the time evolution of the system's density operator and its correspondent thermodynamics entropy, in terms of a formalism grounded on Gaussian states. This becomes evident when recognizing that the DCE, induced by our idealized system, can be described for all times in terms of quadratic Hamiltonians. Consequently, if one consider the system to be initially prepared in the vacuum state $|0; in\rangle$ (which is a Gaussian state), then the evolved state must also be of a Gaussian character and, therefore, be completely characterized with the latter formalism.

But as exemplified by the previous section —where the time evolution of the density operator was restricted to the short-time regime— the intrinsic complexity of the system prevents us to perform a multimode description in the lines of a Gaussian state formalism. For this reason, we will proceed by restricting attention to the dynamics of the density operator (its diagonal terms in the energy eigenbasis) for a single-mode of the field.

More specifically, for a given pure Gaussian state described by a density matrix $\hat{\rho}$ (to be initially the vacuum state $|0; in\rangle \langle 0; in|$) we will consider the system to be decomposed into other two subsystems A and B: where A corresponds to a particular mode of the field, say the m-th field mode, while B consists in its complement, *i.e.*, all the rest of the field modes with $k \neq m$. In this respect, we define the reduced density operator of the mode m by

$$\hat{\rho}^{(m)} = \mathrm{Tr}_B\, \hat{\rho}, \tag{4.68}$$

where we have traced the degrees of freedom of B. Under these considerations, the most general single-mode Gaussian state can be characterized in terms of the following Wigner function for the m-th field mode [109, 110]

$$W_G(\mathbf{r}_m) = \frac{1}{\sqrt{\det \boldsymbol{\Sigma}_m}} \exp\Big\{-\frac{1}{2}\left(\mathbf{r}_m - \langle \hat{\mathbf{r}}_m \rangle\right)^T \boldsymbol{\Sigma}_m^{-1} \left(\mathbf{r}_m - \langle \hat{\mathbf{r}}_m \rangle\right)\Big\}, \tag{4.69}$$

where the m-th mode covariance matrix $\boldsymbol{\Sigma}_m$ stands for

$$\boldsymbol{\Sigma}_m \equiv \begin{pmatrix} \sigma_q^{(m)} & \sigma_{qp}^{(m)} \\ \sigma_{qp}^{(m)} & \sigma_p^{(m)} \end{pmatrix} \tag{4.70}$$

with

$$\sigma_q^{(m)} := \mathrm{Cov}(\hat{q}_m, \hat{q}_m) = \omega_m \left[\langle \hat{q}_m^2 \rangle - \langle \hat{q}_m \rangle^2\right], \tag{4.71a}$$

$$\sigma_p^{(m)} := \mathrm{Cov}(\hat{p}_m, \hat{p}_m) = \omega_m^{-1} \left[\langle \hat{p}_m^2 \rangle - \langle \hat{p}_m \rangle^2\right] \tag{4.71b}$$

$$\sigma_{qp}^{(m)} := \mathrm{Cov}(\hat{q}_m, \hat{p}_m) = \frac{1}{2}\langle \hat{p}_m \hat{q}_m + \hat{q}_m \hat{p}_m \rangle - \langle \hat{q}_m \rangle \langle \hat{p}_m \rangle, \tag{4.71c}$$

whose averages are computed in terms of initial state of the system. In this particular case of a single-mode system, the uncertainty relations of Eq. (4.40) constrain the covariance matrix to satisfy the following inequality

$$\det \boldsymbol{\Sigma}_m = \sigma_q^{(m)} \sigma_p^{(m)} - \left(\sigma_{qp}^{(m)}\right)^2 \geq \frac{1}{4}. \tag{4.72}$$

Moreover, since we are interested in the system's thermodynamic entropy, we focus on the diagonal components of the density operator in the initial energy eigenbasis, which is defined by

$$\rho_{nn}^{(m)} = \mathrm{Tr}\Big\{\hat{\rho}^{(m)}\, |n_m; in\rangle \langle n_m; in|\Big\}. \tag{4.73}$$

The problem then becomes, how can we extract these diagonal contributions of the reduced density operator from our Wigner function (4.69). This task was already performed in Refs. [109, 110], but for completeness we follow their steps for the special case of an initial vacuum state $|0; in\rangle$, where we expect the averages $\langle\hat{q}_m\rangle$, $\langle\hat{p}_m\rangle$ and σ_m^{qp} to vanish. For this purpose, we begin computing the elements of $\hat{\rho}^{(m)}$ in the coherent basis (from definition (4.42)) as

$$\langle\beta|\,\hat{\rho}^{(m)}\,|\alpha\rangle = e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}\sum_{nm}\frac{\beta^{*m}\alpha^m}{\sqrt{m!n!}}\rho_{nm}^{(m)}. \tag{4.74}$$

Another representation for the last expression can be introduced in terms of an overlap integral between Wigner functions

$$\langle\beta|\,\hat{\rho}^{(m)}\,|\alpha\rangle = \frac{1}{2\pi}\int \mathrm{d}q_m\mathrm{d}p_m W_G(q_m,p_m)W_{\alpha\beta}(q_m,p_m), \tag{4.75}$$

where $W_{\alpha\beta}(q_m,p_m)$ is the Wigner-Weyln transform of the operator $|\alpha\rangle\langle\beta|$ and has the explicit expression given by

$$\begin{aligned} W_{\alpha\beta}(q_m,p_m) &= \int_{-\infty}^{\infty}\mathrm{d}v\; e^{iq_m v}\langle p_m - v/2|\alpha\rangle\langle\beta|p_m+v/2\rangle \\ &= 2\exp\left\{-\frac{1}{2}(|\alpha|^2+|\beta|^2) - \alpha\beta_m^* + p_m^2 - q_m^2 - 2\alpha\gamma_m^* + 2\beta^*\gamma_m\right\}, \end{aligned} \tag{4.76}$$

with $\gamma_m = (q_m + ip_m)/\sqrt{2}$.

By putting Eq. (4.76) into Eq. (4.75) and computing the correspondent integral, one obtains

$$\langle\beta|\,\hat{\rho}_m\,|\alpha\rangle = 2\mathcal{P}_0 e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}\exp\left\{-\frac{1}{2}\boldsymbol{\alpha}^T\mathbf{R}\boldsymbol{\alpha}\right\}, \quad \text{with} \quad \mathcal{P}_0 = (1+2\,\mathrm{Tr}\{\boldsymbol{\Sigma}_m\}+4\det\boldsymbol{\Sigma}_m)^{-\frac{1}{2}}, \tag{4.77}$$

where $\boldsymbol{\alpha} = (\beta^*,\alpha)^T$and $\mathbf{R}$ is a 2×2 symmetric matrix with components

$$R_{11} = R_{22}^* = 2\left(\sigma_p^{(m)} - \sigma_q^{(m)}\right)\mathcal{P}_0^2, \tag{4.78}$$

$$R_{21} = R_{22} = (1-4\det\boldsymbol{\Sigma}_m)\,\mathcal{P}_0^2. \tag{4.79}$$

On the other hand, one can recognize the exponential term in Eq. (4.77) as the generating function of the Hermite polynomials of two variables $H_{nn}^{\boldsymbol{R}}$, as in

$$\exp\left\{-\frac{1}{2}\boldsymbol{\alpha}^T\mathbf{R}\boldsymbol{\alpha}\right\} = \sum_{nm}\frac{H_{nm}^{\boldsymbol{R}}}{n!m!}\alpha^n\beta^m. \tag{4.80}$$

Inserting Eq. (4.80) into Eq. (4.77) and comparing the resulting expression with Eq. (4.74) we can identify the diagonal elements of $\hat{\rho}^{(m)}$ as simply

$$\rho_{nn}^{(m)} = \mathcal{P}_0\frac{\mathrm{H}_{nn}^{\boldsymbol{R}}}{n!} = \mathcal{P}_0\,(-\det\boldsymbol{R})^{\frac{n}{2}}\,\mathrm{P}_n\left(-\frac{R_{12}}{\sqrt{-\det\boldsymbol{R}}}\right), \tag{4.81}$$

where the explicit expression for $H^{\boldsymbol{R}}_{nn}$ in terms of the Legendre polynomials P_n was derived in Ref. [109]. Expressing the last equation in terms of the quadrature variances of the field, we finally obtain the following expression for the diagonal terms of the reduced density operator [97, 98, 109, 110]

$$\rho^{(m)}_{nn} = \frac{2\left[\left(2\sigma^{(m)}_q - 1\right)\left(2\sigma^{(m)}_p - 1\right)\right]^{n/2}}{\left[\left(2\sigma^{(m)}_q + 1\right)\left(2\sigma^{(m)}_p + 1\right)\right]^{(n+1)/2}} \mathrm{P}_n\left(\frac{4\sigma^{(m)}_q\sigma^{(m)}_p - 1}{\sqrt{(4(\sigma^{(m)}_q)^2 - 1)(4(\sigma^{(m)}_p)^2 - 1)}}\right). \tag{4.82}$$

4.2.4 Obtaining differential equations for the quadrature operators

With expression (4.82) at hands, our task to calculate the diagonal entropy for a given field mode is severely facilitated. Now, we only need to obtain explicit expressions for the field variances $\sigma^{(m)}_q$ and $\sigma^{(m)}_p$. To do this we begin considering the special case in which the cavity returns (after an interval of time T) to its initial cavity length l_0, such that the initial and final frequencies must be equal $\omega^{in}_k = \omega^{out}_k = \omega_k$. Expressing the quadrature operators (4.35) in terms of the initial operators $\hat{a}^{in}_k$ and $\hat{a}^{in\dagger}_k$ with the Bogoliubov transformations (3.55), the variances can be directly computed, resulting in

$$\sigma^{(m)}_q = \frac{1}{2}\sum_k |\alpha_{km} + \beta_{km}|^2\,, \tag{4.83a}$$

$$\sigma^{(m)}_p = \frac{1}{2}\sum_k |\alpha_{km} - \beta_{km}|^2 \tag{4.83b}$$

where m is an odd integer and the cross term $\sigma^{(m)}_{qp}$ is identically zero for the choice of the initial state. By taking the time derivatives of these last equations in respect to the dimensionless time τ one obtains

$$\left.\begin{matrix}\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma^{(m)}_q \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\sigma^{(m)}_p\end{matrix}\right\} = \sum_k \mathrm{Re}\left[(\alpha_{km} \pm \beta_{km})\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\alpha_{km} \pm \frac{\mathrm{d}}{\mathrm{d}\tau}\beta_{km}\right)\right]. \tag{4.84}$$

Inserting the recursive relations (3.118a) and (3.118b) in Eqs. (4.84), one can shown that

$$\begin{aligned}\left.\begin{matrix}\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma^{(m)}_q \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\sigma^{(m)}_p\end{matrix}\right\} = \mp\,|\alpha_{1m} \pm \beta_{1m}|^2 - \sqrt{3}\,\mathrm{Re}\left[(\alpha_{1m} \pm \beta_{1m})(\alpha_{3m} \pm \beta_{3m})\right] \\ + \sum_{k=3}^{\infty}\sqrt{k(k-2)}\,\mathrm{Re}\left[(\alpha_{km} \pm \beta_{km})\left(\alpha_{(k-2)m} \pm \beta_{(k-2)m}\right)\right] \\ - \sum_{k=3}^{\infty}\sqrt{k(k+2)}\,\mathrm{Re}\left[(\alpha_{km} \pm \beta_{km})\left(\alpha_{(k+2)m} \pm \beta_{(k+2)m}\right)\right].\end{aligned} \tag{4.85}$$

By making the substitution $k \to k+2$ in the second term of Eq. (4.85), one can verify that almost all terms of the expression are canceled, and the final expression takes the following form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma_q^{(m)} = -|\alpha_{1m} + \beta_{1m}|^2 \tag{4.86a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma_p^{(m)} = +|\alpha_{1m} - \beta_{1m}|^2, \tag{4.86b}$$

which depends only on Bogoliubov coefficients with the first index equals to 1 (as we have pointed out in section 3.3.2). Moreover, because the definitions (4.35), the differential equations (4.86) need to satisfy the initial conditions $\sigma_q^{(m)}(0) = \sigma_p^{(m)}(0) = 1/2$. We now will analyze the solutions of these equations in two distinct regimes, the short-time and the long-time.

4.2.5 Short-time regime

In the short time limit, $\tau \ll 1$, one can obtain expressions for the quadrature variances by inserting the Bogoliubov coefficients (3.119) into Eq. (4.86) and integrating the resulting differential equation, such as in,

$$\begin{aligned}\left.\begin{matrix}\sigma_q^{(2\mu+1)} \\ \sigma_p^{(2\mu+1)}\end{matrix}\right\} &= \frac{1}{2} \mp J_\mu^2 \int_0^\tau \mathrm{d}t \left[(2\mu+1)t^{2\mu} \mp 2(\mu+1)K_\mu^2 t^{2\mu+1} + \mathcal{O}(t^{2\mu+2})\right] \Bigg\} \\ &= \frac{1}{2} \mp \tau^{2\mu+1} J_\mu^2 \left[1 \mp K_\mu^2 \tau + \mathcal{O}(\tau^2)\right],\end{aligned} \tag{4.87}$$

with J_μ and K_μ defined in Eq. (3.120).

By introducing expressions (4.87) into Eq. (4.82) we obtain the following expression for the diagonal components of the density operator

$$\begin{aligned}\rho_{nn}^{(2\mu+1)} = (-1)^n i^n J_\mu^n \tau^{n(2\mu+1)} \left(1 - K_\mu^4 \tau^2\right)^{n/2} \left[1 - (n+1)J_\mu^2 \tau^{2\mu+2}\left(K_\mu^2 - \frac{1}{2}J_\mu^2 \tau^{2\mu}\right)\right] \\ \times P_n\left[i\tau\left(K_\mu^2 - J_\mu^2 \tau^{2\mu}\right)\right] + \mathcal{O}(\tau^{2\mu+3}).\end{aligned} \tag{4.88}$$

Using the last expression, we compute the diagonal entropy at short-times for the $(2\mu+1)$-th mode. Consider only contributions up to second order in τ we obtain, for $\mu = 0$, the following result

$$\begin{aligned}S_d^1(\tau \ll 1) &= -\rho_1^{(0)} \ln \rho_1^{(0)} = -\left(1 - \frac{1}{2}K_0^2 J_0^2 \tau^2\right) \ln\left(1 - \frac{1}{2}K_0^2 J_0^2 \tau^2\right) \\ &= \frac{1}{2}N_1(\tau)\left[1 - \ln \frac{1}{2}N_1(\tau)\right],\end{aligned} \tag{4.89}$$

while for any other value of μ, we have

$$S_d^{2\mu+1}(\tau \ll 1) = -\rho_{2\mu+1}^{(0)} \ln \rho_{2\mu+1}^{(0)} = -\left(1 - K_\mu^2 J_\mu^2 \tau^2\right) \ln\left(1 - K_\mu^2 J_\mu^2 \tau^2\right)$$
$$= N_{2\mu+1}(\tau)\Big[1 - \ln N_{2\mu+1}(\tau)\Big] + \mathcal{O}(\tau^{2\mu+3}), \tag{4.90}$$

where the number of particle created in a certain mode $m = 2\mu + 1$ is given by the expression

$$N_{2\mu+1}(\tau) = K_\mu^2 J_\mu^2 \tau^{2\mu+2} + \mathcal{O}(\tau^{2\mu+3}).$$

Therefore, at short times, the entropy for each mode, grows directly proportional to the number of created particles, which is fully consistent with the results presented in the previous section. Such alignment is more explicitly demonstrated by considering the parametric resonance case ($p = 2$) in Eq. (4.34), which was derived using the effective Hamiltonian approach. By doing so, one can recover the exact expression (4.89) obtained through the Gaussian state formalism. This agreement arises because, in the case of the mirror's oscillatory motion (where the effective Hamiltonian analysis is restricted to the short-time regime), only the first field mode contributes to the particle creation process at the second order of the perturbation parameter ($\mathcal{O}(\epsilon^2)$).

As anticipated, the present approach also allows us to investigate the long-time behavior of the entropy production, and we proceed with such an analysis in what follows.

4.2.6 Long-time regime

In the long-time limit, $\tau \gg 1$, by substituting Eqs. (3.121) into Eqs. (4.86), we obtain the time derivatives of the system's quadrature variances as

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma_q^{(2\mu+1)} \approx 0 \tag{4.91a}$$
$$\frac{\mathrm{d}}{\mathrm{d}\tau}\sigma_p^{(2\mu+1)} \approx \frac{16}{\pi^2(2\mu+1)}. \tag{4.91b}$$

Consequently, their solutions can be expressed by the following set of equations:

$$\sigma_q^{(2\mu+1)} \approx A_{2\mu+1} \tag{4.92a}$$
$$\sigma_p^{(2\mu+1)} \approx \frac{16\tau}{\pi^2(2\mu+1)} + B_{2\mu+1}, \tag{4.92b}$$

where $A_{2\mu+1}$ and $B_{2\mu+1}$ are the correspondent constants of integration. The specific values for $A_{2\mu+1}$ and $B_{2\mu+1}$ varies from mode to mode, and it is dependent on the complete form of the Bogoliubov coefficients. However, their general behavior exhibit similarities: both quadrature variances begin with the same value 1/2 at $t = 0$ and evolve to assume distinct asymptotic behavior at $\tau \gg 1$. More specifically, $\sigma_q^{(m)}$ decrease to a constant value whereas $\sigma_p^{(m)}$ increases almost linearly in time.

In particular, explicit expressions for the first two modes were derived in Ref. [97, 98] and take the following form

$$\sigma_q^{(1)} \approx \frac{2}{\pi^2}; \qquad \sigma_p^{(1)} \approx \frac{16}{\pi^2}\tau + \frac{16}{\pi^2}\ln 2 - \frac{2}{\pi^2} \tag{4.93a}$$

$$\sigma_q^{(3)} \approx \frac{38}{9\pi^2}; \qquad \sigma_p^{(3)} \approx \frac{16}{3\pi^2}\tau + \frac{16}{3\pi^2}\ln 2 + \frac{10}{9\pi^2}. \tag{4.93b}$$

With this in mind, by considering that $\sigma_q^{(m)} \sim 1$ and $\sigma_p^{(m)} \sim \tau$, one can compute the single mode reduced density operator in the long-time regime by expanding Eq. (4.82) over orders $1/\tau$ such that

$$\rho_{nn}^{(m)}(\tau \gg 1) = C_n^{(m)} \left[\det \mathbf{\Sigma}_m(\tau)\right]^{-1/2} + \mathcal{O}(1/\tau) \tag{4.94}$$

where

$$C_n^{(m)} = \frac{1}{\sqrt{1+T_m}} \left(\frac{1-T_m}{\sqrt{1-T_m^2}}\right)^n \mathrm{P}_n\left(\frac{1}{\sqrt{1-T_m^2}}\right) \tag{4.95}$$

is a positive real coefficient with $T_m = 1/2\sigma_m^q$.

From Eq. (4.94), we can then compute the diagonal entropy for the m-th field mode by decomposing the logarithmic term in two parts

$$S_d^{(m)}(\tau \gg 1) \approx S_R^{(m)}(\tau) \sum_n \rho_{nn}^{(m)}(\tau) + \left[\det \mathbf{\Sigma}_m(\tau)\right]^{-1/2} \mathcal{S}(C_n^{(m)}), \tag{4.96}$$

where

$$S_R^{(m)}(\tau) = \frac{1}{2}\ln\left[\det \mathbf{\Sigma}_m(\tau)\right] \tag{4.97}$$

is the system's Rényi-2 entropy for the m-th mode as defined in Eq. (4.67) and

$$\mathcal{S}(C_n^{(m)}) = -\sum_n C_n^{(m)} \ln C_n^{(m)}. \tag{4.98}$$

But before interpreting the last expression in more details, let us check the convergence of each one of the summing terms. Consider the formula for the Legendre polynomial in the limit in which $n \gg 1$

$$\mathrm{P}_n\left(\frac{1}{\sqrt{1-e^2}}\right) = \frac{1}{\sqrt{2\pi n e}} \frac{(1+e)^{\frac{n+1}{2}}}{(1-e)^{n/2}} + \mathcal{O}(1/n). \tag{4.99}$$

If we introduce the last expression into Eq. (4.95), one can show that the coefficients $C_n^{(m)}$ take the following asymptotic limit

$$C_{n\gg 1}^{(m)} = \frac{1}{\sqrt{\pi n}}. \tag{4.100}$$

By employing the integral test for convergence, we introduce an upper bound for the infinite sum over n: $\sum_n C_n^{(m)} < \sum_n C_{n\gg 1}^{(m)} < \frac{1}{\sqrt{\pi}} \int_0^{\mathcal{N}} \frac{dn}{\sqrt{n}} = \frac{2}{\sqrt{\pi}}\sqrt{\mathcal{N}}$, where $\mathcal{N} \to \infty$ is the total number of modes. As a result, the sum over all the diagonal components of the reduced density operator (at the long-time

regime) can be shown to satisfy the following inequality

$$\sum_n \rho_{nn}^{(m)}(\tau \gg 1) = \sum_n C_n^{(m)} \left[\det \mathbf{\Sigma}_m\right]^{-1/2} < \frac{2}{\sqrt{\pi}} \frac{\sqrt{\mathcal{N}}}{\sqrt{\det \mathbf{\Sigma}_m}} \sim 1, \tag{4.101}$$

where we have considered that for the long-time regime, both the total number of modes $\mathcal{N}$ and the covariance matrix determinant $\det \mathbf{\Sigma}_m$ grow with the same order of magnitude (remember that $\det \mathbf{\Sigma}_m \sim \tau$ for $\tau \to \infty$ since $\sigma_q^{(m)} \sim 1$ and $\sigma_p^{(m)} \sim \tau$). As expected, Eq (4.101) is telling us that the trace of the reduced density operator converges, and it is bounded to be less than $2/\sqrt{\pi}(\approx 1{,}128\ldots)$. Indeed, by using the generating function for the Legendre polynomial, one can very easily prove that $\mathrm{Tr}\left\{\hat{\rho}^{(m)}\right\}$ given by Eq. (4.82) is exactly identical to unity.

In regard to the second term in Eq. (4.96), by substituting Eq. (4.100) into expression $S(C_n^{(m)}) \det \mathbf{\Sigma}_m^{-1/2}$, one can show that the correspondent summation can be restricted to satisfy the following inequality

$$\begin{aligned} S(C_n^{(m)}) \det \mathbf{\Sigma}_m^{-1/2} &< \frac{1}{2} \frac{1}{\sqrt{\det \mathbf{\Sigma}_m}} \int_0^{\mathcal{N}} \frac{\ln \pi n}{\sqrt{\pi n}} dn \\ &= \frac{1}{2} \left(\frac{2}{\sqrt{\pi}} \frac{\sqrt{\mathcal{N}}}{\sqrt{\det \mathbf{\Sigma}_m}} \ln \pi \mathcal{N} - \frac{4}{\sqrt{\pi}} \frac{\sqrt{\mathcal{N}}}{\sqrt{\det \mathbf{\Sigma}_m}} \right) \sim \frac{1}{2} \ln (\pi \mathcal{N}) - 1, \end{aligned}$$

meaning that the last series diverges logarithmically with the system's dimensionality. Indeed, this last fact is somehow expected since we are considering a field theory and the number of degrees of freedom of the system is infinite. However, as we are considering a finite period of time for the mirror motion, the energy injected into the system must be finite. As a result the field high energy modes are not expected to be excited and contribute to the correspondent divergence. Moreover, we must remember that the thermodynamic entropy is defined up to a multiplicative and an additive constant. So, this last term is not fundamental for the dynamical behavior of the entropy.

As a consequence we can identify the system's thermodynamic entropy for the m-th field mode to be simply identified as the Rényi-2 entropy

$$S_d^{(m)}(\tau \gg 1) \approx S_R^{(m)}(\tau). \tag{4.102}$$

Since the global state of the field is pure —initial pure state under unitary evolution— $S_R^{(m)}(\tau)$ then quantifies the amount of entanglement between the m-th mode and all the remaining modes. Therefore, what Eq. (4.96) is saying to us is that the asymptotic behavior of the diagonal entropy is fundamentally determined by the entanglement between the field modes. In particular, from the expressions given by Eq. (4.93) one can compute the system's thermodynamic entropy for the resonant mode $m = 1$,

$$S_d^{(1)}(\tau) \approx \frac{1}{2} \ln \frac{4}{\pi^4} \left(8\tau + 2 \ln 16 - 1\right) \sim \frac{1}{2} \ln \frac{32}{\pi^4} \tau,$$

which is in agreement with Ref. [59] [2]. In the case of the subsequent mode $m = 3$, we obtain

$$S_d^{(3)}(\tau) \approx \frac{1}{2} \ln \frac{380}{81\pi^4} \left(\frac{24}{5}\tau + \frac{6}{5} \ln 16 + 1 \right) \sim \frac{1}{2} \ln \frac{608}{27\pi^4}\tau.$$

[2] Here, the argument in the Rényi-2 entropy differs from Ref. [59] by a factor of 4. This occurs because the variances defined in the last reference are twice as large as the ones in Eq. (4.71).

Chapter 5

Conclusions and Perspectives

Throughout this work, we have investigated the thermodynamic entropy production in the DCE induced by the confinement of a quantum scalar field through a moving cavity. In practice, such calculations were performed within two distinct approaches: (i) using the dynamical evolution generated by an effective Hamiltonian, it was possible to connect the entropy production in the short-time regime with the generation of quantum coherence in the energy eigenbasis of the system; (ii) by means of a reduced density operator, we were also able to connect the entropy growth in a given mode (for all times) with the generation of entanglement between such mode and the rest of the field.

Together, both approaches provide us with distinct but complementary insights into the dynamical evolution of the production of thermodynamic entropy in the DCE. In summary, such irreversible process is determined by the generation of quantum coherence in the field's mode basis and the quantum entanglement between such modes. Further conclusions can be drawn by the properties of the diagonal entropy [78], which guarantees —because our choice of a diagonal initial state (vacuum state)— that the entropy production, as well as the coherence and the entanglement generation, can only either increase or remain constant, never decrease.

A better physical understanding on the last results can be enunciated as follows. Due to the process of particle creation (where we have transitions between instantaneous energy levels) caused by the non-trivial boundary conditions, one expects an initial state with definite energy value to rapidly evolve into a coherent superposition of excited states: a process called *delocalization of energy* [82]. Even though the process is unitary, since we are generally constrained by a limited set of possible measurements, information quickly spread throughout the Hilbert space, becoming increasingly inaccessible. This uncertainty on the energy measurement induced by the generation of coherence acts as the root of irreversibility, as predicted by the non-vanishing value for the system's diagonal entropy production [78, 82, 86, 87]. On top of that, in the one-dimensional case where the energy spectrum is equidistant, the time-dependent boundary conditions introduce a non-trivial coupling between the modes of the field. Due to this strong inter-mode interaction, the diagonal entropy production also predicts, at the long-time regime, the generation of entanglement between the field modes. Not only information about the energy value of the system is becoming inaccessible due

to the generation of coherence, but the entanglement among different parts of the system means that recovering the system's state must become increasingly difficult due to the strong correlations being created among the field modes. In this context, reversible processes are defined in the limit where the cavity motion is so slow that no particle creation, inter-mode scattering, or entropy production takes place. It is also interesting to notice, that for the special case involving resonant cavity trapping the field, there might exist motions by which no particles are created, and consequently, no entropy is produced. This aspect warrants deeper investigation.

Saying few words about the work as whole, we think our research enhances the understanding of thermodynamic considerations in quantum field theories under non-trivial conditions. The general message is that the source of irreversibility in the DCE can be traced back to the (almost unavoidable) transitions between the instantaneous energy values (due to the generation of coherence) and the strong intermode interaction which generate entanglement among different parts of the system. Despite this, our treatment leaves open several questions to be answered.

One immediate question that emerges when thinking about the thermodynamics of our system, is how exactly is the splitting of energy into work and heat contributions, with the latter being associated with the irreversible nature of the process, while the former characterizing the amount of energy that can be extracted from the field [111, 112]. The statistical description of the field in terms of the production of stochastic entropy and their correspondent fluctuation theorems [113] are also natural issues derived from our considerations. Since we only analyzed the long-time behavior of the diagonal entropy for a particular mode of the field, the role of multipartite entanglement and multiple quantum coherence to the process of entropy production is a relevant problem to be addressed. Finally, the specific thermalization properties of the field dynamics constitute another interesting question that remains unanswered.

Appendix A

Alternative derivation for the effective Hamiltonian

A.1 Dynamical equations for the instantaneous creation and annihilation operators

From Eqs. (3.4) and (3.5), the dynamical equation of motion for a quantum scalar field and its conjugated momentum can be written as

$$\partial_t \hat{\phi}(x,t) = \hat{\pi}(x,t) \tag{A.1a}$$

$$\partial_t \hat{\pi}(x,t) = \nabla^2 \hat{\phi}(x,t). \tag{A.1b}$$

By introducing the instantaneous decomposition (3.79) into the expansion Eqs. (3.61) and (3.66) and taking its time derivatives, one obtains

$$\hat{\phi} = \sum_k \frac{1}{\sqrt{2\omega_k(t)}} \left[\hat{a}_k(t) + \hat{a}_k^\dagger(t)\right] \varphi_k(x,t); \tag{A.2a}$$

$$\hat{\pi} = i \sum_k \sqrt{\frac{\omega_k(t)}{2}} \left[\hat{a}_k^\dagger(t) - \hat{a}_k(t)\right] \varphi_k(x,t); \tag{A.2b}$$

$$\dot{\hat{\phi}} = \sum_k \frac{1}{\sqrt{2\omega_k}} \left[\hat{a}_k + \hat{a}_k^\dagger\right] \left(\dot{\varphi}_k - \frac{\dot{\omega}_k}{2\omega_k}\varphi_k\right) + \sum_k \frac{1}{\sqrt{2\omega_k}} \left[\dot{\hat{a}}_k + \dot{\hat{a}}_k^\dagger\right] \varphi_k \tag{A.2c}$$

$$\dot{\hat{\pi}} = i \sum_k \sqrt{\frac{\omega_k}{2}} \left[\hat{a}_k^\dagger - \hat{a}_k\right] \left(\dot{\varphi}_k + \frac{\dot{\omega}_k}{2\omega_k}\varphi_k\right) + i \sum_k \sqrt{\frac{\omega_k}{2}} \left[\dot{\hat{a}}_k^\dagger - \dot{\hat{a}}_k\right] \varphi_k, \tag{A.2d}$$

where for conciseness we have suppressed the notation of time and spatial dependence in Eqs. s(A.2c) and (A.2d) and the upper dot convey time derivative. Comparing (A.1) with (A.2c) and (A.2d), we

can isolate the time derivative of the ladder operators by computing

$$\int_0^L \mathrm{d}x\ \varphi_j \left(\dot{\hat{\phi}} - \dot{\hat{\pi}}\right) = \sum_k \frac{1}{\sqrt{2\omega_k}} \left[\dot{\hat{a}}_k^\dagger + \dot{\hat{a}}_k\right] \delta_{kj} - \sum_k \frac{1}{\sqrt{2\omega_k}} \left[\hat{a}_k + \hat{a}_k^\dagger\right] \left(G_{jk} + \frac{\dot{\omega}_k}{2\omega_k}\delta_{kj}\right) - i \sum_k \sqrt{\frac{\omega_k}{2}} \left[\hat{a}_k^\dagger - \hat{a}_k\right] \delta_{kj} = 0 \tag{A.3}$$

$$\int_0^L \mathrm{d}x \varphi_j \left(\dot{\hat{\pi}} - \partial_x^2 \hat{\phi}\right) = i \sum_k \sqrt{\frac{\omega_k}{2}} \left(\dot{\hat{a}}_k^\dagger - \dot{\hat{a}}_k\right) \delta_{kj} + i \sum_k \sqrt{\frac{\omega_k}{2}} \left[\hat{a}_k^\dagger - \hat{a}_k\right] \left(G_{kj} + \frac{\dot{\omega}_j}{2\omega_j}\delta_{kj}\right) + \sum_k \frac{\omega_k^2}{\sqrt{2\omega_k}} \left[\hat{a}_k + \hat{a}_k^\dagger\right] \delta_{kj} = 0 \tag{A.4}$$

where it was used $\int_0^L dx \varphi_k \varphi_j = \delta_{kj}$ and $G_{jk} \coloneqq \int_0^L \varphi_k \partial_t \varphi_j$. By defining $\mu_{kj} = \sqrt{\frac{\omega_j}{\omega_k}} \left(G_{jk} + \frac{\dot{\omega}_k}{2\omega_k}\delta_{kj}\right)$ we obtain from (A.3) and (A.4) the following equations

$$\dot{\hat{a}}_j(t) + \dot{\hat{a}}_j^\dagger(t) - i\omega_j(t) \left[\hat{a}_j^\dagger(t) - \hat{a}_j(t)\right] = \sum_k \mu_{kj}(t) \left[\hat{a}_k^\dagger(t) + \hat{a}_k(t)\right], \tag{A.5a}$$

$$\dot{\hat{a}}_j(t) - \dot{\hat{a}}_j^\dagger(t) + i\omega_j(t) \left[\hat{a}_j(t) + \hat{a}_j^\dagger(t)\right] = \sum_k \mu_{jk}(t) \left[\hat{a}_k^\dagger(t) - \hat{a}_k(t)\right]. \tag{A.5b}$$

From the last system, it is easy to isolate $\dot{\hat{a}}_j(t)$ and $\dot{\hat{a}}_j^\dagger(t)$ as

$$\dot{\hat{a}}_j(t) = -i\omega_j(t)\hat{a}_j(t) + \sum_k \left[\mu_{[k,j]}(t) a_k(t) + \mu_{(k,j)}(t) a_k^\dagger(t)\right], \tag{A.6a}$$

$$\dot{\hat{a}}_j^\dagger(t) = +i\omega_j(t)\hat{a}_j^\dagger(t) + \sum_k \left[\mu_{[k,j]}(t) a_k^\dagger(t) + \mu_{(k,j)}(t) a_k(t)\right], \tag{A.6b}$$

with

$$\mu_{[k,j]}(t) = \frac{1}{2} \left[\mu_{kj}(t) - \mu_{jk}(t)\right] e^{-i[\Omega_k(t) - \Omega_j(t)]}, \tag{A.7a}$$

$$\mu_{(k,j)}(t) = \frac{1}{2} \left[\mu_{kj}(t) + \mu_{jk}(t)\right] e^{-i[\Omega_k(t) + \Omega_j(t)]}. \tag{A.7b}$$

Since $\omega_k(t) = k\pi/L(t)$ and using the definition (3.58) we can calculate

$$\begin{aligned} G_{kj}(t) &= \sqrt{\frac{2}{L(t)}}\int_{x_0}^{x_0+L(t)} dx \sin\left[\frac{j\pi(x-x_0)}{L(t)}\right]\frac{d}{dt}\left\{\sqrt{\frac{2}{L(t)}}\sin\left[\frac{k\pi(x-x_0)}{L(t)}\right]\right\} \\ &= -\frac{1}{L(t)}\frac{\dot{L}(t)}{L(t)}\int_{x_0}^{x_0+L(t)} dx \sin\left[\frac{j\pi(x-x_0)}{L(t)}\right]\sin\left[\frac{k\pi}{L(t)}(x-x_0)\right] \\ &- \frac{2k\pi}{L^2(t)}\frac{\dot{L}(t)}{L(t)}\int_{x_0}^{x_0+L(t)} dx\ x \sin\left[\frac{j\pi(x-x_0)}{L(t)}\right]\cos\left[\frac{k\pi(x-x_0)}{L(t)}\right] \\ &= \frac{1}{2}\frac{\dot{L}(t)}{L(t)}\delta_{kj} + \begin{cases} (-1)^{j-k}\frac{2kj}{j^2-k^2}\frac{\dot{L}(t)}{L(t)} & j \neq k \\ -\frac{1}{2}\frac{\dot{L}(t)}{L(t)} & j = k \end{cases} = g_{kj}\frac{\dot{L}(t)}{L(t)}, \end{aligned}$$

and

$$\frac{\dot{\omega}_k(t)}{\omega_k(t)} = -\frac{\dot{L}(t)}{L(t)}, \tag{A.8}$$

where g_{kj} has the same form as expressed in (3.64).

A.2 Effective Hamiltonian

To find the effective Hamiltonian that generates the dynamical equations (A.7) we begin by considering the most general quadratic operator

$$\hat{H}(t) = \sum_{kl}\left[\mathcal{A}_{kl}(t)\hat{a}_k^\dagger(t)\hat{a}_l^\dagger(t) + \mathcal{B}_{kl}(t)\hat{a}_k^\dagger(t)\hat{a}_l(t) + \mathcal{C}_{kl}(t)\hat{a}_l^\dagger(t)\hat{a}_k(t) + \mathcal{D}_{kl}(t)\hat{a}_k(t)\hat{a}_l(t)\right], \tag{A.9}$$

which is: (i) hermitian, by satisfying the conditions $\mathcal{A}_{kl}(t) = \mathcal{D}_{kl}^*(t)$, $\mathcal{B}_{kl}(t) = \mathcal{C}_{kl}^*(t)$ and (ii) invariant over an index change, with the conditions $\mathcal{A}_{kl}(t) = \mathcal{A}_{lk}(t)$, $\mathcal{D}_{kl}(t) = \mathcal{D}_{lk}(t)$, $\mathcal{B}_{kl}(t) = \mathcal{C}_{lk}(t)$ and $\mathcal{B}_{lk}(t) = \mathcal{C}_{kl}(t)$.

Suppressing the notation for time dependence, the correspondent Heisenberg equation of motion for the annihilation and creation operators is therefore

$$\begin{aligned} \dot{\hat{a}}_j = i\left[\hat{H}, \hat{a}_j\right] &= i\sum_{kl}\left(\mathcal{A}_{kl}\left[\hat{a}_k^\dagger\hat{a}_l^\dagger, \hat{a}_j\right] + \mathcal{B}_{kl}\left[\hat{a}_k^\dagger\hat{a}_l, \hat{a}_j\right] + \mathcal{C}_{kl}\left[\hat{a}_l^\dagger\hat{a}_k, \hat{a}_j\right] + \mathcal{D}_{kl}[\hat{a}_k\hat{a}_l, \hat{a}_j]\right) \\ &= -i\sum_k\left[\left(\mathcal{A}_{kj} + \mathcal{A}_{jk}\right)\hat{a}_k^\dagger + \left(\mathcal{B}_{jk} + \mathcal{C}_{kj}\right)\hat{a}_k\right] \end{aligned} \tag{A.10}$$

and

$$\begin{aligned}\dot{\hat{a}}_j^\dagger = i\left[\hat{H}, \hat{a}_j^\dagger\right] &= i\sum_{kl}\left(\mathcal{A}_{kl}\left[\hat{a}_k^\dagger\hat{a}_l^\dagger, \hat{a}_j^\dagger\right] + \mathcal{B}_{kl}\left[\hat{a}_k^\dagger\hat{a}_l, \hat{a}_j^\dagger\right] + \mathcal{C}_{kl}\left[\hat{a}_l^\dagger\hat{a}_k, \hat{a}_j^\dagger\right] + \mathcal{D}_{kl}\left[\hat{a}_k\hat{a}_l, \hat{a}_j^\dagger\right]\right) \\ &= i\sum_k\left[\left(\mathcal{D}_{kj} + \mathcal{D}_{jk}\right)\hat{a}_k + \left(\mathcal{B}_{kj} + \mathcal{C}_{jk}\right)\hat{a}_k^\dagger\right]. \end{aligned} \tag{A.11}$$

Comparing (A.6a) with (A.10) and (A.6b) with (A.11), we obtain the following system

$$\begin{aligned} -i\left[\mathcal{A}_{kj}(t) + \mathcal{A}_{jk}(t)\right] &= -2i\mathcal{A}_{kj}(t) = \mu_{(k,j)}(t), \\ -i\left[\mathcal{C}_{kj}(t) + \mathcal{B}_{jk}(t)\right] &= -2i\mathcal{C}_{kj}(t) = -i\omega_k\delta_{kj} + \mu_{[k,j]}(t), \\ i\left[\mathcal{D}_{kj}(t) + \mathcal{D}_{jk}(t)\right] &= 2i\mathcal{D}_{kj}(t) = \mu_{(k,j)}(t), \\ i\left[\mathcal{B}_{kj}(t) + \mathcal{C}_{jk}(t)\right] &= 2i\mathcal{B}_{kj}(t) = i\omega_k\delta_{kj} + \mu_{[k,j]}(t). \end{aligned}$$

Inserting the last coefficients into Eq. (A.9), one obtains the following expression for the effective Hamiltonian

$$\hat{H}_H(t) = \sum_k \omega_k(t)\hat{a}_k^\dagger(t)\hat{a}_j(t) - \frac{i}{2}\sum_{jk}\left[\mu_{[k,j]}(t)\hat{a}_j^\dagger(t)\hat{a}_k(t) + \mu_{(k,j)}(t)\hat{a}_j^\dagger(t)\hat{a}_k^\dagger(t) - \text{H.c.}\right], \tag{A.12}$$

where the subscript H conveys that the operator is represented in the Heisenberg picture of quantum mechanics. Moving to the Schrodinger picture, the last Hamiltonian takes the form

$$\hat{H}_S(t) = \sum_k \omega_k(t)\hat{a}_k^{in\dagger}\hat{a}_j^{in} - \frac{i}{2}\sum_{jk}\left[\mu_{[k,j]}(t)\hat{a}_j^{in\dagger}\hat{a}_k^{in} + \mu_{(k,j)}(t)\hat{a}_j^{in\dagger}\hat{a}_k^{in\dagger} - \text{H.c.}\right], \tag{A.13}$$

where the Heisenberg annihilation (and creations) operator is defined as $\hat{a}_k(t) = \hat{U}_S^\dagger(t)\hat{a}_k^{in}\hat{U}_S(t)$, with $\hat{U}_S(t)$ being the time evolution operator generated by the Hamiltonian (A.13).

Appendix B

Coupled differential equations for the Fourier coefficients

Let the differential equation for the generalized function $Q_k^n(t)$

$$\ddot{Q}_k^n + \omega_k^2(t)Q_k^n = 2\lambda(t)\sum_j g_{kj}\dot{Q}_j^n + \dot{\lambda}(t)\sum_j g_{kj}Q_j^n + \mathcal{O}(\lambda^2).$$

Here we will concentrate in the parametric amplification of the fundamental cavity mode. Therefore, we impose the following equation of motion for the moving mirror

$$l(t) = l_0\left[1 + \epsilon\sin\left(\omega_p t\right)\right]. \tag{B.1}$$

We search for solutions

$$Q_k^n(t) = \xi_k^n(t)e^{-i\omega_k t} + \eta_k^n(t)e^{i\omega_k t}. \tag{B.2}$$

Here we consider $\ddot{\xi}, \ddot{\eta}, \dot{l} \sim \epsilon^2$, up to second order in ϵ one obtains

$$\lambda(t) = \frac{\dot{l}(t)}{l(t)} = \epsilon\omega_p\cos\omega_p t = \frac{\epsilon\omega_p}{2}\left[e^{i\omega_p t} + e^{-i\omega_p t}\right] \tag{B.3}$$

$$\dot{\lambda}(t) = \frac{\ddot{l}(t)}{l(t)} = -\epsilon\omega_p^2\sin\omega_p t = i\frac{\epsilon\omega_p^2}{2}\left[e^{i\omega_p t} - e^{-i\omega_p t}\right] \tag{B.4}$$

as well as

$$\dot{Q}_k^n = \left[\dot{\xi}_k^n e^{-i\omega_k t} + \dot{\eta}_k^n e^{i\omega_k t}\right] - i\omega_k\left[\xi_k^n e^{-i\omega_k t} - \eta_k^n e^{i\omega_k t}\right] \tag{B.5a}$$

$$\ddot{Q}_k^n \approx -2i\omega_k\left[\dot{\xi}_k^n e^{-i\omega_k t} - \dot{\eta}_k^n e^{i\omega_k t}\right] - \omega_k^2\left[\xi_k^n e^{-i\omega_k t} + \eta_k^n e^{i\omega_k t}\right]. \tag{B.5b}$$

Inserting (B.5) into the left side of Eq. (3.105) and multiplying both sides by $\exp\{\pm i\omega_k t\}$, we end up with

$$\begin{aligned}\mathcal{E}_1^{\pm} &:= \left[\ddot{Q}_k^n + \omega_k^2(t)Q_k^n\right]\times e^{\pm i\omega_k t} \\ &= -2i\omega_k\left[\dot{\xi}_k^n e^{-i(\omega_k\mp\omega_k)t} - \dot{\eta}_k^n e^{i(\omega_k\pm\omega_k)t}\right] - 2\epsilon\sin\omega_p t\left[\xi_k^n e^{-i(\omega_k\mp\omega_k)t} + \eta_k^n e^{i(\omega_k\mp\omega_k)t}\right]\end{aligned} \tag{B.6}$$

as for the right-side, by considering $\epsilon\dot{\xi}, \epsilon\dot{\eta} \sim \epsilon^2$ we have

$$\begin{aligned}\mathcal{E}_2^{\pm} &:= \left[2\lambda \sum_j g_{kj}\dot{Q}_j^n + \dot{\lambda}\sum_j g_{kj}Q_j^n\right] \times e^{\pm i\omega_k t} \\ &= -i\epsilon\omega_p \sum_j g_{kj}\omega_j \left\{\left[e^{i(\omega_p-\omega_j\pm\omega_k)t} + e^{-i(\omega_p+\omega_j\mp\omega_k)t}\right]\xi_j^n - \left[e^{i(\omega_p+\omega_j\pm\omega_k)t} + e^{-i(\omega_p-\omega_j\mp\omega_k)t}\right]\eta_j^n\right\} \\ &+ i\frac{\epsilon\omega_p^2}{2}\sum_j g_{kj}\left\{\left[e^{i(\omega_p-\omega_j\pm\omega_k)t} - e^{-i(\omega_p+\omega_j\mp\omega_k)t}\right]\xi_j^n + \left[e^{i(\omega_p+\omega_j\pm\omega_k)t} - e^{-i(\omega_p-\omega_j\mp\omega_k)t}\right]\eta_j^n\right\}.\end{aligned} \tag{B.7}$$

We then perform the method of averaging over fast oscillations with the frequencies proportional to ω_k. For the case $k \geq p$ all the terms $e^{\pm 2i\omega_k}$, $e^{i[\omega_p\pm(\omega_k+\omega_j)]t}$, $e^{-i[\omega_p\pm(\omega_k+\omega_j)]t}$, $\sin\omega_p t \to 0$ whereas $e^{i[\omega_p\mp(\omega_j-\omega_k)]t}$, $e^{-i[\omega_p\mp(\omega_j-\omega_k)]t} \to \delta_{k,k\pm p}$, meaning we obtain

$$\mathcal{E}_1^{\pm} = \begin{cases} -2i\omega_k\dot{\xi}_k^n & \text{for} \quad \times e^{+i\omega_k t} \\ +2i\omega_k\dot{\eta}_k^n & \text{for} \quad \times e^{-i\omega_k t}\end{cases} \tag{B.8}$$

and

$$\begin{aligned}\mathcal{E}_2^{\pm} &= \begin{cases} -i\epsilon\omega_p\left\{g_{k,k+p}\omega_{k+p}\xi_{k+p}^n + g_{k,k-p}\omega_{k-p}\xi_{k-p}^n\right\} + i\frac{\epsilon\omega_p^2}{2}\left[g_{k,k+p}\xi_{k+p}^n - g_{k,k-p}\xi_{k-p}^n\right] & \text{for} \quad \times e^{+i\omega_k t} \\ +i\epsilon\omega_p\left\{g_{k,k+p}\omega_{k+p}\eta_{k+p}^n + g_{k,k-p}\omega_{k-p}\eta_{k-p}^n\right\} + i\frac{\epsilon\omega_p^2}{2}\left[g_{k,k-p}\eta_{k-p}^n - g_{k,k+p}\eta_{k+p}^n\right] & \text{for} \quad \times e^{-i\omega_k t}\end{cases} \\ &= \begin{cases} -i\epsilon\omega_k\omega_1(-1)^p\left[(k+p)\xi_{k+p}^n - (k-p)\xi_{k-p}^n\right] & \text{for} \quad \times e^{+i\omega_k t} \\ +i\epsilon\omega_k\omega_1(-1)^p\left[(k+p)\eta_{k+p}^n - (k-p)\eta_{k-p}^n\right] & \text{for} \quad \times e^{-i\omega_k t}\end{cases}\end{aligned} \tag{B.9}$$

where were used the identities

$$\begin{aligned} g_{k,k\pm p} &= (-1)^{\mp p}\frac{2k(k\pm p)}{p(p\pm 2k)} \\ \omega_{k\pm p} &= \omega_k \pm \omega_p. \end{aligned}$$

From Eqs. (B.8) and (B.9) one then obtain the following set of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\xi_k^{(n)} = (-1)^p\left[(k+p)\xi_{k+p}^n - (k-p)\xi_{k-p}^n\right], \tag{B.10a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\eta_k^{(n)} = (-1)^p\left[(k+p)\eta_{k+p}^n - (k-p)\eta_{k-p}^n\right], \tag{B.10b}$$

with the small time $\tau = \frac{1}{2}\epsilon\omega_1 t$.

On the other hand, for the case in which $k \leq p$, when applying the method of averaging over fast oscillations for Eqs. (B.6) and (B.7) we expect $e^{\pm 2i\omega_k}$, $e^{i[\omega_p+(\omega_j\pm\omega_k)]t}$, $e^{-i[\omega_p+(\omega_j\pm\omega_k)]t}$, $\sin\omega_p t \to 0$

whereas $e^{i[\omega_p-(\omega_j\mp\omega_k)]t}$, $e^{-i[\omega_p-(\omega_j\mp\omega_k)]t} \to \delta_{k,p\pm k}$. Following the same procedure as before we find

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\xi_k^{(n)} = (-1)^p \left[(p+k)\xi_{k+p}^n - (p-k)\eta_{p-k}^n\right], \tag{B.11a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\eta_k^{(n)} = (-1)^p \left[(p+k)\eta_{k+p}^n - (p-k)\xi_{p-k}^n\right]. \tag{B.11b}$$

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